

Twisted quadratic moments for Dirichlet L -functions at $s = 2$

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Abstract. Let c, n be given positive integers. Let $q > 2$ be coprime with c . Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q . Set

$$M(q, c, n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1) = (-1)^n}} \chi(c) |L(n, \chi)|^2.$$

The goal of this paper is to explain how one can compute explicit formulas for $M(q, c, n)$ for given small integers n and c . As an example, we give explicit formulas for $M(q, c, 2)$ for $c \in \{1, 2, 3, 4, 6\}$, and for $M(p, 5, 2)$ for p a prime integer. As a consequence, we show that a previously published formula for $M(p, 3, 2)$ is false.

1. Introduction

Let c, n be given positive integers. Let $q > 2$ be coprime with c . Let X_q be the multiplicative group of order $\phi(q)$ of the Dirichlet characters modulo q . Set

$$M(q, c, n) := \frac{2}{\phi(q)} \sum_{\substack{\chi \in X_q \\ \chi(-1) = (-1)^n}} \chi(c) |L(n, \chi)|^2.$$

In [4], we developed a method for obtaining formulas for $M(q, 1, n)$. For example, by [4, Theorem 2], we have

$$M(q, 1, 2) = \frac{\pi^2}{90} \times \left\{ \phi_4(q) + \frac{10}{q^2} \phi_2(q) \right\}, \quad (1)$$

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where

$$\phi_k(q) = \prod_{p|q} \left(1 - \frac{1}{p^k}\right) \quad (k \in \mathbb{Z}_{\geq 1}).$$

In [5], for a given $c > 1$, we developed a method for obtaining formulas for $M(q, c, 1)$. For example, in the cases where the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is trivial of order 1, i.e. for $c \in \{1, 2\}$, by [5, (1) and (3)] and [6, Theorem 2], we have

$$M(q, c, 1) = \frac{\pi^2}{6c} \times \left\{ \phi_2(q) - \frac{3c\phi_1(q)}{q} \right\}. \quad (2)$$

In the cases where the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$ is of order 2, i.e. for $c \in \{3, 4, 6\}$, by [5, (4) and (5)] and [6, Theorem 4], we have

$$M(q, c, 1) = \frac{\pi^2}{6c} \times \left\{ \phi_2(q) - \frac{3c\phi_1(q)}{q} - \frac{(c-1)(c-2)\chi_c(q)}{q} \prod_{p|q} \left(1 - \frac{\chi_c(p)}{p}\right) \right\},$$

where χ_c is the non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^*$, i.e. where $\chi_c(n) = 1$ if $n \equiv 1 \pmod{c}$, and $\chi_c(n) = -1$ if $n \equiv -1 \pmod{c}$. We point out that according to [6, Section 4], as c gets bigger, such explicit formula become very complicated, there is no known closed formula for $M(q, c, 2)$ and we gave formulas for $c \in \{1, 2, 3, 4, 5, 6, 8, 10\}$. Restricting himself to prime moduli, in [2, Corollary 1.1], H. LIU gives explicit formulas for $M(p, c, 2)$ for $c = 1, 2, 3, 4$, without explaining why he did not consider neither non-prime moduli nor values of $c > 4$. Here, focussing on the particular example of $M(q, c, 2)$, we explain how to extend Liu's results to non-prime moduli and values of $c > 4$. We will prove the following result and show that the formula for $M(p, 3, 2,)$ in [2, Corollary 1.1] is false (compare with [6, Theorem 4]):

Theorem 1. *Let $q > 1$ be coprime with $c \in \{1, 2, 3, 4, 6\}$. Define*

c	u_c	v_c
1	10	0
2	70	0
3	210	80
4	490	360
6	1830	2240

Let X_q^+ be the group of order $\phi(q)/2$ of the even Dirichlet characters modulo q . For $c \in \{3, 4, 6\}$, let χ_c be the only non-trivial character on the multiplicative group $(\mathbb{Z}/c\mathbb{Z})^ = \{\pm 1\}$. Then*

$$\begin{aligned}
M(q, c, 2) &:= \frac{2}{\phi(q)} \sum_{\chi \in X_q^+} \chi(c) |L(2, \chi)|^2 \\
&= \frac{\pi^4}{90c^2} \times \left\{ \phi_4(q) + \frac{u_c \phi_2(q)}{q^2} + \frac{v_c \chi_c(q)}{q^3} \prod_{p|q} \left(1 - \frac{\chi_c(p)}{p}\right) \right\}.
\end{aligned}$$

