

Pseudo-random subsets constructed by using Fermat quotients

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Abstract. Let p be a prime, and let n be an arbitrary integer with $(n, p) = 1$. The Fermat quotient $q_p(n)$ is defined as the unique integer with

$$q_p(n) \equiv \frac{n^{p-1} - 1}{p} \pmod{p}, \quad 0 \leq q_p(n) \leq p - 1.$$

We also define $q_p(kp) = 0$ for $k \in \mathbb{Z}$. In this paper, we study the pseudo-randomness of subsets constructed by Fermat quotients, by using the estimates for exponential sums and character sums with Fermat quotients.

1. Introduction

For a prime p and an integer n with $(n, p) = 1$, the Fermat quotient $q_p(n)$ is defined as

$$q_p(n) \equiv \frac{n^{p-1} - 1}{p} \pmod{p}, \quad 0 \leq q_p(n) \leq p - 1.$$

We also define $q_p(kp) = 0$, $k \in \mathbb{Z}$. It is easy to show that $q_p(u) = q_p(p^2 \pm u)$ for $p > 2$, and

$$q_p(a + kp) \equiv q_p(a) - ka^{-1} \pmod{p} \quad (1.1)$$

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for any $a, k \in \mathbb{Z}$ with $(a, p) = 1$. Fermat quotients arose from the study of the first case of Fermat's Last Theorem, which claimed that the equation

$$x^n + y^n + z^n = 0, \quad 2 \nmid n, \quad (n, xyz) = 1, \quad n > 1 \quad (1)_n$$

has no solutions in integers x, y and z .

In 1823, Sophie Germain proved that if n and $2n+1$ are both prime, then $(1)_n$ has no solutions in integers. In 1909, Wieferich showed that if $(1)_p$ has solutions in integers x, y and z , then $q_p(2) = 0$. Shortly later, Mirimanoff extended the result to $q_p(3) = 0$. In 1917, Pollaczek proved that if prime p is sufficiently large and $(1)_p$ has solutions in integers x, y and z , then $q_p(u) = 0$ for each prime $u \leq 31$. In 1988, Granville and Monagan showed that if $(1)_p$ has solutions in integers x, y and z , then $q_p(u) = 0$ for all primes $u \leq 89$. See [13] for an excellent survey of the first case of Fermat's Last Theorem.

Fermat quotients have numerous applications in computational and algebraic number theory (see [1]–[2], [4], [6], [12], [16]–[19]). For example, we define the sets

$$I_p(N) = |\{q_p(u) : 1 \leq u \leq N, (u, p) = 1\}|,$$

$$M(p) = |\{q_p(u) : 1 \leq u \leq p-1\}|,$$

$$F(p) = |\{u : 0 \leq u \leq p-1, q_p(u) = u\}|,$$

$$R(p, a) = |\{u : 0 \leq u \leq p-1, q_p(u) = a\}|,$$

$$\vartheta(p) = \frac{1}{p^2} \cdot |\{(u, v) : 0 \leq u, v \leq p-1, q_p(u) = q_p(v)\}|,$$

$$U(p; k, h) = |\{u : 0 \leq u \leq p-1, q_p(u) \equiv z \pmod{p} \text{ for } z \in [k+1, \dots, k+h]\}|.$$

VANDIVER [20] showed that $\sqrt{p} - 1 \leq M(p) \leq p - \sqrt{\frac{p-1}{2}}$. FOUCHÉ [11] proved $R(p, a) \leq p^{1/2+o(1)}$. OSTAFE and SHPARLINSKI [16] obtained the following estimations:

$$\begin{aligned} M(p) &\geq (1+o(1)) \frac{p}{(\log p)^2}, & F(p) &\ll p^{11/12+o(1)}, \\ \vartheta(p) &\leq p^{-3/4+o(1)}, & U(p; k, h) &\leq h^{1/2} p^{1/2+o(1)}. \end{aligned}$$

CHEN and WINTERHOF [6] gave certain generalization and improvement on Ostafe and Shparlinski's result.

Proposition 1.1. *Let $P(x) \in \mathbb{F}_p[x]$ be of degree $1 \leq d < p$. Then we have*

$$\#\{u : 1 \leq u \leq p-1, q_p(u) = P(u)\} \ll \begin{cases} d^{1/4} p^{5/6}, & 1 \leq d \leq p^{1/3}, \\ d^{1/8} p^{7/8}, & d > p^{1/3}, \end{cases}$$

as $p \rightarrow \infty$.

SHPARLINSKI [17] proved some lower bounds on the image size $I_p(N)$.

Proposition 1.2. *For every p and $N < p$, we have*

$$I_p(N) \geq (1 + o(1)) \frac{N^2}{p(\log N)^2}.$$

For every p and arbitrary fixed $\epsilon > 0$, there exists some $\delta > 0$ such that, for $p^\epsilon < N < p$, we have $I_p(N) \geq p^\delta$.

Shparlinski [17] also studied the primitive roots and power non-residues among the values of the Fermat quotients.

