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Powerful numbers in the product of consecutive integer values of a polynomial

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Abstract. Let n and r be positive integers. Also let k be an odd positive integer and d be a non-negative integer. In this paper, we prove that if k has at most four distinct prime factors, then the product $((d+1)^k+r^k)((d+2)^k+r^k)\cdots((d+n)^k+r^k)$ is not a powerful number for $n \geq \max\{r+d, 59-r-d\}$. As a consequence, we prove that if k has at most four distinct prime factors, then the product $(1^k+1)(2^k+1)\cdots(n^k+1)$ is not a powerful number.

1. Introduction

A positive integer a is called a powerful number if $p \mid a$ implies $p^2 \mid a$ for any prime p. For example, perfect powers are powerful numbers. Consider a polynomial $f(x) \in \mathbb{Z}[x]$ of degree $k \geq 1$. Then one can ask: what are the perfect powers or powerful numbers in the product of consecutive integer values of $f(x)$, i.e, in the product

$$f(d+1)f(d+2)\cdots f(d+n)$$

for some integers $d \geq 0$ and $n \geq 1$? Denote the product $f(d+1)f(d+2)\cdots f(d+n)$ by $A_d(n)$ corresponding to a given polynomial $f(x)$ with integer coefficients. For $d=0$, we denote $A_0(n)$ simply by $A(n)$.

This question arises from a famous result of Erdős and Selfridge [5], which states that there are no perfect powers in the product of consecutive integers.

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Hence, for $f(x) = x$, they proved that $A_d(n)$ can never be a perfect power for $n > 1$. There are several other results in the literature concerning perfect powers in the product of consecutive integer values of a linear polynomial. For example, Győry, Hajdu and Pintér [9] proved that if $f(x) = ax + b$ for some coprime fixed positive integers a and b, then $A(n)$ can never be a perfect power for $3 < n < 35$.

Now, consider $f(x)$ is a polynomial of degree $k \geq 2$. Corresponding to the polynomial $x^2 + 1$, Amdeberhan, Medina and Moll [1] conjectured that the product $A(n)$ is not a square for any integer $n > 3$. In 2008, J. Cilleruelo [4] confirmed this conjecture by using Chebyshev's upper bound inequality for prime numbers. In fact, he proved that $A(n)$ is not powerful for $n > 3$. His technique was applied to the product $A(n)$ corresponding to the polynomials such as $4x^2 + 1$ and $2x^2 - 2x + 1$ by Fang [6]. Corresponding to any irreducible quadratic polynomial $f(x) \in \mathbb{Z}[x]$, Zhang and Yuan [17] proved that the product $A(n)$ is not a square for $n > C(f)$, where $C(f)$ is a constant depending on $f(x)$. In 2011, Yang, Togbé and He [14] proved that corresponding to any irreducible quadratic polynomial $f(x) \in \mathbb{Z}[x]$, $A(n)$ is not a perfect power for $n \geq C$, where C is a computable constant depending only on the coefficients of $f(x)$ and $d \geq 1$. Recently, Gürel [7] considered the polynomials x^k+1 for $k = 2$ and 3. He proved that corresponding to these polynomials there exists a positive real number N_d such that for $n \geq N_d$, $A_d(n)$ is not a square.

Now consider the polynomial $x^k + 1$ for some positive integer k. Corresponding to this polynomial, Amdeberhan, Medina and Moll [1] also claimed that if $n > 12$, then $A(n)$ is not a square for any odd prime k. Gürel and Kişisel [8] proved that $A(n)$ is never powerful for $k = 3$. Later, using an idea due to W. Zudilin, Zhang and Wang [16] proved that $A(n)$ is not a powerful number for any odd prime $k \geq 5$. Recently, Chen *et al.* [3] proved that $A(n)$ is not a powerful number for any odd prime power k. In the same paper, they also proved that for any positive odd number k, there exists an integer N such that for any positive integer $n \geq N$, $A(n)$ is not a powerful number. In 2013, Chen and Gong [2] proved that $A(n)$ is not a powerful number, when k is a product of at most two odd prime factors. In the same paper, they also proved that for any integer $t \geq 1$, there exists a positive integer T_t such that if k is a positive integer composed with t distinct odd prime factors and n is an integer with $n \geq T_t$, then $A(n)$ is not a powerful number.

In 2017, Niu and Liu [11] considered the polynomial $x^3 + r^3$ for some fixed positive integer r. They proved that for any positive integers r and $n \geq \max\{r, 1198-r\}$, the product $A(n)$ is not a powerful number. Very recently, Yang

and JHAO [15] proved that $A(n)$ is not a powerful number for $n \geq \max\{r, 11-r\}$ and any odd prime power k, corresponding to the more generalized polynomial $x^k + r^k$, where r is a fixed positive integer. They [15] also proved that, for any positive integer r and odd positive integer k, there exists an integer $N_{r,k}$ such that $A(n)$ is not a powerful number for $n \geq N_{r,k}$.

In this paper, we generalize the results of [15] and [2] considering k as a product of at most four distinct odd primes.

2. Main results

Let r be a fixed positive integer and k be a positive integer. Consider the polynomial

$$f(x) = x^k + r^k.$$

Now we consider the product of consecutive integer values of $f(x)$. For any positive integer n and non-negative integer d, we have

$$A_d(n) = \prod_{x=d+1}^{d+n} f(x) = ((d+1)^k + r^k)((d+2)^k + r^k) \cdots ((d+n)^k + r^k).$$

For $d = 0$, we have

$$A(n) = \prod_{x=1}^{n} f(x) = (1^k + r^k)(2^k + r^k) \cdots (n^k + r^k).$$

Then we prove the following results.

Theorem 2.1. *If k is an odd positive integer composed of at most four distinct primes, then for any integer $n \geq \max\{r + d, 59 - r - d\}$, $A_d(n)$ is not a powerful number.*

Remark 2.2. If k is composed of at most three distinct odd primes, then for any integer $n \geq \max\{r + d, 29 - r - d\}$, $A_d(n)$ is not a powerful number. If k is composed of at most two distinct odd primes, then for any integer $n \geq \max\{r + d, 23 - r - d\}$, $A_d(n)$ is not a powerful number. If k is an odd prime or odd prime power, then for any integer $n \geq \max\{r + d, 11 - r - d\}$, $A_d(n)$ is not a powerful number.

Remark 2.3. By the methods in [2], it is easy to prove that for any integer $n \geq \max\{r+d, C(t)-r-d\}$, $A_d(n)$ is not a powerful number, where k is an odd positive integer composed of t distinct primes, and $C(t)$ is a constant depending only on t.