As a Corollary, we recover the formulas in [2, Corollary 1.1], correct the formula for $M(p, 3, 2)$ and give a new formula, the one for $M(p, 6, 2)$:

Corollary 2. *Let $p > 3$ be a prime integer. We have*

$$M(p, c, 2) = \frac{\pi^4}{90c^2} \times \frac{(p^2 - 1)(p^2 + u_c + 1)}{p^4} \quad (c \in \{1, 2\}),$$

and

$$M(p, c, 2) = \frac{\pi^4}{90c^2} \times \frac{p^4 + u_c p^2 + v_c \chi_c(p)p - (u_c + v_c + 1)}{p^4} \quad (c \in \{3, 4, 6\}).$$

We also have the explicit formulas

$$M(p, 3, 2) = \frac{\pi^4}{810} \times \frac{p^4 + 210p^2 + 80p\chi_3(p) - 291}{p^4},$$

$$M(p, 4, 2) = \frac{\pi^4}{1440} \times \frac{p^4 + 490p^2 + 360\chi_4(p) - 851}{p^4}$$

and

$$M(p, 6, 2) = \frac{\pi^4}{3240} \times \frac{p^4 + 1830p^2 + 2240p\chi_6(p) - 4071}{p^4}.$$

Let us explain how we performed some numerical computation to check our formulas for $M(p, c, 2)$, and let us justify that the formula for $M(p, 3, 2)$ in [2, Corollary 1.1] is not correct. Let $p > 3$ be a prime integer. We refer to [7, Chapter 4] for the justification of what follows. For χ_0 the trivial character modulo p , we have $L(s, \chi_0) = (1 - p^{-s})\zeta(s)$ and $L(2, \chi_0) = \frac{p^2-1}{p^2} \frac{\pi^2}{6}$. For $\chi_0 \neq \chi \in X_p^+$, we have

$$L(2, \chi) = -W_\chi \frac{2\pi^2}{p^{3/2}} L(-1, \bar{\chi}) = W_\chi \frac{\pi^2}{p^{3/2}} B_{2, \bar{\chi}} = W_\chi \frac{\pi^2}{p^{5/2}} \sum_{a=1}^{p-1} \bar{\chi}(a) a^2,$$

where the root number W_χ is a complex number of absolute value equal to 1.

Hence,

$$M(p, c, 2) = \frac{2\pi^4}{p-1} \times \left\{ \frac{(p^2-1)^2}{36p^4} + \sum_{\chi_0 \neq \chi \in X_p^+} \chi(c) \frac{\left| \sum_{a=1}^{p-1} \chi(a)a^2 \right|^2}{p^5} \right\}.$$

Taking $p = 5$, for which the only $\chi_0 \neq \chi \in X_5^+$ is the Legendre symbol $\left(\frac{\bullet}{5}\right)$, for which $\sum_{a=1}^4 \chi(a)a^2 = 1 - 2^2 - 3^2 + 4^2 = 4$, we obtain

$$M(5, c, 2) = \frac{8\pi^4}{5^5} \left(5 + \left(\frac{c}{5}\right) \right).$$

For $c \in \{1, 2, 3, 4\}$, this formula yields the same value as the ones in Corollary 2. In particular, they give $M(5, 3, 2) = 32\pi^4/5^5$, whereas the formula in [2, Corollary 1.1] gives the wrong estimation $M(5, 3, 2) = 238\pi^4/(9 \cdot 5^5)$.

