Publ. Math. Debrecen
66/3-4 (2005), 303–311

Polynomials with weighted sum

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Abstract. In this paper, we study the equation $z^n = \sum_{k=0}^{n-1} a_k z^k$, where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k. We show that, given $p > 1$, there exist $C(1/p)$-polynomials with the degree of weighted sum $n-1$. However, we obtain sufficient conditions for nonexistence of certain lacunary $C(1/p)$-polynomials. In case of the degree of weighted sum $n-2$, we see that, by giving an example, our sufficient condition is best possible in a certain sense.

1. Introduction

Throughout this paper, n is an integer ≥ 3, $p > 1$, and we denote $C(r)$ by the circle of radius r with center the origin.

If z is a complex number inside $C(1)$ which is not a positive real number, then there is an integer n such that z^n is a convex combination of lower integral powers $\{z^k : 0 \leq k < n\}$. Moreover the convex hull of the sequence $1, z, z^2, z^3, \dots$ is a polygon; if n is the number of vertices of this polygon, then these vertices are precisely the first n powers of z. For the proofs of the above, see Lemma 2.1 and Theorem 2.2 of [1]. Conversely, if

$$z^n = \sum_{k=0}^{n-1} a_k z^k, \tag{1}$$

Mathematics Subject Classification: Primary: 26C10; Secondary: 30C15.
Key words and phrases: weighted sum, zeros, polynomials.
This study was supported (in part) by research funds from Chosun University, 2004.

where $\sum_{k=0}^{n-1} a_k = 1$, $a_k \geq 0$ for each k, then it follows from Eneström–Kakeya theorem (see p. 136 of [2] for the statement and its proof) to

$$\frac{z^n - \sum_{k=0}^{n-1} a_k z^k}{z-1}$$

that all zeros of (1) do not lie outside $C(1)$. More precisely, the zeros of (1) are strictly inside $C(1)$ except for $z = 1$ since the average of points on $C(1)$ is strictly inside $C(1)$ unless all of the points are equal.

Whether or not certain polynomials have all their zeros on a circle is one of the most fundamental questions in the theory of distribution of polynomial zeros. Hence, in this paper, we study polynomials of type (1), $z^n - \sum_{k=0}^{n-1} a_k z^k$, whose all zeros except for $z = 1$ lie on $C(1/p)$. For convenience, we call these polynomials $C(1/p)$-polynomials, and $\sum_{k=0}^{n-1} a_k z^k$ in $C(1/p)$-polynomials their weighted sums, respectively.

In Section 2, we start to find $C(1/p)$-polynomials. In fact, we show that, given $p > 1$, there exist $C(1/p)$-polynomials whose the degree of weighted sum is $n-1$. However, by estimating some coefficients of lacunary polynomials with our purpose, we obtain sufficient conditions for nonexistence of certain lacunary $C(1/p)$-polynomials: If $p > n-1$, then there does not exist $C(1/p)$-polynomials whose the degree of weighted sums is $n-2$. Also, if $2p^4 - (n-1)(n-2)p^2 - 2(n-1)p - (n-1)(n-2) > 0$, then there does not exist $C(1/p)$-polynomials whose the degree of weighted sum is $n-3$. In case of the degree of weighted sum $n-2$, we show that, by giving an example, our sufficient condition is best possible in the sense that, for all $n \geq 3$, there exist $C(1/2)$-polynomials with the degree of the weighted sums $n-2$.

2. Proofs

The coefficients of the weighted sum of $C(1/p)$-polynomials are nonnegative. This follows that the constant term of $C(1/p)$-polynomials is $-\frac{1}{p^{n-1}}$. Hence if the weights in $C(1/p)$-polynomials are rational with the same denominator, then p^{n-1} is the smallest possible denominator.

The proposition below shows the existence of $C(1/p)$-polynomials.

Proposition 1. *Given $p > 1$, there exist $C(1/p)$-polynomials (whose the degree of weighted sum is $n-1$).*

Proof. For $p > 1$, consider a polynomial

$$K_{p,n}(z) = z^n - \frac{1}{p^{n-1}} H_{p,n}(z),$$

where

$$H_{p,n}(z) = 1 + (p-1)z\frac{(pz)^{n-1} - 1}{pz - 1}.$$

A simple calculation about $K_{p,n}(z) = 0$ yields that

$$p^n z^{n+1} - p^n z^n - z + 1 = 0. \tag{2}$$

Using change of variable from z to z/p in (2) and multiplying by p, we have

$$z^{n+1} - pz^n - z + p = (z-p)(z^n - 1) = 0,$$

which proves the result. $\square$

It is natural to ask the existence of lacunary $C(1/p)$-polynomials. To get some results for this, we first need the following proposition.