Proposition 1.3. *For every p , there exists $n \leq p^{3/4+o(1)}$ such that $q_p(n)$ is a primitive root modulo p . For every p and positive integer $d \mid p-1$, there exists $n \leq p^{3/4+o(1)}$ such that $q_p(n)$ is a d -th power non-residue modulo p .*

In this paper, we further study the pseudo-randomness of subsets constructed by using Fermat quotients. Let $\mathcal{R} \subset \{1, 2, \dots, N\}$, and define the sequence

$$E_N = E_N(\mathcal{R}) = \{e_1, e_2, \dots, e_N\} \in \left\{1 - \frac{|\mathcal{R}|}{N}, -\frac{|\mathcal{R}|}{N}\right\}^N$$

by

$$e_n = \begin{cases} 1 - \frac{|\mathcal{R}|}{N}, & \text{for } n \in \mathcal{R}, \\ -\frac{|\mathcal{R}|}{N}, & \text{for } n \notin \mathcal{R}. \end{cases}$$

DARTYGE and SÁRKÖZY [8] introduced the measures of pseudo-randomness: The *well-distribution measure* of the subset $\mathcal{R}$ is defined by

$$W(\mathcal{R}, N) = \max_{a, b, t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

where the maximum is taken over all $a, b, t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq N$. The *correlation measure of order k* of the subset $\mathcal{R}$ is defined by

$$C_k(\mathcal{R}, N) = \max_{M, D} \left| \sum_{n=1}^M e_{n+d_1} \cdots e_{n+d_k} \right|,$$

where the maximum is taken over all $D = (d_1, \dots, d_k)$ and M with $0 \leq d_1 < \dots < d_k \leq N - M$.

The subset $\mathcal{R}$ is considered as a pseudo-random subset if both $W(\mathcal{R}, N)$ and $C_k(\mathcal{R}, N)$ (at least for small k) are “small” in terms of N . Later many pseudo-random subsets were given and studied. For example, DARTYGE, MOSAKI and SÁRKÖZY [7] proved the following results.

Proposition 1.4. Assume that p is an odd prime number, $f(x) \in \mathbb{F}_p[x]$ is of degree $d \geq 2$. Let $r \in \mathbb{Z}, s \in \mathbb{N}, s < p/2$. Define $\mathcal{R} \subset \{1, \dots, p\}$ by

$$\mathcal{R} = \{n : 1 \leq n \leq p, \exists h \in \{r, r+1, \dots, r+s-1\} \text{ with } f(n) \equiv h \pmod{p}\}.$$

Then we have

$$W(\mathcal{R}, p) < 2d\sqrt{p}(\log p)^2.$$

And for $2 \leq k \leq d-1$, we have

$$C_k(\mathcal{R}, p) < 2d\sqrt{p}(1 + \log p)^{k+1}.$$

DARTYGE and SÁRKÖZY [9] presented large families of pseudo-random subsets formed by power residues.

Proposition 1.5. Let $p \geq 2$ be a prime number, $d \mid p-1$ and $f \in \mathbb{F}_p[x]$ of degree k . Define

$$\mathcal{R} = \{x \in \mathbb{F}_p : \exists y \in \mathbb{F}_p^* \text{ with } f(x) \equiv y^d \pmod{p}\}.$$

Write $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ for the factorization into irreducible factors in $\overline{\mathbb{F}}_p$, and suppose that $(d, \alpha_1, \dots, \alpha_r) = 1$. Then

$$W(\mathcal{R}, p) \leq 20k\sqrt{p} \log p.$$

Let f be a polynomial of degree k with no multiple roots in $\overline{\mathbb{F}}_p$, and let $k, l \in \mathbb{N}$ such that one of the following conditions is satisfied:

- (i) $l=2$;
- (ii) d is a prime divisor of $p-1$, and $(4l)^k < p$;
- (iii) the polynomial $x^{p-1} + \cdots + x + 1$ is irreducible in $\mathbb{F}_d[x]$ and $\max(k, l) < p$.

Then we have

$$C_l(\mathcal{R}, p) \leq lk \left(1 + \frac{20k \log p}{\sqrt{p}}\right)^l + 9 \left(\frac{d-1}{d}\right)^l \left(1 + \frac{10k \log p}{\sqrt{p}}\right)^l lk\sqrt{p} \log p.$$

DARTYGE, SÁRKÖZY and SZALAY [10] studied the pseudo-randomness of certain subsets related to primitive roots.

Proposition 1.6. Let p be an odd prime, $k, r \in \mathbb{N}$, $f(x) \in \mathbb{F}_p[x]$ of degree D . Let

$$\mathcal{G}_p = \{g_1, g_2, \dots, g_{\phi(p-1)}\}, \quad (0 < g_1 < g_2 < \cdots < g_{\phi(p-1)} < p)$$

be the set of the primitive roots modulo p . Define the subset $\mathcal{R} \subset \mathbb{F}_p$ by

$$\mathcal{R} = \{g^k : g \in \mathcal{G}_p, \exists x \in \mathbb{F}_p^* \text{ with } f(g^k) = x^r\}.$$

If $f(x)$ is irreducible over $\mathbb{F}_p$, then we have

$$W(\mathcal{R}, p) \ll D 2^{\omega(\frac{p-1}{k})} p^{1/2} \log p.$$

Moreover, if we also assume that $D \geq 2$ or $D = r = 1$, we also have

$$C_K(\mathcal{R}, p) \ll KD \left((1 + O(Dp^{-1/2})) 2^{\omega(\frac{p-1}{k})} \right)^K p^{1/2} \log p.$$

CHEN [3] constructed a family of pseudo-random subsets modulo pq , where p, q are two distinct primes satisfying ‘‘RSA type’’ with $2 < p < q < 2p$, and studied the well-distribution and correlation measures. LIU and SONG [15] further studied the correlation measures of the subsets.

In this paper, we study the pseudo-random subsets constructed by using Fermat quotients. Our results are the following.