Corollary 2.4. *If k is an odd positive integer composed of at most four distinct primes, then for any integer $n \geq \max\{r, 59-r\}$, $A(n)$ is not a powerful number.*

As a consequence, we have the following result.

Theorem 2.5. *If k is an odd positive integer composed of at most four distinct primes, then for any positive integer n, $(1^k+1)(2^k+1)\cdots(n^k+1)$ is not a powerful number.*

3. Preliminaries

In this section, we provide some useful lemmas which are essential to prove our main results.

Lemma 3.1. *Let r be a positive integer and p be an odd prime such that $p \nmid r$. Also, let a, k be positive integers with k odd. If*

$$p \mid \frac{a^k + r^k}{a+r},$$

then $(p(p-1), k) > 1$.

Proof. On the contrary, consider $(p(p-1), k) = 1$. As $p \mid (a^k + r^k)$, we have $p \mid (a^{2k} - r^{2k})$. By Fermat's little theorem, $p \mid (a^{p-1} - r^{p-1})$. Hence $p \mid (a^{(2k,p-1)} - r^{(2k,p-1)})$. So we have $p \mid (a^2 - r^2)$. If $p \mid (a-r)$, then $p \mid (a^k - r^k)$. Since $p \mid (a^k + r^k)$, we have $p \mid 2r^k$, which is not possible as $p \nmid r$. Hence, $p \nmid (a-r)$ as $p \nmid r$. Thus, it follows that $p \mid (a+r)$. Thus,

$$\frac{a^k + r^k}{a+r} \equiv a^{k-1} - a^{k-2}r + \cdots + r^{k-1} \equiv kr^{k-1} \pmod{p}.$$

Therefore, we have

$$kr^{k-1} \equiv 0 \pmod{p}, \quad \text{as } p \mid \frac{a^k + r^k}{a+r},$$

which is a contradiction to $(p(p-1), k) = 1$, since $p \nmid r$. Hence $(p(p-1), k) > 1$. □

Lemma 3.2. *Let p be a prime, and r, k be positive integers with k odd and $(k, p-1) = 1$. Then the congruence equation $x^k + r^k \equiv 0 \pmod{p}$ has only one solution $x \equiv -r \pmod{p}$.*

PROOF. If $p \mid r$, then $x \equiv 0 \equiv -r \pmod{p}$ is the only solution for the congruence equation $x^k + r^k \equiv 0 \pmod{p}$. Thus, we can assume that $p \nmid r$. So, $p \nmid x$ as $x^k + r^k \equiv 0 \pmod{p}$. Hence, we have $(-\frac{x}{r})^k \equiv 1 \pmod{p}$. So, the order of the element $(-\frac{x}{r})$ in the group $(\mathbb{Z}/p\mathbb{Z})^*$ divides k. Since $(\mathbb{Z}/p\mathbb{Z})^*$ contains exactly $(p-1)$ elements and $(p-1, k) = 1$, we see that the order of $(-\frac{x}{r})$ must be 1 in $(\mathbb{Z}/p\mathbb{Z})^*$. Hence $(-\frac{x}{r}) \equiv 1 \pmod{p}$, which further implies $x \equiv -r \pmod{p}$. □

Corollary 3.3. *Let r be a positive integer and $k = \prod_{i=1}^{t} p_i^{e_i}$, where e_i's are non-negative integers with at least one $e_i > 0$, and p_i's are distinct odd primes. If p is a prime with $p_i \nmid (p-1)$ for all $1 \le i \le t$, then the congruence equation $x^k + r^k \equiv 0 \pmod{p}$ has only one solution $x \equiv -r \pmod{p}$.*

For any non-negative integer d and positive integers n, k, ℓ with $(k, \ell) = 1$, define

$$P_d(n) = \left\{ q : q \text{ is a prime, } \frac{n+d}{2} < q \le n + d \right\},$$

$$P_d(n; k, \ell) = \left\{ q : q \text{ is a prime, } \frac{n+d}{2} < q \le n + d, \quad q \equiv \ell \pmod{k} \right\}.$$

For a nonzero integer u and a prime p, let $\nu_p(u)$ denote the smallest non-negative integer t such that $p^t \mid u$ but $p^{t+1} \nmid u$.

Lemma 3.4. *Let $k = \prod_{i=1}^{t} p_i^{e_i}$ be an odd positive integer with at least one $e_i > 0$, where p_i's are distinct odd primes. Let p be an odd prime such that $p \ne p_i$ and $p_i \nmid p-1$ for all $i = 1, 2, \ldots, t$. Also let n, r be positive integers, and d be a non-negative integer such that $n \ge r + d$. If $p \in P_{r+d}(n)$, then $A_d(n)$ is not a powerful number.*

PROOF. Since $p \in P_{r+d}(n)$ and $n \ge r + d$, we have $p > \frac{n+d+r}{2} \ge r + d$. By Corollary 3.3, the smallest two positive integers x satisfying the congruence equation $(x+d)^k + r^k \equiv 0 \pmod{p}$ are $p - d - r$ and $2p - d - r$, as $p > r + d$.

Since $p \ne p_i$ and $p_i \nmid p-1$ for all $i = 1, 2, \ldots, t$, we have $(p(p-1), k) = 1$. Also we see that $(p-r)$ is a positive integer and $p \nmid r$ as $p > r + d$. Hence, by Lemma 3.1, we have $p \nmid \frac{(p-r)^k + r^k}{(p-r)+r}$. Thus, $p^2 \nmid ((p-r)^k + r^k)$. Hence, if $p - d - r \le n < 2p - d - r$, then

$$\nu_p(A_d(n)) = \nu_p((p-r)^k + r^k) = 1.$$

Thus, $A_d(n)$ is not a powerful number. □

Corollary 3.5. *Let $k > 1$ be an odd positive integer. Also let n, r be positive integers and d be a non-negative integer such that $n \geq r + d$. Suppose that $A_d(n)$ is a powerful number. If an odd prime $p \in P_{r+d}(n)$, then $(p(p-1), k) > 1$.*

Corollary 3.6. *Let k, n be positive integers with $k(\neq 1)$ odd. Suppose that $(1^k + 1)(2^k + 1) \cdots (n^k + 1)$ is a powerful number. If an odd prime $p \in P_1(n)$, then $(p(p-1), k) > 1$.*

Lemma 3.7. *Let $k > 1$ be an odd positive integer. Also let n, r be positive integers, and d be a non-negative integer such that $n \geq r + d$. Suppose there exist t distinct primes $p_1, p_2, \ldots, p_t \in P_{r+d}(n)$, such that no two of $p_1 - 1, p_2 - 1, \ldots, p_t - 1$ have common odd prime factors. If $A_d(n)$ is a powerful number, then k has at least t distinct prime factors.*

Proof. Since

$$\frac{n + d + r}{2} < p_i \leq n + d + r,$$

we see that $(p_i, p_j - 1) = 1$, for all $1 \leq i, j \leq t$.

