*Publ. Math. Debrecen*
**93/1-2** (2018), 163–170
DOI: 10.5486/PMD.2018.8125

# Characterization of the Euler gamma function with the aid of an arbitrary mean

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**Abstract.** We prove that a continuous function $f : (0,\infty) \to (0,\infty)$ satisfying the functional equation

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1,$$

is the Euler gamma function iff for some $a > 0$ and a strict and continuous mean $M : (a,\infty)^2 \to (a,\infty)$, the following inequality holds:

$$f(M(x,y))\, f\left(\frac{xy}{M(x,y)}\right) \leq f(x)\, f(y), \quad x,y \in (a,\infty).$$

Taking for $M$ the geometric mean $G(x,y) = \sqrt{xy}$, we obtain the result of [2] generalizing the classical Bohr–Mollerup theorem [1]. For $M = A$, where $A(x,y) = \frac{x+y}{2}$ is the arithmetic mean, the assumed inequality reduces to $f(A(x,y))\, f(H(x,y)) \leq f(x)\, f(y)$ for all $x,y > a$, where $H$ is the harmonic mean, and the result gives a new characterization of the gamma function, involving the arithmetic and harmonic means.

## 1. Introduction

According to the celebrated result of Bohr and Mollerup [1], the Euler gamma function $\Gamma$ is the only function $f : (0,\infty) \to (0,\infty)$ satisfying the functional equation

$$f(x+1) = xf(x) \quad \text{for all } x > 0,\ f(1) = 1, \tag{1}$$

and such that $\log \circ f$ is convex.

*Mathematics Subject Classification:* Primary: 33B15, 26E30; Secondary: 39B12.
*Key words and phrases:* gamma function, mean, convexity, iteration, functional equation.

This theorem has been improved in [2], where it is shown that $\Gamma$ *is the only function* $f:(0,\infty)\to(0,\infty)$ *satisfying* (1) *and such that* $\log\circ f\circ\exp$ *is convex in a vicinity of* $\infty$. Interpreting this result, note that [2], for a positive real function $f$ defined in an interval $I\subset(0,\infty)$ and continuous at least at one point, the function $\log\circ f\circ\exp$ is convex in the interval $\log(I)$, iff $f$ is *Jensen geometrically convex* in $I$, i.e., iff

$$f(G(x,y))\leq G(f(x)f(y)),\quad x,y\in I,$$

where $G(x,y):=\sqrt{xy}$ is the geometric mean.

In the present paper, we show that any two-variable strict and continuous mean defined in a vicinity of $\infty$ can be used in a characterization of the Euler gamma function. The main results imply that: *a continuous function* $f:(0,\infty)\to(0,\infty)$ *satisfying* (1) *is the Euler gamma function iff there is an* $a>0$ *and a strict continuous mean* $M:(a,\infty)^2\to(a,\infty)$ *such that, for all* $x,y>a$,

$$f(M(x,y))\,f\left(\frac{xy}{M(x,y)}\right)\leq f(x)\,f(y).$$

Taking $M=G$, one gets the result of [2]; taking $M=A$, where $A$ is the arithmetic mean, we obtain the following characterization of the gamma function: *a continuous function* $f:(0,\infty)\to(0,\infty)$ *satisfying* (1) *is the Euler gamma function, if and only if there is an* $a>0$ *such that*

$$f(A(x,y))\,f(H(x,y))\leq f(x)\,f(y),\quad x,y\in(a,\infty),$$

*where* $H$ *is the harmonic mean.*

## 2. Auxiliary results

Let $I\subset\mathbb{R}$ be an interval. A function $M:I\times I\to I$ is called a *bivariable mean in* $I$ if

$$\min(x,y)\leq M(x,y)\leq\max(x,y),\quad x,y\in I,\tag{2}$$

and the mean $M$ is called *strict*, if these inequalities are sharp for all distinct $x,y\in I$.

