

The topology of semilocal connectedness and s -continuity of multivalued maps

By JANINA EWERT (Slupsk)

A topological space $(Y, \mathcal{T})$ is said to be semilocally connected if for each open set $U \subset Y$ and each point $x \in U$ there exists an open set V such that $x \in V \subset U$ and $Y \setminus V$ consists of a finite number of components [2]. This notion was introduced by WHYBURN [9. p. 19] under the additional assumption that $(Y, \mathcal{T})$ is a connected T_1 -space. Similarly as in [2, 3, 4, 8] we consider the semilocal connectedness of general topological spaces.

In a topological space $(Y, \mathcal{T})$ the family $\{U \in \mathcal{T} : Y \setminus U \text{ is connected}\}$ is a subbase of a topology denoted as $\mathcal{T}^*$. Obviously, $\mathcal{T}^* \subset \mathcal{T}$. Further $(Y, \mathcal{T}^*)$ is semilocally connected [2, Th. 3.1] and $(Y, \mathcal{T})$ is a semilocally connected space iff $\mathcal{T} = \mathcal{T}^*$ [2, Th. 3.3].

Let X, Y be topological spaces. A function $f : X \rightarrow Y$ is called s -continuous at a point $x \in X$ if for each open set $V \subset Y$ containing $f(x)$ and having connected complement there is an open set U satisfying $x \in U \subset f^{-1}(V)$. A function f is called s -continuous if it is s -continuous at each point [2, 3, 4]. One can readily check that:

(A) A function $f : X \rightarrow (Y, \mathcal{T})$ is s -continuous if and only if
 $f : X \rightarrow (Y, \mathcal{T}^*)$ is continuous [8, Prop. 9].

Now, let $F : X \rightarrow Y$ be a multivalued map. For a set $B \subset Y$ we denote $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. A multivalued map $F : X \rightarrow Y$ is said to be upper (lower) s -continuous at a point $x \in X$ if for each open set $V \subset Y$ with $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) and $Y \setminus V$ connected there exists an open set $U \subset X$ for which $x \in U \subset F^+(V)$, (resp. $x \in U \subset F^-(V)$) holds, [6]. A multivalued map F is called upper (lower) s -continuous if it is upper (lower) s -continuous at each point.

Theorem (A) simply rewritten for multivalued maps is not true in general. For a multivalued map $F : X \rightarrow (Y, \mathcal{T})$ the upper (lower) $\mathcal{T}^*$ -semicontinuity implies the upper (lower) s -continuity, but the inverse does not hold.

Example 1. Let $(R, \mathcal{T})$ be the space of real numbers with the natural topology and let $D \subset R$ be a non-Borel set. We define a multivalued map $F : R \rightarrow R$ assuming

$$F(x) = \begin{cases} [1, 2] \cup [5, 7], & \text{if } x \in D. \\ [1, 3] \cup [6, 7], & \text{if } x \notin D. \end{cases}$$

The map F is lower s -continuous. For $V_1 = (5, 6)$, $V_2 = (0, 4) \cup (5, 8)$ we have $F^-(V_1) = D$, $F^+(V_2) = R \setminus D$, thus F is not lower semicontinuous. Since $\mathcal{T} = \mathcal{T}^*$ the map F is not lower $\mathcal{T}^*$ -semicontinuous. (Moreover it is not upper nor lower Borel measurable).

Example 2. Let $(R^2, \mathcal{T})$ be the plane with the natural topology, B_n the closed ball with center p_n and radius $\frac{1}{2}$, where $p_n = (0, 2n - 1)$ for $n \geq 1$. Let us put

$$A = \bigcup_{n=0}^{\infty} ([-1, 1] \times \{2n\}) \cup \{-1, 1\} \times R$$

and $A_n = A \cup \{0\} \times [2n - 1, \infty)$ for $n \geq 1$. Then we define the multivalued map $F : R \rightarrow R^2$ as follows

$$F(x) = \begin{cases} A_n, & \text{if } x = \frac{1}{n}, n \geq 1 \\ A, & \text{if } x \notin \{\frac{1}{n} : n \geq 1\}. \end{cases}$$

If $V \subset R^2$ is an open set such that $R^2 \setminus V$ is connected, then $F^+(V) = R$ or $F^+(V) = R \setminus \{1, \frac{1}{2}, \dots, \frac{1}{k}\}$ for some $k \geq 1$, hence F is upper s -continuous. On the other hand the set $W = R^2 \setminus \bigcup_{n=1}^{\infty} B_n$ is open and $F^+(W) = R \setminus \{\frac{1}{n} : n \geq 1\}$, so F is not upper semicontinuous. Since we have $\mathcal{T} = \mathcal{T}^*$, the map F is not upper $\mathcal{T}^*$ -semicontinuous, either.