Now by Corollary 3.5,

$$(p_i(p_i - 1), k) > 1, \quad \text{for all } 1 \leq i \leq k.$$

Since no two of $p_1 - 1, p_2 - 1, \ldots, p_t - 1$ have common odd prime factors, it follows that k has at least t distinct prime factors. □

Lemma 3.8. *Let k be an odd positive integer composed of t distinct primes. Also let n, r be positive integers, and d be a non-negative integer such that $n \geq r + d$. Suppose that there exist $t+1$ distinct primes $p_1, p_2, \ldots, p_{t+1}$ with $3 \leq p_1 < p_2 < \cdots < p_{t+1}$ such that no two of $p_1 - 1, p_2 - 1, \ldots, p_{t+1} - 1$ have common odd prime factors. Then, for $p_{t+1} - d - r \leq n < 2p_1 - d - r$, $A_d(n)$ is not a powerful number.*

Proof. For all integers n with $p_{t+1} - d - r \leq n < 2p_1 - d - r$, we have $p_1, p_2, \ldots, p_{t+1} \in P_{r+d}(n)$.

If $A_d(n)$ is a powerful number, then by Lemma 3.7, k has at least $t+1$ prime factors, which is a contradiction. □

Lemma 3.9. *Let k be an odd positive integer and n, r be positive integers. Let p be an odd prime with $p \nmid r$ such that $p || (a^{(k,p-1)} + r^{(k,p-1)})$ for some $2 \leq a \leq n$, and $p \nmid (b^{(k,p-1)} + r^{(k,p-1)})$ for all $2 \leq b \leq n$ with $b \neq a$. If $p^2 \mid A(n)$, then $p \mid k$.*

PROOF. Let, $(k, p-1) = u$ and $uv = k$ for some integers u, v. Since $p^2 \mid A(n)$, there exists an integer c with $2 \leq c \leq n$ such that $p \mid (c^k + r^k)$. Hence, we have $p \mid (c^{2k} - r^{2k})$. By Fermat's little theorem, we have $p \mid (c^{p-1} - r^{p-1})$. Thus, we have $p \mid (c^{(2k,p-1)} + r^{(2k,p-1)})$, which further implies that $p \mid (c^{2u} - r^{2u})$.

If $p \mid (c^u - r^u)$, then $p \mid (c^k - r^k)$, which shows that $p \mid r$, a contradiction. Hence $p \nmid (c^u - r^u)$. Thus we see that $p \mid (c^u + r^u)$. Since $p||(a^{(k,p-1)} + r^{(k,p-1)})$ for some $2 \leq a \leq n$ and $p \nmid (b^{(k,p-1)} + r^{(k,p-1)})$ for all $2 \leq b \leq n$ with $b \neq a$, we conclude that $c = a$.

Since $p^2 \mid A(n)$, it follows that $p^2 \mid (a^k + r^k)$. Hence, $p \mid \frac{a^k + r^k}{a^u + r^u}$ as $p||(a^u + r^u)$. Thus,

$$\frac{a^{uv} + r^{uv}}{a^u + r^u} \equiv a^{u(v-1)} - a^{u(v-2)} r^u + \cdots + r^{u(v-1)} \equiv v r^{u(v-1)} \pmod{p}.$$

Hence, we have

$$v r^{u(v-1)} \equiv 0 \pmod{p}, \text{ as } p \mid \frac{a^{uv} + r^{uv}}{a^u + r^u}.$$

It follows that $p \mid v$ as $p \nmid r$. Therefore, $p \mid k$. □

If $n + d + r = m$, then $P_{r+d}(m - d - r) = \{p \text{ is a prime }, \frac{m}{2} < p \leq m\}$ and $P_{r+d}(m - d - r; k, \ell) = \{p \text{ is a prime }, \frac{m}{2} < p \leq m, p \equiv \ell \pmod{k}\}$. Denote $P_{r+d}(m - d - r)$ by $P(m)$ and $P_{r+d}(m - d - r; k, \ell)$ by $P(m; k, \ell)$.

Also let $\mathcal{P}$ be the set of all primes. Then for any positive real number x, denote

$$\pi(x) = |\{p \in \mathcal{P} : p \leq x\}|, \qquad \pi(x; k, \ell) = |\{p \in \mathcal{P} : p \leq x, p \equiv \ell \pmod{k}\}|,$$

and

$$\theta(x; k, \ell) = \sum_{p \equiv \ell \pmod{k}}^{p \leq x} \log p,$$

where k, ℓ are relatively prime positive integers.

Hence,

$$|P(m)| = \pi(m) - \pi(m/2)$$

and

$$|P(m; k, \ell)| = \pi(m; k, \ell) - \pi(m/2; k, \ell).$$

Remark 3.10. For positive real numbers x and y, we have

$$(\pi(x + y; k, \ell) - \pi(x; k, \ell)) \log x \leq \theta(x + y; k, \ell) - \theta(x; k, \ell) \leq (\pi(x + y; k, \ell) - \pi(x; k, \ell)) \log(x + y),$$

where k and ℓ are relatively coprime integers.

Hence, for $x = \frac{m}{2}$ and $y = \frac{m}{2}$, we have

$$\frac{\theta(m;k,\ell) - \theta(m/2;k,\ell)}{\log m} \leq |P(m;k,\ell)| \leq \frac{\theta(m;k,\ell) - \theta(m/2;k,\ell)}{\log \frac{m}{2}}. \tag{3.1}$$

Lemma 3.11 ([10])**.** *Let x and y be positive real numbers with $y \leq x$. Also let k and ℓ be positive integers with $(k,\ell) = 1$. For $y > k$, we have*

$$\pi(x+y;k,\ell) - \pi(x;k,\ell) < \frac{2y}{\varphi(k)\log(y/k)}, \tag{3.2}$$

where $\varphi(k)$ is the Euler totient function.