2. Proof of Theorem 1

Lemma 3. Assume that $\gcd(c, d) = 1$. Set

$$S_{\cot^k}(d) = \sum_{a=1}^{d-1} \cot^k \left(\frac{\pi a}{d} \right),$$

with the convention that $S_{\cot^k}(1) = 0$, and

$$S_2(c, d) := \sum_{a=1}^{d-1} \cot^2 \left(\frac{\pi a}{d} \right) \cot^2 \left(\frac{\pi ac}{d} \right), \quad (3)$$

with the convention that $S_2(c, 1) = 0$ for $c \geq 1$. Then

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{d|q} \mu(q/d) \left((d-1) + 2S_{\cot^2}(d) + S_2(c, d) \right). \quad (4)$$

In particular,

$$M(p, c, 2) = \frac{\pi^4}{2q^4} \left((p-1) + 2S_{\cot^2}(p) + S_2(c, p) \right). \quad (5)$$

PROOF. From [4, Proposition 3], we have

$$L(2, \chi) = -\frac{\pi^2}{2q^2} \sum_{a=1}^{q-1} \chi(a) \cot'(\pi a/q) \quad (\chi \in X_q^+),$$

and as in the proof of [4, Proposition 3], we obtain

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{\substack{a=1 \\ \gcd(a, q)=1}}^{q-1} \cot'(\pi a/q) \cot'(\pi ac/q),$$

with $\cot' = -1 - \cot^2$ the derivative of $\cot$. Using $\sum_{d|a \text{ and } q} \mu(d) = 0$ or 1 for either $\gcd(a, q) > 1$ or $\gcd(a, q) = 1$, respectively, we obtain

$$M(q, c, 2) = \frac{\pi^4}{2q^4} \sum_{\substack{d>1 \\ d|q}} \mu(q/d) \sum_{a=1}^{d-1} \left(1 + \cot^2\left(\frac{\pi a}{d}\right)\right) \left(1 + \cot^2\left(\frac{\pi ac}{d}\right)\right), \quad (6)$$

and the desired result, thanks to the conventions, $S_{\cot^2}(1) = S_2(c, 1) = 0$. $\square$

Lemma 4. *Let $c > 1$ be an integer. It holds that*

$$\begin{aligned} (\cot^2 x)(\cot^2(cx)) &= \frac{1}{c^2} \cot^4 x - \frac{2(c^2 - 1)}{3c^2} \cot^2 x + \frac{c^4 - 1}{15c^2} \\ &\quad - \frac{2}{c^2} \sum_{k=1}^{c-1} \frac{\frac{2 \cot^3(k\pi/c) + \cot(k\pi/c)}{\sin^2(k\pi/c)}}{\cot(k\pi/c) - \cot x} + \frac{1}{c^2} \sum_{k=1}^{c-1} \frac{\frac{\cot^2(k\pi/c)}{\sin^4(k\pi/c)}}{(\cot(k\pi/c) - \cot x)^2}. \end{aligned}$$

PROOF. Adapt the proof of [5, Lemma 4]. $\square$

Lemma 5. *Let $d > 1$ be an integer and $\theta \in (0, \pi) \setminus \{\pi a/d; 1 \leq a \leq d-1\}$.*

Set

$$T_m(\theta, d) := \frac{1}{d} \sum_{a=1}^{d-1} \frac{1}{(\cot \theta - \cot(\pi a/d))^m}.$$

Then

$$T_1(\theta, d) = (\sin^2 \theta) (\cot \theta - \cot(d\theta)),$$

$$T_2(\theta, d) = (\sin^4 \theta) \left(d \cot^2(d\theta) - 2(\cot \theta)(\cot(d\theta)) + \cot^2 \theta + (d-1) \right)$$

and

$$\begin{aligned} &- 2(2 \cot^3 \theta + \cot \theta) \frac{T_1(\theta, d)}{\sin^2 \theta} + (\cot^2 \theta) \frac{T_2(\theta, d)}{\sin^4 \theta} \\ &= (d-3) \cot^2 \theta - 3 \cot^4 \theta + 2(\cot^3 \theta + \cot \theta) \cot(d\theta) + d(\cot^2 \theta)(\cot^2(d\theta)). \end{aligned}$$