Theorem 3. *Let $F : X \rightarrow (Y, \mathcal{T})$ be a multivalued map with connected values. Then F is lower s -continuous if and only if $F : X \rightarrow (Y, \mathcal{T}^*)$ is lower semicontinuous.*

PROOF. Let $\mathcal{B}$ be a base of $\mathcal{T}^*$ consisting of $\mathcal{T}$ -open sets whose complements have a finite number of components in $(Y, \mathcal{T})$. If $W \in \mathcal{B}$, then $Y \setminus W = \bigcup_{j=1}^n M_j$ where the M_j are pairwise disjoint connected closed subsets of $(Y, \mathcal{T})$. For a point $x \in F^+(Y \setminus W)$ we have $F(x) \subset M_j$ for some $j \leq n$, thus $F^+(Y \setminus W) = \bigcup_{j=1}^n F^+(M_j)$. It follows from the lower s -continuity that $F^+(M_j)$ is a closed set, so $F^-(W)$ is open and the proof is completed.

As Example 2 shows, for upper s -continuity the theorem analogous to the above one is not true. We remind that a multivalued map $F : X \rightarrow Y$ is said to be of lower (upper) Baire class 1 if for each open set $W \subset Y$ the set $F^-(W)$ (resp. $F^+(W)$) is an F_σ set, [5].

Now let $S(Y)$ denote the family of all non-empty subsets of Y and let $\tau_{\mathcal{T}}$ be the Vietoris topology on $S(Y)$ given by a topology $\mathcal{T}$ on Y , [1]. For multivalued maps $F, F_n : X \rightarrow Y$, $n \geq 1$, we will write $F = \tau_{\mathcal{T}}\text{-}\lim F_n$ iff for each $x \in X$ the sequence $\{F_n(x) : n \geq 1\}$ converges to $F(x)$ in the space $(S(Y), \tau_{\mathcal{T}})$. Then we have the following result:

Theorem 4. *Let X be a topological space, $(Y, \mathcal{T})$ a locally connected separable metrizable space, and let $F, F_n : X \rightarrow Y$, $n \geq 1$, be multivalued maps with $F = \tau_{\mathcal{T}}\text{-}\lim F_n$. Then*

- (a) *if F_n are upper s -continuous maps, then F is of lower Baire class 1;*
- (b) *if $F(x)$ is a compact set for each $x \in X$ and the F_n are lower $\mathcal{T}^*$ -semicontinuous, then F is of upper Baire class 1.*

PROOF. Let $W \subset Y$ be a non-empty open set; there exists a sequence $\{G_n : n \geq 1\}$ of connected open sets satisfying $W = \bigcup_{n=1}^{\infty} \bar{G}_n$. Assume $W_1 = G_1$. It follows from our assumptions on Y that we can choose a connected open set U_1 satisfying $\bar{G}_1 \subset U_1 \subset \bar{U}_1 \subset W$, then let us put $W_2 = G_1 \cup U_1$. Thus W_1, W_2 are open sets having finitely many components and $\bar{W}_1 \subset W_2 \subset \bar{W}_2 \subset W$. Assume that we have constructed open sets $W_1, \dots, W_m$ such that each of them has a finite number of components and $\bar{W}_i \subset W_{i+1} \subset W$ for $i \leq m-1$. Let $S_1, \dots, S_k$ be components of the set W_m . We take open connected sets $V_1, \dots, V_k$ satisfying $\bar{S}_j \subset V_j \subset \bar{V}_j \subset W$ for $j \leq k$ and we denote $W_{m+1} = G_{m+1} \cup \bigcup_{j=1}^k V_j$. Thus W_{m+1} is open, it has at most $k+1$ components and $\bar{W}_m \subset W_{m+1} \subset W$. So we have shown:

- (1) each open set $W \subset Y$ is of the form $W = \bigcup_{n=1}^{\infty} W_n$, where the W_n are open sets having finitely many components and $\bar{W}_n \subset W_{n+1}$ for $n \geq 1$.