Corollary 3.12. *Let m be a positive integer. Then for any positive integers k and ℓ with $(k,\ell) = 1$,*

$$|P(m,k,\ell)| < \frac{m}{\varphi(k)\log(m/2k)},$$

for $m > 2k$.

Proof. Put $x = y = m/2$ in inequality (3.2). Then,

$$|P(m;k,\ell)| = \pi(m;k,\ell) - \pi\left(\frac{m}{2};k,\ell\right) < \frac{m}{\phi(k)\log(m/2k)},$$

where $m > 2k$. □

Lemma 3.13 ([12])**.** *For any triple (k,ϵ,x_0) given in* [12, Table 1] *and any ℓ prime to k, we have*

$$\max_{1\leq y\leq x}\left|\theta(y;k,\ell) - \frac{y}{\phi(k)}\right| \leq \epsilon\frac{x}{\phi(k)},$$

for $x \geq x_0$.

In particular, ϵ is given explicitly in [12, Table 1] *for $x \geq 10^{10}$ and $k \leq 72$.*

Corollary 3.14. *If m, k and ℓ are positive integers with $(k,\ell) = 1$, then*

$$\frac{m}{\phi(k)\log m}(1/2 - 2\epsilon) \leq |P(m;k,\ell)| \leq \frac{m}{\phi(k)\log\frac{m}{2}}(1/2 + 2\epsilon),$$

for $m \geq 10^{10}$ and $k \leq 72$, where ϵ is given in [12, Table 1].

PROOF. Taking $x = y = m$ in Lemma 3.13, we have

$$\frac{m(1-\epsilon)}{\phi(k)} \leq \theta(m;k,\ell) \leq \frac{m(1+\epsilon)}{\phi(k)}. \tag{3.3}$$

Again, by taking $x = m$ and $y = m/2$ in Lemma 3.13, we have

$$\frac{m(1-2\epsilon)}{2\phi(k)} \leq \theta(m/2;k,\ell) \leq \frac{m(1+2\epsilon)}{2\phi(k)}. \tag{3.4}$$

We now use the inequality (3.1) to obtain the result. □

Lemma 3.15 ([12])**.** *For any positive real number $x \leq 10^{10}$ and integers k, ℓ with $(k,\ell) = 1$, we have*

$$\left|\theta(x;k,\ell) - \frac{x}{\phi(k)}\right| < \theta\sqrt{x},$$

where θ is given in [12, Table 2].

Corollary 3.16. *For any positive integers m, k, ℓ with $m \leq 10^{10}$ and $(k,\ell) = 1$, we have*

$$\frac{m}{\log m}\left(\frac{1}{2\phi(k)} - \frac{\theta}{\sqrt{m}}\left(1+\frac{1}{\sqrt{2}}\right)\right) < |P(m;k,\ell)| < \frac{m}{\log \frac{m}{2}}\left(\frac{1}{2\phi(k)} + \frac{\theta}{\sqrt{m}}\left(1+\frac{1}{\sqrt{2}}\right)\right),$$

where θ is given in [12, Table 2].

PROOF. It follows from the inequality (3.1) and Lemma 3.15. □

Corollary 3.17. *For any positive integer m with $m \geq 94000$, we have*

(1) $0.4885\frac{m}{\log m} < |P(m)| < 0.5445\frac{m}{\log m}$;
(2) $0.2399\frac{m}{\log m} < |P(m;3,\ell)| < 0.2768\frac{m}{\log m}$;
(3) $|P(m;5,\ell)| < 0.1415\frac{m}{\log m}$;
(4) $|P(m;7,\ell)| < 0.0953\frac{m}{\log m}$;
(5) $|P(m;11,\ell)| < 0.0591\frac{m}{\log m}$;
(6) $|P(m;13,\ell)| < 0.0497\frac{m}{\log m}$;
(7) $|P(m;15,\ell)| > 0.0563\frac{m}{\log m}$.

PROOF. First, we notice that $\log \frac{x}{2}/\log x$ is an increasing function in any positive real number x. Hence, $\log \frac{m}{2} \geq 0.9394 \log m$ for any positive integer $m \geq 94000$, and $\log \frac{m}{2} \geq 0.9699 \log m$ for any positive integer $m \geq 10^{10}$. Then we use Corollaries 3.14 and 3.16 to obtain the result. □

Corollary 3.18. *Let q be a prime. Then for any positive integer m with $m \geq 94000$, we have*

(1) $|P(m;q,\ell)| < 0.0392\frac{m}{\log m}$, *for primes* $17 \leq q \leq 71$;

(2) $|P(m;q,\ell)| < 0.0246\frac{m}{\log m}$, *for primes* $q \geq 73$.

PROOF. For $m \geq 94000$, we have $\log \frac{m}{2} \geq 0.9394 \log m$. Then by using Corollaries 3.14 and 3.16, for any $17 \leq q \leq 71$ we can show that $|P(m;q,\ell)| < 0.0392\frac{m}{\log m}$.

Now take $q \geq 73$. Consider the function

$$f(x) = (x-1)\log\frac{m}{2x}.$$

This is an increasing function in $x \in [73, \frac{m}{2e}]$.

If $q \leq \frac{m}{2e}$, then by Corollary 3.12,

$$|P(m;q,\ell)| < \frac{m}{(q-1)\log(m/2q)} < \frac{m}{72\log(m/146)} < 0.0246\frac{m}{\log m},$$

since for $m \geq 94000$, $\log(m/146) > 0.56479 \log m$ and $q \geq 73$.

If $q > \frac{m}{2e}$, then

$$|P(m;q,\ell)| \leq 2 < 0.0246\frac{m}{\log m},$$

for $m \geq 94000$. □

4. Proof of Theorem 2.1

By Bertrand's postulate, there exists a prime $p \in (\frac{n+d+r}{2}, n+d+r]$. Now, $\frac{n+d+r}{2} \geq r+d$, as $n \geq r+d$. Hence, for $k=1$, $\nu_p(A_d(n)) = \nu_p((1+d+r)(2+d+r)\cdots(n+d+r) = 1$, which shows that $A_d(n)$ is not powerful. Thus, we can assume that $k > 1$.

Let $k = \prod_{i=1}^{4} p_i^{e_i}$, where p_i's are distinct odd primes and $e_i \geq 0$ with at least one $e_i > 0$. Then, by Lemma 3.4, it is enough to prove that there exists a prime $p \in P_{r+d}(n)$ such that $p \neq p_i$ and $(p-1, p_i) = 1$ for all $i = 1, 2, 3, 4$. Hence, in other words, we need to prove that there exists a prime $p \in P(m)$ such that $p \neq p_i$ and $(p-1, p_i) = 1$ for all $i = 1, 2, 3, 4$.