We shall need the following:

**Lemma 1** ([5, Theorem 1])**.** *Let $M : I^2 \to I$ and $N : I^2 \to I$ be continuous means in $I$. If, for all $x, y \in I$, $x \neq y$,*

$$\max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \max(x,y) - \min(x,y), \quad (3)$$

*then there exists a unique mean $K : I^2 \to I$ that is invariant with respect to the mean-type mapping $(M, N) : I^2 \to I^2$, i.e., such that*

$$K(M(x,y), N(x,y)) = K(x,y), \quad x, y \in I;$$

*moreover, the sequence $((M,N)^n)_{n\in\mathbb{N}}$ of iterates of the mapping $(M, N)$ converges pointwise in $I^2$ and*

$$\lim_{n\to\infty} (M,N)^n (x,y) = (K(x,y), K(x,y)), \quad (x,y) \in I^2.$$

*Remark 1.* It is obvious that condition (3) is satisfied if one of the means $M$ or $N$ is strict.

From Lemma 1, we obtain

**Lemma 2.** *Let $I \subseteq (0,\infty)$ be an interval, $M : I^2 \to I$ be a (strict) mean, and let $N : I^2 \to I$ be given by*

$$N(x,y) := \frac{xy}{M(x,y)}, \quad x, y \in I.$$

*Then*

(i) *the function $N$ is a (strict) mean in $I$;*

(ii) *the geometric mean $G$ is invariant with respect the mean-type mapping $(M, N)$, i.e., $G \circ (M, N) = G$;*

(iii) *for every $n \in \mathbb{N}$, the mapping $(M, N)^n$, the $n$-th iterate of $(M, N)$, is a mean-type mapping;*

(iv) *if $M$ is a continuous and strict mean, then the sequence $((M,N)^n)_{n\in\mathbb{N}}$ of iterates of $(M, N)$ converges pointwise in $I^2$ and*

$$\lim_{n\to\infty} (M,N)^n (x,y) = (\sqrt{xy}, \sqrt{xy}), \quad (x,y) \in I^2.$$

Proof. To prove (i), it is enough to observe that condition (2) is equivalent to

$$\min(x,y) \leq \frac{xy}{M(x,y)} \leq \max(x,y), \quad x, y \in I,$$

and these inequalities are sharp iff so are inequalities (2).

To verify (ii), note that for all $x, y \in I$, we have

$$G \circ (M(x,y), N(x,y)) = \sqrt{M(x,y)\frac{xy}{M(x,y)}} = \sqrt{xy} = G(x,y).$$

Part (iii) is an obvious consequence of the definition of mean. Part (iv) follows from (ii) and Lemma 1. □

*Remark 2.* In [4], the mean $N$ such that $G \circ (M, N) = G$ is referred to as the complementary to $M$ with respect to $M$.

Let us also quote

**Lemma 3** ([2, Corollary 1])**.** *If a function $f : (0,\infty) \to (0,\infty)$ satisfying (1) is continuous at a point (or bounded above at a point) and there is $a > 0$ such that $f$ is Jensen geometrically convex in the interval $(a.\infty)$, i.e.,*

$$f(G(x,y)) \leq G(f(x), f(y)), \quad x, y > a,$$

*then $f = \Gamma$.*

In view of the Bernstein–Doetsch theorem, if $f$ is bounded from above in a neighborhood of a point ([3, Kuczma, p. 145]), the function $\log \circ f \circ \exp$ is Jensen convex in the interval $\log(I)$ iff

$$f\left(x^t y^{1-t}\right) \leq [f(x)]^t [f(y)]^{1-t} \quad \text{for all } x, y \in I, \quad t \in (0,1),$$

that is, iff $f$ is convex with respect to the family of weighted geometric means $I$, and referred to as the *geometric convexity* of $f$ ([2]).

