Title: Sharp inequalities for sine polynomials

Author(s): Horst Alzer and Man Kam Kwong

Let $F_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}$ and $C_n(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}$.

The classical inequalities

\[0 < F_n(x) < \int_{0}^{\pi} \frac{\sin(t)}{t} \, dt = 1.85193 \ldots \quad \text{and} \quad 0 < C_n(x) \leq 1\]

are valid for all $n \geq 1$ and $x \in (0, \pi)$. All constant bounds are sharp. We present the following refinements of the lower bound for $F_n(x)$ and the upper bound for $C_n(x)$.

(i) Let $\mu \geq 1$. The inequality $\frac{\sin(x)}{\mu - \cos(x)} < F_n(x)$ holds for all odd $n \geq 1$ and $x \in (0, \pi)$ if and only if $\mu \geq 2$.

(ii) For all $n \geq 2$ and $x \in [0, \pi]$, we have $C_n(x) \leq 1 - \lambda \sin(x)$ with the best possible constant factor $\lambda = \sqrt{9} - 2$.

Moreover, we offer a companion to the inequality $C_n(x) > 0$.

(iii) Let $n \geq 1$. The inequality $0 \leq \sum_{k=1}^{n} (\delta(n) - (k - 1)k) \sin((2k-1)x)$ holds for all $x \in [0, \pi]$ if and only if $\delta(n) \geq (n^2 - 1)/2$.

This extends a result of Dimitrov and Merlo, who proved the inequality for the special case $\delta(n) = n(n + 1)$. The following inequality for the Chebyshev polynomials of the second kind plays a key role in our proof of (iii).

(iv) Let $m \geq 0$. For all $t \in \mathbb{R}$, we have $(m^2(1 - t^2) - 1) U_{n+1}(t) + (m+1) U_{2m}(t) \leq m(m + 1)$. The upper bound is sharp.

Address:
Horst Alzer
Morsbacher Straße 10
51545 Waldbröl
Germany

Address:
Man Kam Kwong
The Polytechnic University of Hong Kong
Hung hom
Hong Kong