Theorem 1.1. *Let p be an odd prime number, and let $r, s \in \mathbb{N}$ with $1 \leq r < r + s - 1 \leq p - 1$. Define $\mathcal{R} \subset \{1, \dots, p^2\}$ by*

$$\mathcal{R} = \{n : 1 \leq n \leq p^2, \exists h \in \{r, r + 1, \dots, r + s - 1\} \text{ with } q_p(n) \equiv h \pmod{p}\}.$$

Then we have

$$W(\mathcal{R}, p^2) \ll p(\log p)^2,$$

and

$$C_2(\mathcal{R}, p^2) \ll p(\log p)^3.$$

Furthermore, if $s = o(p)$, then

$$C_3(\mathcal{R}, p^2) \ll s^2 + p^{\frac{3}{2}} (\log p)^3.$$

While $s = \left\lceil \frac{p}{3} \right\rceil$, we have

$$C_3(\mathcal{R}, p^2) \geq \frac{1}{54} p^2 + O\left(p(\log p)^4\right).$$

Theorem 1.2. Let p be an odd prime number, and $d \mid p-1$. Define $\mathcal{R} \subset \{1, \dots, p^2\}$ by

$$\mathcal{R} = \{n : 1 \leq n \leq p^2, \exists y \text{ such that } 1 \leq y \leq p-1 \text{ and } q_p(n) \equiv y^d \pmod{p}\}.$$

Then we have

$$W(\mathcal{R}, p^2) \ll p^{\frac{3}{2}} \log p,$$

and

$$C_k(\mathcal{R}, p^2) \ll kp^{\frac{5}{3}},$$

where the implied constant depends on d .

Theorem 1.3. Let p be an odd prime number, and let $\mathcal{G}_p$ be the set of the primitive roots modulo p . Define $\mathcal{R} \subset \{1, \dots, p^2\}$ by

$$\mathcal{R} = \{n : 1 \leq n \leq p^2, q_p(n) \in \mathcal{G}_p\}.$$

Then we have

$$W(\mathcal{R}, p^2) \ll 2^{\omega(p-1)} p^{\frac{3}{2}} \log p.$$

and

$$C_k(\mathcal{R}, p^2) \ll k2^{k\omega(p-1)} p^{\frac{5}{3}}.$$

Remark 1.1. Our results show that the subsets in Theorems 1.2 and 1.3 enjoy good pseudo-random properties. However, the subset in Theorem 1.1 is not pseudo-random.

2. Exponential sums and character sums with Fermat quotients

The exponential sums with Fermat quotients have been studied by many authors. Write $e(y) = e^{2\pi i y}$. HEATH-BROWN [14] proved the following estimation by using the Pólya–Vinogradov bound and the Burgess bound.

Proposition 2.1. For any integer a coprime to p , we have

$$\sum_{\substack{M < n \leq M+N \\ p \nmid n}} e\left(\frac{aq_p(n)}{p}\right) \ll N^{1/2} p^{3/8},$$

uniformly for $M, N \geq 1$. In particular,

$$\sum_{n=1}^{p-1} e\left(\frac{aq_p(n)}{p}\right) \ll p^{7/8},$$

uniformly for $p \nmid a$.

OSTAFE and SHPARLINSKI [16] obtained a non-trivial bound of exponential sums with linear combinations of Fermat quotients.

Proposition 2.2. *For any integer $s \geq 1$, we have*

$$\max_{(a_0, \dots, a_s, p)=1} \left| \sum_{u=M+1}^{M+N} e \left(\frac{\sum_{j=0}^{s-1} a_j q_p(u+j)}{p} \right) \right| \ll sp \log p,$$

uniformly over M and $p^2 > N \geq 1$.

Using Proposition 2.2, they studied the distribution of the points

$$\left(\frac{q_p(u)}{p}, \dots, \frac{q_p(u+s-1)}{p} \right), \quad u = M+1, \dots, M+N,$$

in the s -dimensional cube. Moreover, CHEN, OSTAFE and WINTERHOF [5] gave the following generalization.

Proposition 2.3. *Let $d_0, d_1, \dots, d_{s-1}$ be integers with $0 \leq d_0 < d_1 < \dots < d_{s-1} < p^2$. Suppose that no triple (d_l, d_h, d_t) satisfies $d_l \equiv d_h \equiv d_t \pmod{p}$ for $0 \leq l < h < t < s$. We have*

$$\max_{(a_0, \dots, a_s, p)=1} \left| \sum_{u=1}^N e \left(\frac{\sum_{j=0}^{s-1} a_j q_p(u+d_j)}{p} \right) \right| \ll s \max \left\{ p \log p, N p^{-\frac{1}{2}} \right\}$$

for $1 \leq N \leq p^2$. If $s = 2$ or $d_{s-1} < p$, the stronger bound $sp \log p$ holds.

Now, we give the following identities on certain exponential sums of Fermat quotients.

Theorem 2.1. *Let $p > 2$ be a prime, and let u be an integer with $(u, p) = 1$. For any integer z , we have*

$$\left| \sum_{n=1}^{p^2} e \left(\frac{u q_p(n)}{p} \right) e \left(\frac{zn}{p^2} \right) \right| = p.$$

PROOF. Write $n = a + bp$, we get

$$\begin{aligned} \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) &= \sum_{a=1}^p \sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + \sum_{b=0}^{p-1} e\left(\frac{z(1+b)}{p}\right). \end{aligned}$$

Assume that $p \nmid z$. By (1.1), we have

$$\begin{aligned} \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) &= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) \\ &= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{u(q_p(a) - ba^{-1})}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) \\ &= \sum_{a=1}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right) \sum_{b=0}^{p-1} e\left(\frac{b(z - ua^{-1})}{p}\right) \\ &= p \sum_{\substack{a=1 \\ z \equiv ua^{-1} \pmod{p}}}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right) \end{aligned}$$