Let W be an open set and $x \in F^-(W)$. Then (1) implies that there exists k_0 such that $F(x) \cap W_k \neq \emptyset$ for $k \geq k_0$. From the $\tau_{\mathcal{T}}$ -convergence of the sequence $\{F_n(x) : n \geq 1\}$ for each $k \geq k_0$ there exists n_k such that $F_n(x) \cap W_k \neq \emptyset$ for $n \geq n_k$. Hence we can choose $k \geq 1$ such that $F_{n+k}(x) \cap \bar{W}_k \neq \emptyset$ for each $n \geq 1$; this leads to the inclusion

$$F^-(W) \subset \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^-(W_k).$$

If $x \notin F^-(W)$, then $F(x) \subset Y \setminus \bar{W}_k$ for each $k \geq 1$. The τ_T -convergence implies that for each k there exists n_k such that $F_n(X) \subset Y \setminus \bar{W}_k$ for $n \geq n_k$, thus for each $k \geq 1$ there exists $n \geq 1$ for which $F_{n+k}(x) \subset Y \setminus \bar{W}_k$ holds. Hence

$$X \setminus F^-(W) \subset \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_{n+k}^+(Y \setminus \bar{W}_k) = X \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^-(\bar{W}_k).$$

So we have shown

$$(2) \quad F^-(W) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^-(\bar{W}_k).$$

Let $S_{k,1}, \dots, S_{k,m_k}$ be components of $\bar{W}_k$. If the F_n are upper s -continuous, then

$$F_{n+k}^-(\bar{W}_k) = \bigcup_{j=1}^{m_k} F_{n+k}^-(S_{k,j});$$

this set is closed, so F is of lower Baire class 1 and (a) is proved.

Now let $W \subset Y$ be an open set and let $x \in F^+(W)$. Since $F(x) \subset \bigcup_{k=1}^{\infty} W_k$ where the W_k are open sets such as in (1) and $F(x)$ is compact, there exists $k_0 \geq 1$ such that $F(x) \subset W_k$ for each $k \geq k_0$. Furthermore for $k \geq k_0$ there is n_k such that $F_n(x) \subset W_k$ for $n \geq n_k$; hence we obtain that there is $k \geq 1$ such that for every $n \geq 1$ there holds $F_{n+k}(x) \subset \bar{W}_k$. In the consequence we have

$$F^+(W) \subset \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^+(\bar{W}_k).$$

Finally let $x \notin F^+(W)$. Then $F(x) \cap (Y \setminus \bar{W}_k) \neq \emptyset$ for each $k \geq 1$. Using the τ_T -convergence of the sequence $\{F_n(x) : n \geq 1\}$ for each $k \geq 1$ we can choose $n \geq 1$ for which $F_{n+k}(x) \cap (Y \setminus \bar{W}_k) \neq \emptyset$. This implies

$$X \setminus F^+(W) \subset \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_{n+k}^-(Y \setminus \bar{W}_k)$$

and then

$$(3) \quad F^+(W) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_{n+k}^+(\bar{W}_k).$$

The sets $\bar{W}_k$ are $\mathcal{T}^*$ -closed. If the F_n are lower $\mathcal{T}^*$ -semicontinuous, then according to (3), F is of upper Baire class 1.

Let us observe that in Theorem 4 (b) the lower $\mathcal{T}^*$ -semicontinuity of the maps F_n cannot be replaced by lower s -continuity. For instance, it suffices to take $F_n = F$ for $n \geq 1$, where F is the map from Example 1.

For a topological space $(Y, \mathcal{T})$ we denote by $2_c^{(Y, \mathcal{T})}$ the family of all non-empty connected closed subsets of Y (for simplicity we will write 2_c^Y). For a set $U \subset Y$ we denote

$$U^+ = \{B \in 2_c^Y : B \subset U\}, \quad U^- = \{B \in 2_c^Y : B \cap U \neq \emptyset\}.$$

Then $\mathcal{T}^+$ and $\mathcal{T}^-$ are topologies on 2_c^Y given by the base $\{U^+ : U \in \mathcal{T}\}$ and the subbase $\{U^- : U \in \mathcal{T}\}$ respectively. Similarly are constructed topologies $(\mathcal{T}^*)^+$ and $(\mathcal{T}^*)^-$ on 2_c^Y by taking the sets $U \in \mathcal{T}^*$.