Case 1. $m \geq 94000$.

Subcase (i). $3 \nmid k$.
Then $p_i > 3$ for all $i = 1, 2, 3, 4$. Hence,

$$|P(m)| > 0.4885\frac{m}{\log m} > \sum_{i=1}^{4} |P(m; p_i, 1)| + 4,$$

by Corollaries 3.17 and 3.18, which shows that there exists a prime $p \in P(m)$ such that $p \neq p_i$ and $p \not\equiv 1 \pmod{p_i}$ for all $i = 1, 2, 3, 4$.

Subcase (ii). $3 \mid k, 5 \nmid k$.
Then $p_i \neq 5$ for all $i = 1, 2, 3, 4$. Let $p_1 = 3$ and $p_i > 5$ for all $i = 2, 3, 4$. Then,

$$|P(m)| > 0.4885\frac{m}{\log m} > \sum_{i=1}^{4} |P(m; p_i, 1)| + 4,$$

by Corollaries 3.17 and 3.18, which shows that there exists a prime $p \in P(m)$ such that $p \neq p_i$ and $p \not\equiv 1 \pmod{p_i}$ for all $i = 1, 2, 3, 4$.

Subcase (iii). $3 \mid k, 5 \mid k$.
Let $p_1 = 3$ and $p_2 = 5$. So, $p_i \geq 7$ for all $i = 3, 4$.

Now, we have

$$\begin{aligned} |P(m; 3, 2)| &> 0.2399\frac{m}{\log m} \\ &> |P(m; 5, 1)| + \sum_{i=3}^{4} |P(m; p_i, 1)| - |P(m; 15, 1)| + 4, \end{aligned} \tag{4.1}$$

by using Corollaries 3.17 and 3.18.

We notice that

$$\begin{aligned} &\sum_{i=1}^{4} |P(m; p_i, 1)| - \sum_{p_i < p_j} |P(m; p_i p_j, 1)| \\ &\quad + \sum_{p_i < p_j < p_k} |P(m; p_i p_j p_k, 1)| - |P(m; p_1 p_2 p_3 p_4, 1)| \\ &< \sum_{i=1}^{4} |P(m; p_i, 1)| - |P(m; p_1 p_2, 1)|. \end{aligned} \tag{4.2}$$

Since $|P(m)| = |P(m;3,1)| + |P(m;3,2)| + 1$, using inequalities (4.1) and (4.2), we have

$$\begin{aligned}|P(m)| &> |P(m;3,1)| + |P(m;5,1)| + \sum_{i=3}^{4} |P(m;p_i,1)| + 5 \\ &> \sum_{i=1}^{4} |P(m;p_i,1)| - \sum_{p_i<p_j} |P(m;p_ip_j,1)| + \sum_{p_i<p_j<p_k} |P(m;p_ip_jp_k,1)| \\ &\quad - |P(m;p_1p_2p_3p_4,1)| + 4. \end{aligned} \tag{4.3}$$

Therefore there exists a prime $p \in P(m)$ such that $p \neq p_i$ and $p \not\equiv 1 \pmod{p_i}$ for all $i = 1, 2, 3, 4$.

Case 2. $59 \leq m \leq 94000$.

Here we use Lemma 3.8. First, we construct a set of prime quintuples $(p_1, p_2, p_3, p_4, p_5)$ with $p_1 < p_2 < p_3 < p_4 < p_5$ in $P(m)$ such that $p_1 - 1, p_2 - 1$, $p_3 - 1, p_4 - 1$ and $p_5 - 1$ have no common odd prime factors. Consider

$$\begin{aligned}&(p_1, p_2, p_3, p_4, p_5) \\ &\quad \in \{\, (37, 41, 47, 53, 59), (47, 53, 59, 71, 73), (59, 71, 73, 83, 89), \\ &\qquad (89, 97, 101, 107, 113), (131, 137, 139, 149, 167), (227, 229, 239, 251, 257), \\ &\qquad (419, 431, 439, 443, 449), (797, 809, 821, 823, 827), \\ &\qquad (1553, 1559, 1571, 1579, 1583), (3023, 3041, 3049, 3083, 3089), \\ &\qquad (5939, 5953, 5987, 6011, 6029), (11807, 11813, 11831, 11863, 11867), \\ &\qquad (23561, 23563, 23567, 23603, 23609), (47057, 47059, 47087, 47093, 47111)\,\}.\end{aligned}$$

For these quintuples we see that $A_{r,k,d}(n)$ is not powerful when m lies in intervals $[59, 73]$,$[73, 93]$,$[89, 117]$,$[113, 177]$,$[167, 261]$,$[257, 453]$,$[449, 837]$,$[827, 1593]$, $[1583, 3105]$,$[3089, 6045]$,$[6029, 11877]$,$[11867, 23613]$,$[23609, 47121]$,$[47111, 94113]$. Combining all these intervals, we can see that $A_d(n)$ is not powerful for $59 \leq m \leq 94000$.

Hence, from both cases we see that $A_d(n)$ is not powerful for $m \geq 59$. Since $n + r + d = m$ and $n \geq r + d$, we observe that $A_d(n)$ is not powerful for $n \geq \max\{r + d, 59 - r - d\}$. This completes the proof of Theorem 2.1. □

5. Proof of Theorem 2.5

Consider $f(x) = x^k + 1$, where k is an odd positive integer composed of at most four distinct primes. Hence $A(n) = (1^k+1)(2^k+1)\cdots(n^k+1)$. Also, denote $P_1(n)$ by $P(n)$. Hence $P(n) = \{q : q \text{ is a prime, } \frac{n+1}{2} < q \leq n+1\}$.

For $n = 1$, $A(n) = 2$, which is not powerful. For $n = 2$, $(n) = 2(2^k + 1)$, which is again not powerful, as $\nu_2(2(2^k + 1)) = 1$. Hence, we can assume that $n \geq 3$.

By Corollary 2.4, we see that $A(n)$ is not powerful for $n \geq 58$. Finally, we consider the case $3 \leq n \leq 57$. Let us assume that $A(n)$ is a powerful number for $3 \leq n \leq 57$.