Hence,

$$\left| \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) \right| = p.$$

Suppose that $p \mid z$. By (1.1), we get

$$\begin{aligned} \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) &= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{uq_p(a+bp)}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + p \\ &= \sum_{a=1}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{u(q_p(a) - ba^{-1})}{p}\right) e\left(\frac{z(a+bp)}{p^2}\right) + p \\ &= \sum_{a=1}^{p-1} e\left(\frac{uq_p(a)}{p}\right) e\left(\frac{za}{p^2}\right) \sum_{b=0}^{p-1} e\left(-\frac{bua^{-1}}{p}\right) + p = p. \end{aligned}$$

This proves Theorem 2.1. $\square$

Theorem 2.2. *Let $p > 2$ be a prime, and let u_1, u_2, u_3 be integers with $(u_1 u_2 u_3, p) = 1$. Let d_1, k_2, k_3 and M be integers such that $0 \leq d_1 < d_1 + k_2 p < d_1 + k_3 p \leq p^2 - M$. Then we have*

$$\begin{aligned} & \sum_{n=1}^M e \left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p} \right) \\ &= \begin{cases} M, & \text{if } u_1 + u_2 + u_3 \equiv 0 \pmod{p} \text{ and } u_2 k_2 + u_3 k_3 \equiv 0 \pmod{p}, \\ O(p), & \text{if } u_1 + u_2 + u_3 \equiv 0 \pmod{p} \text{ and } u_2 k_2 + u_3 k_3 \not\equiv 0 \pmod{p}, \\ O(p \log p), & \text{if } u_1 + u_2 + u_3 \not\equiv 0 \pmod{p}. \end{cases} \end{aligned}$$

PROOF. We may suppose that $M \geq p$, since otherwise the claim is trivial. By (1.1), we get

$$\begin{aligned} & \sum_{n=1}^M e \left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p} \right) \\ &= \sum_{\substack{n=1 \\ (n+d_1, p)=1}}^M e \left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p} \right) + \sum_{\substack{n=1 \\ (n+d_1, p)>1}}^M 1 \\ &= \sum_{\substack{n=1 \\ (n+d_1, p)=1}}^M e \left(\frac{(u_1 + u_2 + u_3) q_p(n + d_1) - (u_2 k_2 + u_3 k_3)(n + d_1)^{-1}}{p} \right) + \sum_{\substack{n=1 \\ (n+d_1, p)>1}}^M 1. \end{aligned}$$

Case 1. $u_1 + u_2 + u_3 \equiv 0 \pmod{p}$ and $u_2 k_2 + u_3 k_3 \equiv 0 \pmod{p}$. We have

$$\sum_{n=1}^M e \left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p} \right) = M.$$

Case 2. $u_1 + u_2 + u_3 \equiv 0 \pmod{p}$ and $u_2 k_2 + u_3 k_3 \not\equiv 0 \pmod{p}$. It is not hard to show that

$$\begin{aligned} & \sum_{n=1}^M e \left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p} \right) \\ &= \sum_{\substack{n=1 \\ (n+d_1, p)=1}}^M e \left(-\frac{(u_2 k_2 + u_3 k_3)(n + d_1)^{-1}}{p} \right) + \sum_{\substack{n=1 \\ (n+d_1, p)>1}}^M 1 = \end{aligned}$$

$$\begin{aligned}
&= \sum_{0 \leq k \leq \left[\frac{M}{p}\right]-1} \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p e\left(-\frac{(u_2 k_2 + u_3 k_3)(kp + v + d_1)^{-1}}{p}\right) + O(p) \\
&= \sum_{0 \leq k \leq \left[\frac{M}{p}\right]-1} \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p e\left(-\frac{(u_2 k_2 + u_3 k_3)(v + d_1)^{-1}}{p}\right) + O(p) \ll p.
\end{aligned}$$

Case 3. $u_1 + u_2 + u_3 \not\equiv 0 \pmod{p}$. By (1.1) and the properties of residue systems, we have

$$\begin{aligned}
&\sum_{n=1}^M e\left(\frac{u_1 q_p(n + d_1) + u_2 q_p(n + d_1 + k_2 p) + u_3 q_p(n + d_1 + k_3 p)}{p}\right) \\
&= \sum_{\substack{n=1 \\ (n+d_1, p)=1}}^M e\left(\frac{(u_1 + u_2 + u_3)q_p(n + d_1) - (u_2 k_2 + u_3 k_3)(n + d_1)^{-1}}{p}\right) + \sum_{\substack{n=1 \\ (n+d_1, p)>1}}^M 1 \\
&= \sum_{0 \leq k \leq \left[\frac{M}{p}\right]-1} \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p e\left(\frac{(u_1 + u_2 + u_3)q_p(kp + v + d_1)}{p} - \frac{(u_2 k_2 + u_3 k_3)(kp + v + d_1)^{-1}}{p}\right) + O(p) \\
&= \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p e\left(\frac{(u_1 + u_2 + u_3)q_p(v + d_1) - (u_2 k_2 + u_3 k_3)(v + d_1)^{-1}}{p}\right) \\
&\quad \times \sum_{0 \leq k \leq \left[\frac{M}{p}\right]-1} e\left(-\frac{(u_1 + u_2 + u_3)(v + d_1)^{-1}k}{p}\right) + O(p) \\
&\ll \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p \left| \sum_{0 \leq k \leq \left[\frac{M}{p}\right]-1} e\left(-\frac{(u_1 + u_2 + u_3)(v + d_1)^{-1}k}{p}\right) \right| + p \\
&\ll \sum_{\substack{v=1 \\ (v+d_1, p)=1}}^p \frac{1}{\left\langle -\frac{(u_1 + u_2 + u_3)(v + d_1)^{-1}}{p} \right\rangle} + p \ll \sum_{0 < |v| \leq \frac{p-1}{2}} \frac{1}{\left\langle \frac{v}{p} \right\rangle} + p \ll p \log p,
\end{aligned}$$

where $\langle \alpha \rangle$ denotes the distance of the real number α to the nearest integer.