Corollary 5. *Let $F : X \rightarrow (Y, \mathcal{T})$ be a multivalued map with values in 2_c^Y . Then F is lower s -continuous if and only if the function $F : X \rightarrow (2_c^Y, (\mathcal{T}^*)^-)$ is continuous.*

According to Example 2 a similar property for upper s -continuity cannot be formulated.

In the sequel we will show that there exists a topology τ on 2_c^Y such that a multivalued map $F : X \rightarrow Y$ with values in 2_c^Y is upper s -continuous if and only if the function $F : X \rightarrow (2_c^Y, \tau)$ is continuous. It will be obtained by constructing of some quasi-uniformities on 2_c^Y (for quasi-uniform spaces see [7]).

For subsets U_1, U_2 of a topological space $(Y, \mathcal{T})$ such that $U_1 \subset U_2$ we put:

$$H_{U_1, U_2} = \{(A, B) \in 2_c^Y \times 2_c^Y : A \cap U_1 \neq \emptyset \Rightarrow B \cap U_2 \neq \emptyset\}$$

$$M_{U_1, U_2} = \{(A, B) \in 2_c^Y \times 2_c^Y : A \subset U_1 \Rightarrow B \subset U_2\}.$$

It can be easily verified that these sets have the following properties:

- (P1) $\Delta \subset H_{U_1, U_2}$ and $\Delta \subset M_{U_1, U_2}$;
- (P2) if $U_1 \subset U_2 \subset U_3$, then $H_{U_1, U_2} \subset H_{U_1, U_3}$ and $M_{U_1, U_2} \subset M_{U_1, U_3}$;
 $H_{U_2, U_3} \subset H_{U_1, U_3}$, $M_{U_2, U_3} \subset M_{U_1, U_3}$;
- (P3) $H_{U_1, U_2}^{-1} = M_{Y \setminus U_2, Y \setminus U_1}$ and $M_{U_1, U_2}^{-1} = H_{Y \setminus U_2, Y \setminus U_1}$;
- (P4) if $U_1 \subset U_2 \subset U_3$ then $H_{U_1, U_2} \cdot H_{U_2, U_3} \subset H_{U_1, U_3}$ and $M_{U_1, U_2} \cdot M_{U_2, U_3} \subset M_{U_1, U_3}$.

Theorem 5. *Let $(Y, \mathcal{T})$ be a locally connected T_4 space and let*

$$\mathcal{G}_1 = \{H_{U_1, U_2} : U_1, U_2 \in \mathcal{T}, \bar{U}_1 \subset U_2 \text{ and } Y \setminus U_1, Y \setminus U_2 \text{ are connected}\},$$

$$\mathcal{G}_2 = \{M_{U_1, U_2} : U_1, U_2 \in \mathcal{T}, \bar{U}_1 \subset U_2 \text{ and } Y \setminus U_1, Y \setminus U_2 \text{ are connected}\},$$

$$\mathcal{P}_1 = \{H_{U_1, U_2} : \bar{U}_1 \subset U_2 \text{ and } U_1, U_2 \text{ are connected open sets}\},$$

$$\mathcal{P}_2 = \{M_{U_1, U_2} : \bar{U}_1 \subset U_2 \text{ and } U_1, U_2 \text{ are connected open sets}\}.$$

Then

- (a) $\mathcal{G}_1, \mathcal{G}_2, \mathcal{P}_1, \mathcal{P}_2$ are subbases of quasi-uniformities $\mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{U}_1$ and $\mathfrak{U}_2$ on 2_c^Y ;
- (b) $\mathfrak{V}_1^{-1} = \mathfrak{U}_2$ and $\mathfrak{V}_2^{-1} = \mathfrak{U}_1$;
- (c) $\tau_{\mathfrak{V}_1} = (\mathcal{T}^*)^-$, $\tau_{\mathfrak{U}_1} = \mathcal{T}^-$, $\tau_{\mathfrak{V}_2} \subset (\mathcal{T}^*)^+$ and $\tau_{\mathfrak{U}_2} = \mathcal{T}^+$;
- (d) Let X be a topological space and $F : X \rightarrow Y$ a multivalued map with connected closed values. Then F is upper s -continuous if and only if the function $F : X \rightarrow (2_c^Y, \mathfrak{V}_2)$ is continuous.