PROOF. For $T_1(\theta, d)$, see [5, Lemma 5] and take $\alpha = \cot \theta$. For $T_2(\theta, d)$, notice that $T_2(\theta, d) = (\sin^2 \theta) \frac{dT_1(\theta, d)}{d\theta}$. $\square$

Now, noticing that $\sum_{d|q} \mu(q/d)d^k = q^k \phi_k(q)$ and $\phi_0(q) = 0$ for $q > 1$, Theorem 1 follows from (4) and the following Proposition:

Proposition 6. *Assume that $\gcd(c, d) = 1$. Set*

$$F_c(d) := \sum_{k=1}^{c-1} (\cot^3(k\pi/c) + \cot(k\pi/c)) \cot(kd\pi/c),$$

with the convention that $F_1(d) = 0$ for $d \geq 1$. Then

$$(d-1) + 2S_{\cot^2}(d) + S_2(c, d) = \frac{Q_c(d)}{45c^2} + \frac{d^2}{c^2} S_2(d, c) + \frac{2d}{c^2} F_c(d),$$

where $Q_c(d) := d^4 + 5(7c^2 - 9c + 4)d^2 - (3c^4 + 5c^2 + 3)$. Notice that $S_2(d, c)$ and $F_c(d)$ depend only on d modulo c . In particular, we have

c	$S_2(d, c)$	$F_c(d)$	$45c^2((d-1) + 2S_{\cot^2}(d) + S_2(c, d))$
1	0	0	$d^4 + 10d^2 - 11$
2	0	0	$d^4 + 70d^2 - 71$
3	$\frac{2}{9}$	$\frac{8}{9}\chi_3(d)$	$d^4 + 210d^2 + 80d\chi_3(d) - 291$
4	2	$4\chi_4(d)$	$d^4 + 490d^2 + 360d\chi_4(d) - 851$
6	$\frac{164}{9}$	$\frac{224}{9}\chi_6(d)$	$d^4 + 1830d^2 + 2240d\chi_6(d) - 4071$

PROOF. Using (3) and Lemma 4, we obtain

$$\begin{aligned} c^2 S_2(c, d) &= S_{\cot^4}(d) - \frac{2(c^2 - 1)}{3} S_{\cot^2}(d) + \frac{c^4 - 1}{15} (d - 1) \\ &\quad - 2d \sum_{k=1}^{c-1} (2 \cot^3(k\pi/c) + \cot(k\pi/c)) \frac{T_1(k\pi/c, d)}{\sin^2(k\pi/c)} \\ &\quad + d \sum_{k=1}^{c-1} (\cot^2(k\pi/c)) \frac{T_2(k\pi/c, d)}{\sin^4(k\pi/c)}. \end{aligned}$$

Using the last assertion in Lemma 5 for $\theta = k\pi/c$, we obtain

$$\begin{aligned} c^2 S_2(c, d) &= S_{\cot^4}(d) - \frac{2(c^2 - 1)}{3} S_{\cot^2}(d) + \frac{c^4 - 1}{15} (d - 1) \\ &\quad + d \left((d - 3) S_{\cot^2}(c) - 3 S_{\cot^4}(c) \right) \end{aligned}$$

$$\begin{aligned}
& + 2d \sum_{k=1}^{c-1} (\cot^3(k\pi/c) + \cot(k\pi/c)) \cot(kd\pi/c) \\
& + d^2 \sum_{k=1}^{c-1} (\cot^2(k\pi/c)) (\cot^2(kd\pi/c)).
\end{aligned}$$

Now, for $n > 1$, $\cot(\pi k/n)$, $1 \leq k \leq n-1$, are the roots of

$$\frac{(X+i)^n - (X-i)^n}{2in} = X^{n-1} - \frac{(n-1)(n-2)}{6} X^{n-3} + \dots \in \mathbb{Q}[X].$$

Hence, $S_{\cot^2}(n) = \frac{(n-1)(n-2)}{3}$ and $S_{\cot^4}(n) = \frac{(n-1)(n-2)(n^2+3n-13)}{45}$. The desired result follows. $\square$