This completes the proof of Theorem 2.2. $\square$

On the other hand, character sums with Fermat quotients have also been studied by many authors. For example, from [12] we know that

$$\sum_{u=0}^{N-1} \chi(q_p(au+b)) \ll N^{1-\frac{1}{v}} p^{\frac{5v+1}{4v^2}} (\log p)^{\frac{1}{v}}, \quad (2.1)$$

for $1 \leq N \leq p^2$ and every fixed integer $v \geq 1$, where χ is a non-trivial multiplicative character modulo p , and a, b are integers with $(a, p^2) \neq p^2$. GOMEZ and WINTERHOF [12] further studied the following character sums.

Proposition 2.4. *Let $\chi_1, \chi_2, \dots, \chi_l$ be nontrivial multiplicative characters modulo p . Then we have*

$$\sum_{u=0}^{N-1} \chi_1(q_p(u+d_1)) \cdots \chi_l(q_p(u+d_l)) \ll \max \left\{ \frac{lN}{p^{\frac{1}{3}}}, lp^{\frac{3}{2}} \log p \right\},$$

for any integers $0 \leq d_1 < \dots < d_l \leq p^2 - 1$ and $1 \leq N \leq p^2$.

3. Proof of Theorem 1.1

For integer h with $(h, p) = 1$, by (1.1), we get

$$\begin{aligned} \sum_{\substack{n=1 \\ q_p(n) \equiv h \pmod{p}}}^{p^2} 1 &= \sum_{\substack{n=1 \\ (n,p)=1 \\ q_p(n) \equiv h \pmod{p}}}^{p^2} 1 = \sum_{\substack{a=1 \\ q_p(a+kp) \equiv h \pmod{p}}}^{p-1} \sum_{k=0}^{p-1} 1 \\ &= \sum_{\substack{a=1 \\ q_p(a)-ka^{-1} \equiv h \pmod{p}}}^{p-1} \sum_{k=0}^{p-1} 1 = \sum_{a=1}^{p-1} \sum_{\substack{k=0 \\ ka^{-1} \equiv q_p(a)-h \pmod{p}}}^{p-1} 1 = p-1. \end{aligned}$$

Suppose that $\mathcal{A} \subset \{1, 2, \dots, p-1\}$. Then

$$\sum_{\substack{n=1 \\ q_p(n) \in \mathcal{A}}}^{p^2} 1 = |\mathcal{A}| \cdot (p-1). \quad (3.1)$$

Noting that $1 \leq r < r+s-1 \leq p-1$, we have

$$|\mathcal{R}| = s(p-1).$$

It is easy to show that, for $1 \leq n \leq p^2$,

$$\frac{1}{p} \sum_{h=r}^{r+s-1} \sum_{|u| \leq \frac{p-1}{2}} e\left(\frac{u(q_p(n) - h)}{p}\right) = \begin{cases} 1, & n \in \mathcal{R}, \\ 0, & n \notin \mathcal{R}. \end{cases}$$

Hence,

$$\begin{aligned} e_n &= \frac{1}{p} \sum_{h=r}^{r+s-1} \sum_{|u| \leq \frac{p-1}{2}} e\left(\frac{u(q_p(n) - h)}{p}\right) - \frac{|\mathcal{R}|}{p^2} \\ &= \frac{1}{p} \sum_{1 \leq |u| \leq \frac{p-1}{2}} \left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right) \right) e\left(\frac{uq_p(n)}{p}\right) + O\left(\frac{1}{p}\right), \end{aligned} \quad (3.2)$$

where the term $\frac{|\mathcal{R}|}{p^2}$ is compensated by the contribution of $u = 0$.

For $a, b, t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq p^2$, from (3.2) and Theorem 2.1, we get

$$\begin{aligned} \sum_{j=0}^{t-1} e_{a+jb} &= \frac{1}{p} \sum_{1 \leq |u| \leq \frac{p-1}{2}} \left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right) \right) \sum_{j=0}^{t-1} e\left(\frac{uq_p(a+jb)}{p}\right) + O(p) \\ &= \frac{1}{p^3} \sum_{1 \leq |u| \leq \frac{p-1}{2}} \left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right) \right) \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) \sum_{j=0}^{t-1} \\ &\quad \times \sum_{z=1}^{p^2} e\left(\frac{z(n - (a+jb))}{p^2}\right) + O(p) \\ &= \frac{1}{p^3} \sum_{1 \leq |u| \leq \frac{p-1}{2}} \left(\sum_{h=r}^{r+s-1} e\left(-\frac{uh}{p}\right) \right) \sum_{z=1}^{p^2} \left(\sum_{j=0}^{t-1} e\left(-\frac{z(a+jb)}{p^2}\right) \right) \\ &\quad \times \sum_{n=1}^{p^2} e\left(\frac{uq_p(n)}{p}\right) e\left(\frac{zn}{p^2}\right) + O(p) \ll p(\log p)^2. \end{aligned}$$