PROOF. (a) For any open sets U_1, U_2 such that $\bar{U}_1 \subset U_2$ and $Y \setminus U_1, Y \setminus U_2$ are connected there exists an open set U with $Y \setminus U$ connected and $\bar{U}_1 \subset U \subset \bar{U} \subset U_2$. Then $H_{U_1, U}, H_{U, U_2} \in \mathcal{G}_1$, $M_{U_1, U}, M_{U, U_2} \in \mathcal{G}_2$ and $H_{U_1, U} \circ H_{U, U_2} \subset H_{U_1, U_2}$, $M_{U_1, U} \circ M_{U, U_2} \subset M_{U_1, U_2}$. These together with (P.1) give that $\mathcal{G}_1, \mathcal{G}_2$ are subbases of some quasi-uniformities $\mathfrak{V}_1, \mathfrak{V}_2$ on 2_c^Y . Using analogous arguments we conclude that $\mathcal{P}_1$ and $\mathcal{P}_2$ are subbases of some quasi-uniformities $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on 2_c^Y .

(b) Let us take open sets U_1, U_2 such that $\bar{U}_1 \subset U_2$ and $Y \setminus U_1, Y \setminus U_2$ are connected. We can choose connected open sets V_1, V_2 satisfying

$$Y \setminus U_2 \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset Y \setminus \bar{U}_1.$$

Then we have $H_{U_1, U_2} \in \mathcal{G}_1$, $M_{U_1, U_2} \in \mathcal{G}_2$ and

$$H_{U_1, U_2}^{-1} = M_{Y \setminus U_2, Y \setminus U_1} \supset M_{V_1, V_2}, \quad M_{U_1, U_2}^{-1} = H_{Y \setminus U_2, Y \setminus U_1} \supset H_{V_1, V_2}.$$

Since $M_{V_1, V_2} \in \mathcal{P}_2$ and $H_{V_1, V_2} \in \mathcal{P}_1$ we obtain $\mathfrak{V}_1^{-1} \subset \mathfrak{U}_2$ and $\mathfrak{V}_2^{-1} \subset \mathfrak{U}_1$. Now, let U_1, U_2 be connected open sets and let $\bar{U}_1 \subset U_2$. We choose connected open sets V_1, V_2 for which $\bar{U}_1 \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset U_2$ holds. Then we have $H_{U_1, U_2} \in \mathcal{P}_1$, $M_{U_1, U_2} \in \mathcal{P}_2$ and

$$\begin{aligned} H_{U_1, U_2}^{-1} &= M_{Y \setminus U_2, Y \setminus U_1} \supset M_{Y \setminus \bar{V}_2, Y \setminus \bar{V}_1} \\ M_{U_1, U_2}^{-1} &= H_{Y \setminus U_2, Y \setminus U_1} \supset H_{Y \setminus \bar{V}_2, Y \setminus \bar{V}_1}. \end{aligned}$$

Since $M_{Y \setminus \bar{V}_2, Y \setminus \bar{V}_1} \in \mathcal{G}_2$ and $H_{Y \setminus \bar{V}_2, Y \setminus \bar{V}_1} \in \mathcal{G}_1$ we obtain $\mathfrak{U}_1^{-1} \subset \mathfrak{V}_2$ and $\mathfrak{U}_2^{-1} \subset \mathfrak{V}_1$; thus (b) has been shown.

(c) For any set $A \subset Y$ we have

$$\begin{aligned} H_{U_1, U_2}[A] &= \begin{cases} U_2^-, & \text{if } A \cap U_1 \neq \emptyset \\ 2_c^Y, & \text{if } A \cap U_1 = \emptyset \end{cases} \\ M_{U_1, U_2}[A] &= \begin{cases} U_2^+, & \text{if } A \subset U_1 \\ 2_c^Y, & \text{if } A \cap (Y \setminus U_1) \neq \emptyset. \end{cases} \end{aligned}$$

Thus

$$(1) \quad \tau_{\mathfrak{V}_1} \subset (\mathcal{T}^*)^-, \quad \tau_{\mathfrak{V}_2} \subset (\mathcal{T}^*)^+, \quad \tau_{\mathfrak{U}_1} \subset \mathcal{T}^- \text{ and } \tau_{\mathfrak{U}_2} \subset \mathcal{T}^+.$$

Let $\mathcal{B}^*$ be a base of the topology $\mathcal{T}^*$ consisting of the open sets whose complements have a finite number of components in $(Y, \mathcal{T})$. Then

$\{U^- : U \in \mathcal{B}^*\}$ is a subbase of the topology $(\mathcal{T}^*)^-$ in 2_c^Y . If $U \in \mathcal{B}^*$, then $Y \setminus U = \bigcup_{i=1}^n D_i$, where the D_i are connected closed pairwise disjoint sets.