The idea is as follows. For different values of $3 < n \leq 57$, first of all we find out all possible primes in $P(n)$. Then by using Corollary 3.6, one can find out some possible odd prime divisors of k. Let q be such a prime divisor. Then we find an odd prime p with $(p-1, k) = q$ such that $p \,||\, (a^q + 1)$ for some $2 \leq a \leq n$ but $p \nmid (b^q + 1)$ for all $2 \leq b \leq n$ with $b \neq a$. Then by using Lemma 3.9, we can conclude $p \mid k$. We continue in this way to find at least five distinct prime divisors of k to get a contradiction, as k has at most four prime factors. We discuss all possible cases in detail below.

Case 1. $52 \leq n \leq 57$.

We see that $31, 37, 41, 43, 47, 53 \in P(n)$. Then by Corollary 3.6, we have $(31\times3\times5, k) > 1, (37\times3, k) > 1, (41\times5, k) > 1, (43\times3\times7, k) > 1, (47\times23, k) > 1$ and $(53 \times 13, k) > 1$. Hence $p_1p_2p_3 \mid k$, where $p_1 \in \{47, 23\}$, $p_2 \in \{13, 53\}$ and $p_3 \in \{5, 41\}$.

Subcase (i). $3 \mid k$.

Now, $(97 - 1, k) = 3$. We notice that $97 \,||\, (36^3 + 1)$ and $97 \nmid (b^3 + 1)$ for all $2 \leq b \leq 57$ with $b \neq 36$. Hence, by Lemma 3.9, $97 \mid k$, which is not possible as k has at most four distinct prime factors.

Subcase (ii). $3 \nmid k$.

Since $(37 \times 3, k) > 1$ and $(43 \times 3 \times 7, k) > 1$, we have $37 \mid k$ and $43 \mid k$ or $7 \mid k$, which is not possible again.

Case 2. $37 \leq n \leq 51$.

We see that $29, 31, 37 \in P(n)$. Then by Corollary 3.6, we have $(29\times7, k) > 1$, $(31 \times 3 \times 5, k) > 1$ and $(37 \times 3, k) > 1$. If $45 \leq n \leq 51$, then $41 \in P(n)$. Then by Corollary 3.6, $(41 \times 5, k) > 1$. Hence for $45 \leq n \leq 51$, $p_1p_2 \mid k$, where $p_1 \in \{41, 5\}$ and $p_2 \in \{7, 29\}$. If $37 \leq n \leq 44$, then $23 \in P(n)$. Then by Corollary 3.6,

$(23 \times 11, k) > 1$. Hence for $37 \leq n \leq 44$, $q_1 q_2 \mid k$, where $q_1 \in \{23, 11\}$ and $q_2 \in \{7, 29\}$.

Subcase (i). $3 \mid k$.
Since $3 \mid k$, we see that $97 \mid k$, from Case 1. Since k has at most four distinct prime factors, we have $13 \nmid k$. Then $(157 - 1, k) = 3$. We notice that $157 \mid\mid (13^3 + 1)$ and $157 \nmid (b^3 + 1)$ for all $2 \leq b \leq 51$ with $b \neq 13$. Hence, by Lemma 3.9, $157 \mid k$, which is a contradiction.

Subcase (ii). $3 \nmid k$.
Since $(37 \times 3, k) > 1$, we have $37 \mid k$. If $53 \nmid k$, then $(3923 - 1, k) = 37$. We notice that $3923 \mid\mid (6^{37} + 1)$ and $3923 \nmid (e^{37} + 1)$ for all $2 \leq e \leq 51$ with $e \neq 6$. Hence, by Lemma 3.9, $53 \mid k$ or $3923 \mid k$. Since $3 \nmid k$, we have $31 \mid k$ or $5 \mid k$. This is not possible for $37 \leq n \leq 44$, as k has at most four distinct prime factors. If $45 \leq n \leq 51$, then $5 \mid k$ and $19 \nmid k$, as k has at most four distinct prime factors. Now, $(191 - 1, k) = 5$. We notice that $191 \mid\mid (7^5 + 1)$ and $191 \nmid (b^5 + 1)$ for all $2 \leq b \leq 15$ with $b \neq 7$. Hence, by Lemma 3.9, $191 \mid k$, which is a contradiction.

Case 3. $25 \leq n \leq 36$.

We see that $19, 23 \in P(n)$. Then by Corollary 3.6, we have $(19 \times 3, k) > 1$ and $(23 \times 11, k) > 1$. Hence $p_1 \mid k$, where $p_1 \in \{11, 23\}$.

Subcase (i). $3 \mid k$.
Since $3 \mid k$, we have $157 \mid k$ or $13 \mid k$, from Case 2. If $7 \nmid k$, then $(43 - 1, k) = 3$. We notice that $43 \mid\mid (7^3 + 1)$ and $43 \nmid (b^3 + 1)$ for all $2 \leq b \leq 36$ with $b \neq 7$. Hence, by Lemma 3.9, $43 \mid k$ or $7 \mid k$. Now, $5 \nmid k$ as k has at most four distinct prime factors. Hence $(61 - 1, k) = 3$. We notice that $61 \mid\mid (14^3 + 1)$ and $61 \nmid (b^3 + 1)$ for all $2 \leq b \leq 36$ with $b \neq 14$. Hence, by Lemma 3.9, $61 \mid k$, which is a contradiction.

Subcase (ii). $3 \nmid k$.
Since $(19 \times 3, k) > 1$, we have $19 \mid k$. Now, $(1217 - 1, k) = 19$. We notice that $1217 \mid\mid (16^{19} + 1)$ and $1217 \nmid (e^{19} + 1)$ for all $2 \leq e \leq 36$ with $e \neq 16$. Hence, by Lemma 3.9, $1217 \mid k$. If $33 \leq n \leq 36$, then $29, 31 \in P(n)$. Then by Corollary 3.6, we have $(29 \times 7, k) > 1$ and $(31 \times 5, k) > 1$. Hence, $p_2 p_3 \mid k$, where $p_2 \in \{29, 7\}$ and $p_3 \in \{31, 5\}$, which is a contradiction. If $25 \leq n \leq 32$, then $17 \in P(n)$. Then by Corollary 3.6, we have $17 \mid k$. If $5 \nmid k$, then $(2551 - 1, k) = 17$. We notice that $2551 \mid\mid (12^{17} + 1)$ and $2551 \nmid (f^{17} + 1)$ for all $2 \leq f \leq 32$ with $f \neq 12$. Hence, by Lemma 3.9, $2551 \mid k$ or $5 \mid k$, which is a contradiction.

Case 4. $13 \leq n \leq 24$.