Hence,

$$W(\mathcal{R}, p^2) \ll p(\log p)^2.$$

For integers d_1, d_2 and M with $0 \leq d_1 < d_2 \leq p^2 - M$, from (3.2) and Proposition 2.3, we get

$$\begin{aligned}
\sum_{n=1}^M e_{n+d_1} e_{n+d_2} &= \sum_{n=1}^M \left(\frac{1}{p} \sum_{1 \leq |u_1| \leq \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1 h_1}{p}\right) \right) e\left(\frac{u_1 q_p(n+d_1)}{p}\right) + O\left(\frac{1}{p}\right) \right) \\
&\quad \times \left(\frac{1}{p} \sum_{1 \leq |u_2| \leq \frac{p-1}{2}} \left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2 h_2}{p}\right) \right) e\left(\frac{u_2 q_p(n+d_2)}{p}\right) + O\left(\frac{1}{p}\right) \right) \\
&= \frac{1}{p^2} \sum_{1 \leq |u_1| \leq \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1 h_1}{p}\right) \right) \sum_{1 \leq |u_2| \leq \frac{p-1}{2}} \left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2 h_2}{p}\right) \right) \\
&\quad \times \sum_{n=1}^M e\left(\frac{u_1 q_p(n+d_1) + u_2 q_p(n+d_2)}{p}\right) + O(p \log p) \ll p(\log p)^3.
\end{aligned}$$

Therefore,

$$C_2(\mathcal{R}, p^2) \ll p(\log p)^3.$$

For integers d_1, d_2, d_3 and M with $0 \leq d_1 < d_2 < d_3 \leq p^2 - M$, by (3.2), we have

$$\begin{aligned}
&\sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} \\
&= \sum_{n=1}^M \left(\frac{1}{p} \sum_{1 \leq |u_1| \leq \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1 h_1}{p}\right) \right) e\left(\frac{u_1 q_p(n+d_1)}{p}\right) + O\left(\frac{1}{p}\right) \right) \\
&\quad \times \left(\frac{1}{p} \sum_{1 \leq |u_2| \leq \frac{p-1}{2}} \left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2 h_2}{p}\right) \right) e\left(\frac{u_2 q_p(n+d_2)}{p}\right) + O\left(\frac{1}{p}\right) \right) \\
&\quad \times \left(\frac{1}{p} \sum_{1 \leq |u_3| \leq \frac{p-1}{2}} \left(\sum_{h_3=r}^{r+s-1} e\left(-\frac{u_3 h_3}{p}\right) \right) e\left(\frac{u_3 q_p(n+d_3)}{p}\right) + O\left(\frac{1}{p}\right) \right) \\
&= \frac{1}{p^3} \sum_{1 \leq |u_1| \leq \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e\left(-\frac{u_1 h_1}{p}\right) \right) \sum_{1 \leq |u_2| \leq \frac{p-1}{2}} \left(\sum_{h_2=r}^{r+s-1} e\left(-\frac{u_2 h_2}{p}\right) \right) \\
&\quad \times \sum_{1 \leq |u_3| \leq \frac{p-1}{2}} \left(\sum_{h_3=r}^{r+s-1} e\left(-\frac{u_3 h_3}{p}\right) \right) \\
&\quad \times \sum_{n=1}^M e\left(\frac{u_1 q_p(n+d_1) + u_2 q_p(n+d_2) + u_3 q_p(n+d_3)}{p}\right) + O(p(\log p)^2).
\end{aligned}$$

We consider d_1, d_2, d_3 in two cases.

Case 1. $d_1 \equiv d_2 \equiv d_3 \pmod{p}$ does not hold. Then from Proposition 2.3, we get

$$\sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} \ll p^{\frac{3}{2}} (\log p)^3.$$

Case 2. $d_1 \equiv d_2 \equiv d_3 \pmod{p}$. Write $d_2 = d_1 + k_2 p$, $d_3 = d_1 + k_3 p$, where $0 < k_2 < k_3 < p$. Then from Theorem 2.2, we have

$$\begin{aligned} \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} &= \frac{M}{p^3} \sum_{1 \leq |u_1| \leq \frac{p-1}{2}} \sum_{1 \leq |u_2| \leq \frac{p-1}{2}} \sum_{1 \leq |u_3| \leq \frac{p-1}{2}} \left(\sum_{h_1=r}^{r+s-1} e \left(-\frac{u_1 h_1}{p} \right) \right) \\ &\quad \times \left(\sum_{h_2=r}^{r+s-1} e \left(-\frac{u_2 h_2}{p} \right) \right) \left(\sum_{h_3=r}^{r+s-1} e \left(-\frac{u_3 h_3}{p} \right) \right) + O(p (\log p)^4) \\ &= \frac{M}{p^3} \sum_{h_1=r}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} \sum_{1 \leq |u| \leq \frac{p-1}{2}} \\ &\quad \times e \left(\frac{u \left((1 - k_3 k_2^{-1}) h_1 + k_3 k_2^{-1} h_2 - h_3 \right)}{p} \right) + O(p (\log p)^4). \end{aligned} \quad (3.3)$$

Assume that $s = o(p)$. Then

$$\begin{aligned} \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} &= \frac{M}{p^2} \sum_{h_1=r}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} 1 - \frac{M s^3}{p^3} + O(p (\log p)^4) \\ &\quad (1 - k_3 k_2^{-1}) h_1 + k_3 k_2^{-1} h_2 - h_3 \equiv 0 \pmod{p} \\ &\ll \frac{M s^2}{p^2} + p (\log p)^4. \end{aligned}$$