For any connected set $C \subset \bigcup_{i=1}^n D_i$ there exists $i \leq n$ such that $C \subset D_i$,

thus $\left(\bigcup_{i=1}^n D_i\right)^+ = \bigcup_{i=1}^n D_i^+$. From this we obtain $\left(\bigcap_{i=1}^n U_i\right)^- = \bigcap_{i=1}^n U_i^-$,

where $U_i = Y \setminus D_i$ and $U = \bigcap_{i=1}^n U_i$. Let $A \in U^-$; then we can choose a

point $y \in A \cap \bigcap_{i=1}^n U_i$ and open sets V_i with $y \in V_i \subset \bar{V}_i \subset U_i$ and $Y \setminus V_i$

connected for $i \leq n$. So $H_{V_i, U_i} \in \mathcal{G}_1$ and $U^- = \bigcap_{i=1}^n U_i^- = \bigcap_{i=1}^n H_{V_i, U_i}[A]$,

which implies $U^- \in \tau_{\mathfrak{U}_1}$. As a consequence we have $(\mathcal{T}^*)^- \subset \tau_{\mathfrak{U}_1}$ hence $\tau_{\mathfrak{U}_1} = (\mathcal{T}^*)^-$.

Now, let $\mathcal{B}$ be a base of the topology $\mathcal{T}$ consisting of connected open sets. The family $\{U^- : U \in \mathcal{B}\}$ is a subbase of the topology $\mathcal{T}^-$ on 2_c^Y . For any $U \in \mathcal{B}$ and $A \in U^-$ we take a point $y \in A \cap U$ and a set $V \in \mathcal{B}$ such that $y \in V \subset \bar{V} \subset U$. Then $H_{V, U} \in \mathcal{P}_1$ and $U^- = H_{V, U}[A]$; it gives $\mathcal{T}^- \subset \tau_{\mathfrak{U}_1}$ and finally $\tau_{\mathfrak{U}_1} = \mathcal{T}^-$.

Let $U \in \mathcal{T}$ and $A \in U^+$. Since A is a connected closed set, it follows from assumptions on $(Y, \mathcal{T})$ that we can take connected open sets V_1, V_2 for which $A \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset U$ holds. Then $M_{V_1, V_2} \in \mathcal{P}_2$ and $M_{V_1, V_2}[A] = V_2^+ \subset U^+$, so $U^+ \in \tau_{\mathfrak{U}_2}$. Since $\{U^+ : U \in \mathcal{T}\}$ is a base of $\mathcal{T}^+$, in virtue of (1) we have $\tau_{\mathfrak{U}_2} = \mathcal{T}^+$.

(d) Let $F : X \rightarrow Y$ be a multivalued map with values in 2_c^Y . Let us assume that F is upper s -continuous, $M_{U_1, U_2} \in \mathcal{G}_2$ and $x_0 \in X$. If $F(x_0) \not\subset U_1$, then $M_{U_1, U_2}[F(x_0)] = 2_c^Y$; hence for each $x \in X$ we have $F(x) \in M_{U_1, U_2}[F(x_0)]$. In the other case, if $F(x_0) \subset U_1$, then according to the upper s -continuity there is a neighbourhood W of x_0 such that $F(x) \subset U_1$ for $x \in W$. Thus $F(x) \in M_{U_1, U_2}[F(x_0)]$ for each $x \in W$ and this implies the continuity of the function $F : X \rightarrow (2_c^Y, \mathfrak{U}_2)$.

Conversely, suppose that $F : X \rightarrow (2_c^Y, \mathfrak{U}_2)$ is continuous. Let $x_0 \in X$, $U \in \mathcal{T}$, $F(x_0) \subset U$ and let $Y \setminus U$ be connected. We choose an open set V such that $Y \setminus V$ is connected and $F(x_0) \subset V \subset \bar{V} \subset U$. Since $M_{V, U} \in \mathcal{G}_2$ there exists a neighbourhood W of x_0 such that $F(x) \in M_{V, U}[F(x_0)]$ for $x \in W$. But $M_{V, U}[F(x_0)] = U^+$, so $F(x) \subset U$ for $x \in W$ and the proof is completed.