## 3. Main results

**Theorem 1.** *Let a function $f : (0,\infty) \to (0,\infty)$ be continuous and such that*

$$f(x+1) = xf(x), \quad x > 0,\ f(1) = 1.$$

*If there is an $a > 0$ and a strict continuous mean $M : (a,\infty)^2 \to (a,\infty)$ such that*

$$f(M(x,y)) f\left(\frac{xy}{M(x,y)}\right) \leq f(x) f(y), \quad x, y \in (a,\infty), \tag{4}$$

*then $f$ is the Euler gamma function.*

PROOF. In view of Lemma 2, the function $N:(a,\infty)^2\to(a,\infty)$ defined by

$$N(x,y):=\frac{xy}{M(x,y)},\quad x,y\in(a,\infty),\tag{5}$$

is a continuous and strict mean. Since

$$M(x,y)\,N(x,y)=xy,\quad x,y\in(a,\infty),$$

taking the square root of both sides, we get

$$G\circ(M,N)=G,$$

where $G:(0,\infty)^2\to(0,\infty)$, $G(x,y)=\sqrt{xy}$, $x,y>0$, is the geometric mean. Thus the geometric mean $G$ is invariant with respect to the mean-type mapping $(M,N):(0,\infty)^2\to(0,\infty)^2$.

For every $n\in\mathbb{N}$, denote by $(M_n,N_n)$ the $n$-th iterate $(M,N)^n$ of the mean-type mapping $(M,N)$.

From (4), we have

$$f(M(x,y))\,f(N(x,y))\le f(x)\,f(y),\quad x,y\in(a,\infty).$$

Replacing here $x$ by $M(x,y)$ and $y$ by $N(x,y)$, we have

$$f(M(M(x,y),N(x,y)))\,f(N(M(x,y),N(x,y)))\le f(M(x,y))\,f(N(x,y))$$

for all $x,y\in(a,\infty)$, that is,

$$f(M_2((x,y)))\,f(N_2((x,y)))\le f(M(x,y))\,f(N(x,y))\le f(x)\,f(y)$$

for $x,y\in(a,\infty)$, whence

$$f(M_2((x,y)))\,f(N_2((x,y)))\le f(x)\,f(y),\quad x,y\in(a,\infty).$$

Similarly, applying the induction, we obtain

$$f(M_n(x,y))\,f(N_n(x,y))\le f(x)\,f(y),\quad n\in\mathbb{N},\ x,y\in(a,\infty).\tag{6}$$

The invariance of $G$ with respect to the mean-type mapping $(M,N)$ and Lemma 2 imply that

$$\lim_{n\to\infty}M_n(x,y)=G(x,y)=\lim_{n\to\infty}N_n(x,y),\quad x,y\in(a,\infty).$$

Therefore, letting $n\to\infty$ in (6), and making use of the continuity of $f$, we obtain

$$[f(G(x,y))]^2\le f(x)\,f(y),\quad x,y\in(a,\infty),$$

or, equivalently,

$$f(G(x,y))\le G(f(x)\,f(y)),\quad x,y\in(a,\infty),$$

which proves that $f$ is Jensen geometrically convex in $(a,\infty)$. Applying Lemma 3, we conclude that $f=\Gamma$. □

**Theorem 2.** *For every $a > a_0 := 1.462$ and every strict mean $M : (a,\infty)^2 \to (a,\infty)$, the Euler gamma function satisfies the inequality*

$$\Gamma\left(M\left(x,y\right)\right)\Gamma\left(\frac{xy}{M\left(x,y\right)}\right) \leq \Gamma\left(x\right)\Gamma\left(y\right), \quad x,y \in (a,\infty).$$

Proof. Since $\Gamma$ is logarithmically convex and increasing in $(a_0,\infty)$, where $a_0 = 1.462$, it is geometrically convex in every interval $(a,\infty)$ with $a > a_0$. Thus, for an arbitrarily fixed $a > a_0$, and for all $x,y \in (a,\infty)$ and $t \in (0,1)$, we have

$$\Gamma\left(x^t y^{1-t}\right) \leq \left[\Gamma\left(x\right)\right]^t \left[\Gamma\left(y\right)\right]^{1+t}. \tag{7}$$