Therefore,

$$C_3(\mathcal{R}, p^2) \ll s^2 + p^{\frac{3}{2}} (\log p)^3.$$

Now taking $M = p^2 - 2p$, $d_1 = 0$, $d_2 = p$ and $d_3 = 2p$ in (3.3), we have

$$\begin{aligned} &\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} \\ &= \frac{p^2 - 2p}{p^3} \sum_{h_1=r}^{r+s-1} \sum_{h_2=r}^{r+s-1} \sum_{h_3=r}^{r+s-1} \sum_{1 \leq |u| \leq \frac{p-1}{2}} e \left(\frac{u(2h_2 - h_1 - h_3)}{p} \right) + O(p (\log p)^4) = \end{aligned}$$

$$\begin{aligned}
&= \frac{p^2 - 2p}{p^3} \sum_{h_1=0}^{s-1} \sum_{h_2=0}^{s-1} \sum_{h_3=0}^{s-1} \sum_{1 \leq |u| \leq \frac{p-1}{2}} e\left(\frac{u(2h_2 - h_1 - h_3)}{p}\right) + O\left(p(\log p)^4\right) \\
&= \frac{p^2 - 2p}{p^3} \sum_{h_1=0}^{s-1} \sum_{h_2=0}^{s-1} \sum_{\substack{h_3=0 \\ h_1+h_3=2h_2}}^{s-1} 1 - \frac{p^2 - 2p}{p^3} s^3 + O\left(p(\log p)^4\right).
\end{aligned}$$

Noting that

$$\begin{aligned}
\sum_{\substack{h_1=0 \\ h_1+h_3=2h_2}}^{s-1} \sum_{h_2=0}^{s-1} \sum_{h_3=0}^{s-1} 1 &= \sum_{h_2=0}^{s-1} \sum_{\substack{h_1=0 \\ h_1+h_3=2h_2}}^{s-1} \sum_{h_3=0}^{s-1} 1 \\
&= \sum_{0 \leq 2h_2 \leq s-1} (2h_2 + 1) + \sum_{s \leq 2h_2 \leq 2s-2} (2s - 1 - 2h_2) \\
&= \sum_{\substack{1 \leq t \leq s \\ 2 \nmid t}} t + \sum_{\substack{1 \leq t \leq s-1 \\ 2 \nmid t}} t = \begin{cases} \frac{s^2 + 1}{2}, & \text{if } 2 \nmid s, \\ \frac{s^2}{2}, & \text{if } 2 \mid s. \end{cases}
\end{aligned}$$

Thus we have

$$\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} = \frac{s^2}{2} - \frac{s^3}{p} + O\left(p(\log p)^4\right).$$

Assume that $s = \left\lfloor \frac{p}{3} \right\rfloor$. We immediately get

$$\sum_{n=1}^{p^2-2p} e_{n+d_1} e_{n+d_2} e_{n+d_3} = \frac{1}{54} p^2 + O\left(p(\log p)^4\right).$$

Therefore,

$$C_3(\mathcal{R}, p^2) \geq \frac{1}{54} p^2 + O\left(p(\log p)^4\right).$$

This proves Theorem 1.1.

4. Proof of Theorem 1.2

From (3.1), we get

$$|\mathcal{R}| = \frac{(p-1)^2}{d}.$$

On the other hand, we denote by χ_0 the principal character modulo p . Assume that $q_p(n) \neq 0$. By the orthogonality of characters of order d , we have

$$\begin{aligned} \frac{1}{d} \sum_{\substack{\chi^d = \chi_0 \\ \chi \bmod p}} \chi(q_p(n)) &= \begin{cases} 1, & \text{if } \exists y \text{ such that } 1 \leq y \leq p-1 \text{ and } q_p(n) \equiv y^d \pmod{p}, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & n \in \mathcal{R}, \\ 0, & n \notin \mathcal{R}. \end{cases} \end{aligned}$$

Hence,

$$e_n = \frac{1}{d} \sum_{\substack{\chi^d = \chi_0 \\ \chi \bmod p}} \chi(q_p(n)) - \frac{|\mathcal{R}|}{p^2} = \frac{1}{d} \sum_{\substack{\chi^d = \chi_0 \\ \chi \neq \chi_0 \\ \chi \bmod p}} \chi(q_p(n)) + O\left(\frac{1}{p}\right). \quad (4.1)$$

For $a, b, t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq p^2$, from (4.1) and (2.1), we get

$$\begin{aligned} \sum_{j=0}^{t-1} e_{a+jb} &= \sum_{j=0}^{t-1} \left(\frac{1}{d} \sum_{\substack{\chi^d = \chi_0 \\ \chi \neq \chi_0 \\ \chi \bmod p}} \chi(q_p(a+jb)) + O\left(\frac{1}{p}\right) \right) \\ &= \frac{1}{d} \sum_{\substack{\chi^d = \chi_0 \\ \chi \neq \chi_0 \\ \chi \bmod p}} \sum_{j=0}^{t-1} \chi(q_p(a+jb)) + O(p) \ll p^{\frac{3}{2}} \log p. \end{aligned}$$