Corollary 6. *Let $(Y, \mathcal{T})$ be a locally connected T_4 space. Then there exist uniformities S_1, S_2 on 2_c^Y such that for any multivalued map $F : X \rightarrow Y$ with closed connected values the following conditions are equivalent:*

- (a) *F is upper semicontinuous and lower s -continuous (lower semicontinuous and upper s -continuous);*
- (b) *the function $F : X \rightarrow (2_c^Y, S_1)$, (resp. $F : X \rightarrow (2_c^Y, S_2)$) is continuous.*

PROOF. Applying Theorem 5 it suffices to take $S_1 = \mathfrak{V}_1 \vee \mathfrak{U}_2$ and $S_2 = \mathfrak{U}_1 \vee \mathfrak{V}_2$.

In theorem 5 (c) the equality $\tau_{\mathfrak{V}_2} = (\mathcal{T}^*)^+$ does not hold in general.

Example 7. Let $(R^2, \mathcal{T})$ be the plane with the natural topology and let A, W be the subsets of R^2 defined in Example 2. Then A is connected closed, W is open and $A \subset W$. Let $M_{U_{1,j}, U_{2,j}} \in \mathcal{G}_2$ for $j \leq n$ and

$$(1) \quad \left(\bigcap_{j=1}^n M_{U_{1,j}, U_{2,j}} \right) [A] \subset W^+;$$

without loss of generality we can assume $A \subset \bigcap_{j=1}^n U_{1,j}$. Since

$$\left(\bigcap_{j=1}^n M_{U_{1,j}, U_{2,j}} \right) [A] = \bigcap_{j=1}^n M_{U_{1,j}, U_{2,j}} [A] = \bigcap_{j=1}^n U_{2,j}^+ = \left(\bigcap_{j=1}^n U_{2,j} \right)^+,$$

for each point $x \in \bigcap_{j=1}^n U_{2,j}$ we have $\{x\} \in \left(\bigcap_{j=1}^n U_{2,j} \right)^+$. Hence

$$(2) \quad \bigcap_{j=1}^n U_{2,j} \subset W.$$

The sets $R^2 \setminus U_{2,j}$, $j \leq n$, are connected, so $R^2 \setminus \bigcap_{j=1}^n U_{2,j}$ has a finite number of components. Further $A \subset \bigcap_{j=1}^n U_{2,j}$, hence $B_m \not\subset \bigcap_{j=1}^n U_{2,j}$ holds only for a finite number of indices m . Thus $\bigcup_{m=1}^{\infty} B_m \not\subset R^2 \setminus \bigcap_{j=1}^n U_{2,j}$ and this contradicts (2). Consequently (1) does not hold and $(\mathcal{T}^*)^+ = \mathcal{T}^+ \not\subset \tau_{\mathfrak{V}_2}$.

References

- [1] R. ENGELKING, General topology, *Warszawa*, 1977.

- [2] L. K. KOHLI, A class of mappings containing all continuous and all semiconnected mappings, *Proc. Amer. Math. Soc.* **72** (1978), 175–181.
- [3] L. K. KOHLI, S -continuous functions and certain weak forms of regularity and complete regularity, *Math. Nachr.* **97** (1980), 189–196.
- [4] L. K. KOHLI, S -continuous mappings, certain weak forms of normality and strongly semilocally connected spaces, *Math. Nachr.* **99** (1980), 69–76.
- [5] K. KURATOWSKI, Some remarks on the relation of classical set-valued mappings to the Baire classification, *Colloq. Math.* **42** (1979), 273–277.
- [6] T. LIPSKI, S -continuous multivalued maps, *Math. Chronicle* **18** (1989), 57–61.
- [7] M. G. MURDESHWAR and S.A. NAIMPALLY, Quasi-uniform topological spaces, *Noordhoff*, 1966.
- [8] I. L. REILLY and M.K. VAMANAMURTHY, On the topology of semilocal connectedness, *Math. Nachr.* **129** (1986), 109–113.
- [9] G. T. WHYBURN, Analytic topology, *Amer. Math. Soc. Colloq. Publ.* **28** Providence R.I. (1963).

JANINA EWERT
DEPARTMENT OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
ARCISZEWSKIEGO 22
76-200 SLUPSK
POLAND

(Received September 10, 1991; revised March 26, 1992)