Corollary 7. For $p \neq 2, 5$ a prime integer, we have

$$M(q, 5, 2) = \frac{\pi^4}{2250p^4} \times \begin{cases} p^4 + 994p^2 + 1008p - 2003 & \text{if } p \equiv 1 \pmod{5}, \\ p^4 + 706p^2 + 144p - 2003 & \text{if } p \equiv 2 \pmod{5}, \\ p^4 + 706p^2 - 144p - 2003 & \text{if } p \equiv 3 \pmod{5}, \\ p^4 + 994p^2 - 1008p - 2003 & \text{if } p \equiv 4 \pmod{5}. \end{cases}$$

PROOF. Follows from (5), Proposition 6 and the computation of $S_2(r, 5)$ and $F_5(r)$ for the value r of d modulo 5 ranging in $\{1, 2, 3, 4\}$. $\square$

Remark 8. As in [8], for q a positive integer, set

$$d(q; a_1, \dots, a_n) = (-1)^{n/2} \sum_{k=1}^{q-1} \cot\left(\frac{ka_1}{q}\right) \cdots \cot\left(\frac{ka_n}{q}\right).$$

In [8, (47)], Zagier gave a reciprocity law for these generalized Dedekind sums under the assumption that q and the a_k 's be pairwise coprime. For $S_2(c, q) = d(q; 1, 1, c, c)$ this assumption is not fulfilled. Hence, Proposition 6, where $F_c(x) := d(c; 1, 1, 1, x) + d(c; 1, x)$, which can be viewed as a reciprocity law for the sums $S_2(c, d)$, does not follow from Zagier's reciprocity law.

3. Conclusion

[5, Lemma 4], which deals with $(\cot x)(\cot(cx))$, and the present Corollary 4, which deals with $(\cot^2 x)(\cot^2(cx))$, could easily be generalized to evaluate

$(\cot^m x)(\cot^n(cx))$ for small values of m and n . Lemma 5 can be very easily generalized to evaluate $T_m(\theta, d)$ for small values of m . Lemma 3 can be generalized to express $M(q, c, n)$ in terms of

$$S_{k,l}(c, d) := \sum_{a=1}^{d-1} \cot^k \left(\frac{\pi a}{d} \right) \cot^l \left(\frac{\pi ac}{d} \right),$$

see [4]. Hence, following the method developed here one could obtain explicit formulas for $M(q, c, n)$ and $M(p, c, n)$ for other small values of n and c . As explained here and in [6], the formulas for $M(q, c, n)$ would get more complicated as $\phi(c)$ increases. We would get $\phi(c)$ twisted quadratic moments formulas, one for each value of d modulo c (notice that $\phi(c) = 1$ if and only if $c = 1, 2$, and $\phi(c) = 2$ if and only if $c = 3, 4, 6$). Only asymptotic estimates as in [1] and [3] could yield explicit formulas for all c 's.

References

- [1] S. BETTIN, On the reciprocity law for the twisted second moment of Dirichlet L -functions, *Trans. Amer. Math. Soc.* **368** (2016), 6887–6914.
- [2] H. LIU, On the mean values of Dirichlet L -functions, *J. Number Theory* **147** (2015), 172–183.
- [3] S. H. LEE and S. LEE, On the twisted quadratic moment for Dirichlet L -functions, *J. Number Theory* **174** (2017), 427–435.
- [4] S. LOUBOUTIN, The mean value of $|L(k, \chi)|^2$ at positive rational integers $k \geq 1$, *Colloq. Math.* **90** (2001), 69–76.
- [5] S. LOUBOUTIN, A twisted quadratic moment for Dirichlet L -functions, *Proc. Amer. Math. Soc.* **142** (2014), 1539–1544.
- [6] S. LOUBOUTIN, Twisted quadratic moments for Dirichlet L -functions, *Bull. Korean Math. Soc.* **52** (2015), 2095–2105.
- [7] L. C. WASHINGTON, Introduction to Cyclotomic Fields, Second Edition, Graduate Texts in Mathematics, Vol. **83**, Springer-Verlag, New York, 1997.
- [8] D. ZAGIER, Higher dimensional Dedekind sums, *Math. Ann.* **202** (1973), 149–172.

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