We see that $13 \in P(n)$. Then by Corollary 3.6, we have $(13 \times 3, k) > 1$. If $16 \leq n \leq 24$, then $17 \in P(n)$. Then by Corollary 3.6, $17 \mid k$ for $16 \leq n \leq 24$.

If $13 \leq n \leq 15$, then $11 \in P(n)$. Then by Corollary 3.6, $(11 \times 5, k) > 1$. Hence for $13 \leq n \leq 15$, $p_1 \mid k$, where $p_1 \in \{11, 5\}$.

Subcase (i). $3 \mid k$.
Since $3 \mid k$, we have $157 \mid k$ or $13 \mid k$ and $7 \mid k$ or $43 \mid k$, from Case 3. If $5 \nmid k$, then $(31 - 1, k) = 3$. We notice that $31 \mid\mid (6^3 + 1)$ and $31 \nmid (b^3 + 1)$ for all $2 \leq b \leq 24$ with $b \neq 6$. Hence, by Lemma 3.9, $5 \mid k$ or $31 \mid k$. This is not possible for $16 \leq n \leq 24$, as k has at most four distinct prime factors. If $13 \leq n \leq 15$, then $5 \mid k$ and $19 \nmid k$, as k has at most four distinct prime factors. Hence, $191 \mid k$ from Case 2, which is a contradiction.

Subcase (ii). $3 \nmid k$.
Then $13 \mid k$. Now, $(157 - 1, k) = 13$. We notice that $157 \mid\mid (3^{13} + 1)$ and $157 \nmid (e^{13} + 1)$ for all $2 \leq e \leq 24$ with $e \neq 3$. Hence, by Lemma 3.9, $157 \mid k$. Again, $(937 - 1, k) = 13$. We see that $937 \mid\mid (6^{13} + 1)$ and $937 \nmid (f^{13} + 1)$ for all $2 \leq f \leq 24$ with $f \neq 6$. Hence, by Lemma 3.9, $937 \mid k$. Also, if $11 \nmid k$, $(859 - 1, k) = 13$. We notice that $859 \mid\mid (10^{13} + 1)$ and $859 \nmid (g^{13} + 1)$ for all $2 \leq g \leq 24$ with $g \neq 10$. Hence, by Lemma 3.9, $11 \mid k$ or $859 \mid k$. This is not possible for $16 \leq n \leq 24$, as k has at most four distinct prime factors. If $13 \leq n \leq 15$, then $11 \mid k$ and $5 \nmid k$, as k has at most four distinct prime factors. Now, $(5501 - 1, k) = 11$, as $5 \nmid k$. We notice that $5501 \mid\mid (9^{11} + 1)$ and $5501 \nmid (b^{11} + 1)$ for all $2 \leq b \leq 24$ with $b \neq 9$. Hence, by Lemma 3.9, $5501 \mid k$, which is a contradiction.

Case 5. $10 \leq n \leq 12$.

We see that $7, 11 \in P(n)$. Then by Corollary 3.6, we have $(11 \times 5, k) > 1$ and $(7 \times 3, k) > 1$. Hence $p_1 \mid k$, where $p_1 \in \{5, 11\}$.

Subcase (i). $3 \mid k$.
Since $3 \mid k$, we have $43 \mid k$ or $7 \mid k$ and $31 \mid k$ or $5 \mid k$, from Case 4. If $5 \nmid k$, then $(31 \times 11) \mid k$. Since $11 \mid k$ and $5 \nmid k$, we have $5501 \mid k$ from Case 4, which is a contradiction. Thus, $5 \mid k$. Hence $19 \mid k$ or $191 \mid k$, from Case 2. Now, $(41 - 1, k) = 5$. We notice that $41 \mid\mid (4^5 + 1)$ and $41 \nmid (c^5 + 1)$ for all $2 \leq c \leq 12$ with $c \neq 4$. Hence, by Lemma 3.9, $41 \mid k$, which is a contradiction.

Subcase (ii). $3 \nmid k$.
Then $7 \mid k$. Now, $(449 - 1, k) = 7$. We notice that $449 \mid\mid (5^7 + 1)$ and $449 \nmid (g^7 + 1)$ for all $2 \leq g \leq 12$ with $g \neq 5$. Hence, by Lemma 3.9, $449 \mid k$. If $13 \nmid k$, then $(547 - 1, k) = 7$. We notice that $547 \mid\mid (3^7 + 1)$ and $547 \nmid (h^7 + 1)$ for all $2 \leq h \leq 12$ with $h \neq 3$. Hence, by Lemma 3.9, $547 \mid k$ or $13 \mid k$. Now, $43 \nmid k$, as k has at most four distinct prime factors. Then $(5419 - 1, k) = 7$. We observe

that $5419 \mid\mid (8^7+1)$ and $5419 \nmid (i^7+1)$ for all $2 \leq i \leq 12$ with $i \neq 8$. Hence, by Lemma 3.9, $5419 \mid k$, which is a contradiction.

Case 6. $7 \leq n \leq 9$.

We see that $7 \in P(n)$. Then by Corollary 3.6, $(7 \times 3, k) > 1$.

Subcase (i). $3 \mid k$.

We see that $43 \mid k$ or $7 \mid k$ and $31 \mid k$ or $5 \mid k$, from Case 4. Now, $(13-1, k) = 3$. We notice that $13 \mid\mid (4^3+1)$ and $13 \nmid (c^3+1)$ for all $2 \leq c \leq 9$ with $c \neq 4$. Hence, by Lemma 3.9, $13 \mid k$. If $5 \mid k$, then $41 \mid k$, from Case 5, which is a contradiction, as k has at most four distinct prime factors. Hence $5 \nmid k$. Also $17 \nmid k$. Now, $(37571-1, k) = 13$. We notice that $37571 \mid\mid (6^{13}+1)$ and $37571 \nmid (e^{13}+1)$ for all $2 \leq e \leq 9$ with $e \neq 6$. Hence, by Lemma 3.9, $37571 \mid k$, which is a contradiction.

Subcase (ii). $3 \nmid k$.

Then $7 \mid k$. Now, we see that $13 \mid k$ or $547 \mid k$ and $449 \mid k$, from Case 5. If $7 \leq n \leq 8$, then $5 \in P(n)$. Then by Corollary 3.6, we have $5 \mid k$ for $7 \leq n \leq 8$. Hence $41 \mid k$, from Case 5, which is not possible for $7 \leq n \leq 8$. For $n = 9$, we have $43 \mid k$ or $5419 \mid k$, from Case 5, as $7 \mid k$. Since k is composed of at most four primes, we have $19 \nmid k$ and $31 \nmid k$. Now, $(16493-1, k) = 7$. We observe that $16493 \mid\mid (9^7+1)$ and $16493 \nmid (i^7+1)$ for all $2 \leq i \leq 9$ with $i \neq 9$. Hence, by Lemma 3.9, $16493 \mid k$, which is a contradiction again.