Therefore,

$$W(\mathcal{R}, p^2) \ll p^{\frac{3}{2}} \log p.$$

For integers $d_1, d_2, \dots, d_k$ and M with $0 \leq d_1 < d_2 < \dots < d_k \leq p^2 - M$, from (4.1) and Proposition 2.4, we have

$$\begin{aligned} &\sum_{n=1}^M e_{n+d_1} \cdots e_{n+d_k} \\ &= \sum_{n=1}^M \left(\frac{1}{d} \sum_{\substack{\chi_1^d = \chi_0 \\ \chi_1 \neq \chi_0 \\ \chi_1 \bmod p}} \chi_1(q_p(n+d_1)) + O\left(\frac{1}{p}\right) \right) \times \cdots \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{1}{d} \sum_{\substack{\chi_k^d = \chi_0 \\ \chi_k \not\equiv \chi_0 \pmod{p}}} \chi_k(q_p(n + d_k)) + O\left(\frac{1}{p}\right) \right) \\
& = \frac{1}{d^k} \sum_{\substack{\chi_1^d = \chi_0 \\ \chi_1 \not\equiv \chi_0 \pmod{p}}} \cdots \sum_{\substack{\chi_k^d = \chi_0 \\ \chi_k \not\equiv \chi_0 \pmod{p}}} \sum_{n=1}^M \chi_1(q_p(n + d_1)) \cdots \chi_k(q_p(n + d_k)) + O(kp) \ll kp^{\frac{5}{3}}.
\end{aligned}$$

Hence,

$$C_k(\mathcal{R}, p^2) \ll kp^{\frac{5}{3}}.$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

By (3.1), we have

$$|\mathcal{R}| = (p-1)\phi(p-1).$$

Suppose that $q_p(n) \neq 0$. Noting that

$$\begin{aligned}
\frac{\phi(p-1)}{p-1} \sum_{s|p-1} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \pmod{p}}} \chi(q_p(n)) &= \begin{cases} 1, & \text{if } q_p(n) \text{ is a primitive root modulo } p, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 1, & n \in \mathcal{R}, \\ 0, & n \notin \mathcal{R}. \end{cases}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
e_n &= \frac{\phi(p-1)}{p-1} \sum_{s|p-1} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \pmod{p}}} \chi(q_p(n)) - \frac{|\mathcal{R}|}{p^2} \\
&= \frac{\phi(p-1)}{p-1} \sum_{\substack{s|p-1 \\ s>1}} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi = s \\ \chi \pmod{p}}} \chi(q_p(n)) + O\left(\frac{1}{p}\right). \tag{5.1}
\end{aligned}$$

For $a, b, t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq p^2$, from (5.1) and (2.1), we get

$$\begin{aligned}
\sum_{j=0}^{t-1} e_{a+jb} &= \sum_{j=0}^{t-1} \left(\frac{\phi(p-1)}{p-1} \sum_{\substack{s|p-1 \\ s>1}} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi=s \\ \chi \bmod p}} \chi(q_p(a+jb)) + O\left(\frac{1}{p}\right) \right) \\
&= \frac{\phi(p-1)}{p-1} \sum_{\substack{s|p-1 \\ s>1}} \frac{\mu(s)}{\phi(s)} \sum_{\substack{\text{ord } \chi=s \\ \chi \bmod p}} \sum_{j=0}^{t-1} \chi(q_p(a+jb)) + O(p) \\
&\ll 2^{\omega(p-1)} p^{\frac{3}{2}} \log p.
\end{aligned}$$

Therefore,

$$W(\mathcal{R}, p^2) \ll 2^{\omega(p-1)} p^{\frac{3}{2}} \log p.$$

For integers $d_1, d_2, \dots, d_k$ and M with $0 \leq d_1 < d_2 < \dots < d_k \leq p^2 - M$, from (5.1) and Proposition 2.4 we have

$$\begin{aligned}
&\sum_{n=1}^M e_{n+d_1} \cdots e_{n+d_k} \\
&= \sum_{n=1}^M \left(\frac{\phi(p-1)}{p-1} \sum_{\substack{s_1|p-1 \\ s_1>1}} \frac{\mu(s_1)}{\phi(s_1)} \sum_{\substack{\text{ord } \chi_1=s_1 \\ \chi_1 \neq \chi_0 \\ \chi_1 \bmod p}} \chi_1(q_p(n+d_1)) + O\left(\frac{1}{p}\right) \right) \times \cdots \\
&\quad \times \left(\frac{\phi(p-1)}{p-1} \sum_{\substack{s_k|p-1 \\ s_k>1}} \frac{\mu(s_k)}{\phi(s_k)} \sum_{\substack{\text{ord } \chi_k=s_k \\ \chi_k \neq \chi_0 \\ \chi_k \bmod p}} \chi_k(q_p(n+d_k)) + O\left(\frac{1}{p}\right) \right) \\
&= \frac{\phi^k(p-1)}{(p-1)^k} \sum_{\substack{s_1|p-1 \\ s_1>1}} \frac{\mu(s_1)}{\phi(s_1)} \sum_{\substack{\text{ord } \chi_1=s_1 \\ \chi_1 \neq \chi_0 \\ \chi_1 \bmod p}} \cdots \sum_{\substack{s_k|p-1 \\ s_k>1}} \frac{\mu(s_k)}{\phi(s_k)} \sum_{\substack{\text{ord } \chi_k=s_k \\ \chi_k \neq \chi_0 \\ \chi_k \bmod p}} \\
&\quad \times \sum_{n=1}^M \chi_1(q_p(n+d_1)) \cdots \chi_k(q_p(n+d_k)) + O\left(k 2^{k\omega(p-1)} p\right) \ll k 2^{k\omega(p-1)} p^{\frac{5}{3}}.
\end{aligned}$$

Hence,

$$C_k(\mathcal{R}, p^2) \ll k 2^{k\omega(p-1)} p^{\frac{5}{3}}.$$

This completes the proof of Theorem 1.3.

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