Let $M : (a,\infty)^2 \to (a,\infty)$ be a strict mean. Taking arbitrary $x,y \in (a,\infty)$, $x \neq y$, and putting

$$t = t\left(x,y\right) := \frac{\log M\left(x,y\right) - \log y}{\log x - \log y},$$

we have

$$0 < t\left(x,y\right) < 1, \quad M\left(x,y\right) = x^{t(x,y)} y^{1-t(x,y)}$$

and, by (5),

$$N\left(x,y\right) = x^{1-t(x,y)} y^{t(x,y)}.$$

Hence, applying (7), we get

$$\Gamma\left(M\left(x,y\right)\right) = \Gamma\left(x^{t(x,y)} y^{1-t(x,y)}\right) \leq \left[\Gamma\left(x\right)\right]^{t(x,y)} \left[\Gamma\left(y\right)\right]^{1-t(x,y)}$$

and

$$\Gamma\left(N\left(x,y\right)\right) = \Gamma\left(x^{1-t(x,y)} y^{t(x,y)}\right) \leq \left[\Gamma\left(x\right)\right]^{1-t(x,y)} \left[\Gamma\left(y\right)\right]^{t(x,y)},$$

whence, multiplying the respective sides of these inequalities, we obtain

$$\begin{aligned}\Gamma\left(M\left(x,y\right)\right)\Gamma\left(N\left(x,y\right)\right) \leq & \left(\left[\Gamma\left(x\right)\right]^{t(x,y)} \left[\Gamma\left(y\right)\right]^{1-t(x,y)}\right)\left(\left[\Gamma\left(x\right)\right]^{1-t(x,y)} \left[\Gamma\left(y\right)\right]^{t(x,y)}\right) \\ = & \Gamma\left(x\right)\Gamma\left(y\right),\end{aligned}$$

that is, for all $x,y \in (a,\infty)$, $x \neq y$,

$$\Gamma\left(M\left(x,y\right)\right)\Gamma\left(\frac{xy}{M\left(x,y\right)}\right) \leq \Gamma\left(x\right)\Gamma\left(y\right).$$

Since this inequality is obvious for all $x,y \in (a,\infty)$ such that $x = y$, the proof is completed. □

*Remark 3.* Theorem 1 with $M = G$ reduces to the main result of [2].

Indeed, taking $M = G$ in inequality (4), we get $[f(G(x,y))]^2 \leq f(x) f(y)$ or, equivalently, $f(G(x,y)) \leq G(f(x) f(y))$ for all $x, y \in (a, \infty)$, which means that $f$ is Jensen geometrically convex.

Finally note that the arithmetic and harmonic means can be used for a characterization of the gamma function. Namely, we have the following:

**Theorem 3.** *A continuous function $f : (0,\infty) \to (0,\infty)$ satisfying the functional equation*

$$f(x+1) = x f(x), \quad x > 0, \quad f(1) = 1,$$

*is the Euler gamma function $\Gamma$, if and only if, there is an $a > 0$ such that*

$$f(A(x,y)) f(H(x,y)) \leq f(x) f(y), \quad x, y \in (a, \infty),$$

*where $A(x,y) = \frac{x+y}{2}$ is the arithmetic mean and $H(x,y) = \frac{2xy}{x+y}$ is the harmonic mean.*

Proof. Define $M : (0,\infty)^2 \to (0,\infty)$ by

$$M(x,y) := A(x,y) = \frac{x+y}{2}, \quad x, y > 0.$$

Then

$$\frac{xy}{M(x,y)} = \frac{2xy}{x+y} = H(x,y), \quad x, y > 0,$$

therefore the result is a consequence of Theorem 1 and Theorem 2. □

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*(Received September 7, 2017; revised November 10, 2017)*