Case 7. $4 \leq n \leq 6$.

We see that $5 \in P(n)$. Then by Corollary 3.6, we have $5 \mid k$. Hence, $41 \mid k$, from Case 8.

Subcase (i). $3 \mid k$.

Since $3 \mid k$, we notice that, $13 \mid k$, from Case 6. Also, if $31 \nmid k$, then $(1613-1, k) = 13$. We observe that $1613 \mid\mid (4^{13}+1)$ and $1613 \nmid (h^{13}+1)$ for all $2 \leq h \leq 6$ with $h \neq 4$. Hence, by Lemma 3.9, either $1613 \mid k$ or $31 \mid k$, which is a contradiction.

Subcase (ii). $3 \nmid k$.

Now, $(61-1, k) = 5$. We notice that $61 \mid\mid (3^5+1)$ and $61 \nmid (e^5+1)$ for all $2 \leq e \leq 6$ with $e \neq 3$. Hence, by Lemma 3.9, $61 \mid k$. Also, $(733-1, k) = 61$. We notice that $733 \mid\mid (4^{61}+1)$ and $733 \nmid (f^{61}+1)$ for all $2 \leq f \leq 6$ with $f \neq 4$. Hence, by Lemma 3.9, $733 \mid k$. Also, if $7 \nmid k$, then $(1709-1, k) = 61$. We observe that $1709 \mid\mid (4^{61}+1)$ and $1709 \nmid (d^{61}+1)$ for all $2 \leq d \leq 6$ with $d \neq 4$. Hence, by Lemma 3.9, $1709 \mid k$ or $7 \mid k$, which is a contradiction.

Case 8. $n = 3$.

$3 \in P(n)$. Then by Corollary 3.6, $3 \mid k$. Now, $(7-1,k) = 3$. We notice that $7 \nmid (2^3+1)$ and $7 \mid\mid (3^3+1)$. Hence, by Lemma 3.9, $7 \mid k$. If $13 \nmid k$, then $(547-1,k) = 21$. We notice that $547 \mid\mid (3^{21}+1)$ and $547 \nmid (2^{21}+1)$. Hence, by Lemma 3.9, $547 \mid k$ or $13 \mid k$. If $43 \nmid k$, then $(5419-1,k) = 21$ or 63. We observe that $5419 \mid\mid (2^{21}+1)$, $5419 \mid\mid (2^{63}+1)$ and $5419 \nmid (3^{63}+1)$. Hence, by Lemma 3.9, either $43 \mid k$ or $5419 \mid k$. If $3^2 \mid k$, then $(127-1,k) = 63$. Now, $127 \mid\mid (3^{63}+1)$ and $127 \nmid (2^{63}+1)$. Hence $127 \mid k$, which is a contradiction. Hence, we conclude that $3 \mid\mid k$. Now, $(2269-1,k) = 21$. We notice that $2269 \mid\mid (3^{21}+1)$ and $2269 \nmid (2^{21}+1)$. Hence, by Lemma 3.9, $2269 \mid k$, which is a contradiction again. □

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References

[1] T. Amdeberhan, L. A. Medina and V. H. Moll, Arithmetical properties of a sequence arising from an arctangent sum, *J. Number Theory* **128** (2008), 1807–1846.

[2] Y.-G. Chen and M.-L. Gong, On the products $(1^l+1)(2^l+1)\cdots(n^l+1)$, II, *J. Number Theory* **144** (2014), 176–187.

[3] Y.-G. Chen, M.-L. Gong and X.-Z. Ren, On the products $(1^l+1)(2^l+1)\cdots(n^l+1)$, *J. Number Theory* **133** (2013), 2470–2474.

[4] J. Cilleruelo, Squares in $(1^2+1)(2^2+1)\cdots(n^2+1)$, *J. Number Theory* **128** (2008), 2488–2491.

[5] P. Erdős and J. Selfridge, The product of consecutive integers is never a power, *Illinois J. Math.* **19** (1975), 292–301.

[6] J.-H. Fang, Neither $\prod_{k=1}^{n}(4k^2+1)$ nor $\prod_{k=1}^{n}(2k(k-1)+1)$ is a perfect square, *Integers* **9** (2009), 177–180.

[7] E. Gürel, A note on the products $((m+1)^2+1)\cdots(n^2+1)$ and $((m+1)^3+1)\cdots(n^3+1)$, *Math. Commun.* **21** (2016), 109–114.

[8] E. Gürel, A. U. O. Kişisel and A. Ulaş, A note on the products $(1^\mu+1)(2^\mu+1)\cdots(n^\mu+1)$, *J. Number Theory* **130** (2010), 187–191.

[9] K. Győry, L. Hajdu and A. Pintér, Perfect powers from products of consecutive terms in arithmetic progression, *Compos. Math.* **145** (2009), 845–864.

[10] H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika* **20** (1973), 119–134.

[11] C. Niu and W. Liu, On the products $(1^3+q^3)(2^3+q^3)\cdots(n^3+q^3)$, *J. Number Theory* **180** (2017), 403–409.

[12] O. Ramaré and R. Rumely, Primes in arithmetic progressions, *Math. Comp.* **65** (1996), 397–425.

[13] J. Sándor, D. S. Mitrinović and B. Crstici, Handbook of Number Theory. I, Second Edition, *Springer*, *Dordrecht*, 2006.

[14] S.-C. Yang, A. Togbé and B. He, Diophantine equations with products of consecutive values of a quadratic polynomial, *J. Number Theory* **131** (2011), 1840–1851.

[15] Q.-H. Yang and Q.-Q. Jhao, Powerful numbers in $(1^k + q^k)(2^k + q^k)\cdots(n^k + q^k)$, *C. R. Acad. Sci. Paris* **356** (2018), 13–16.

[16] W. Zhang and T. Wang, Powerful numbers in $(1^k + 1)(2^k + 1)\cdots(n^k + 1)$, *J. Number Theory* **132** (2012), 2630–2635.

[17] Z. F. Zhang and P. Z. Yuan, Squares in $\prod_{k=1}^{n}(ak^2 + bk + c)$, *Acta Math. Sinica (Chin. Ser.)* **53** (2010), 199–204.

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