Proposition 2. *Let r be an integer with $1 \le r \le \lfloor \frac{n-1}{2} \rfloor$. Suppose*

$$f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k$$

is a $C(1/p)$-polynomial, where $a_{n-1} = a_{n-2} = \cdots = a_{n-r} = 0$. Then, for $1 \le k \le r$,

$$a_k = \frac{1}{p^{n-2k+1}}(p^2 - 1), \tag{3}$$

and, for $r + 1 \le k \le \lfloor \frac{n-1}{2} \rfloor$,

$$\begin{aligned} a_k = {} & (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_{n-k})\frac{1}{p^{n-2k-1}} \\ & - (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_{n-k+1})\frac{1}{p^{n-2k+1}}. \end{aligned} \tag{4}$$

Proof. Suppose $f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k$ is a $C(1/p)$-polynomial, where $a_{n-1} = a_{n-2} = \cdots = a_{n-r} = 0$. Then the equation $\frac{f(z)}{z-1} = 0$, i.e.,

$$\begin{aligned} &z^{n-1} + z^{n-2} + \cdots + z^{n-r-1} + (1 - a_{n-r-1})z^{n-r-2} \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2})z^{n-r-3} + \dots \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_2)z \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_2 - a_1) = 0 \end{aligned} \tag{5}$$

should have all zeros on $C(1/p)$. Now we let $z = \zeta/p$. Then (5) becomes

$$\begin{aligned} &\frac{\zeta^{n-1}}{p^{n-1}} + \frac{\zeta^{n-2}}{p^{n-2}} + \cdots + \frac{\zeta^{n-r-1}}{p^{n-r-1}} + (1 - a_{n-r-1}))\frac{\zeta^{n-r-2}}{p^{n-r-2}} \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2})\frac{\zeta^{n-r-3}}{p^{n-r-3}} + \dots \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_2)\frac{\zeta}{p} \\ &\quad + (1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_2 - a_1) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{\zeta^{n-1}}{p^{n-1}} + \frac{\zeta^{n-2}}{p^{n-2}} + \cdots + \frac{\zeta^{n-r-1}}{p^{n-r-1}} + (a_0 + a_1 + \cdots + a_{n-r-2})\frac{\zeta^{n-r-2}}{p^{n-r-2}} \\ &+(a_0 + a_1 + \cdots + a_{n-r-3})\frac{\zeta^{n-r-3}}{p^{n-r-3}} + \cdots + (a_0 + a_1)\frac{\zeta}{p} + a_0 = 0. \end{aligned} \tag{6}$$

We observe that the equation (6) of ζ has all zeros on $C(1)$, and its coefficients are all real. So, if ζ is a zero of (6) then so is $1/\zeta$. This follows that the left of (6) is self-reciprocal. Hence we have

$$\begin{aligned} a_0 &= \frac{1}{p^{n-1}} \\ \frac{a_0 + a_1}{p} &= \frac{1}{p^{n-2}} \\ \frac{a_0 + a_1 + a_2}{p^2} &= \frac{1}{p^{n-3}} \\ &\vdots \\ \frac{a_0 + a_1 + a_2 + \cdots + a_r}{p^r} &= \frac{1}{p^{n-r-1}} \end{aligned}$$

and

$$\frac{a_0 + a_1 + a_2 + \cdots + a_{r+1}}{p^{r+1}} = \frac{1 - a_{n-r-1}}{p^{n-r-2}}$$

$$\frac{a_0 + a_1 + a_2 + \cdots + a_{r+2}}{p^{r+2}} = \frac{1 - a_{n-r-1} - a_{n-r-2}}{p^{n-r-3}}$$

$$\vdots$$

$$\frac{a_0 + a_1 + a_2 + \cdots + a_{\lfloor \frac{n-1}{2} \rfloor}}{p^{\lfloor \frac{n-1}{2} \rfloor}} = \frac{1 - a_{n-r-1} - a_{n-r-2} - \cdots - a_{n-\lfloor \frac{n-1}{2} \rfloor}}{p^{n-\lfloor \frac{n-1}{2} \rfloor - 1}}.$$

From the above, we get (3) and (4). □

Remark 3. Suppose $f(z) = z^n - \sum_{k=0}^{n-1} a_k z^k$ is a $C(1/p)$-polynomial, where $a_{n-1} = a_{n-2} = 0$ and $a_{n-3} \neq 0$. Then, by applying $r = 2$ to Proposition 2, we have

$$a_0 = \frac{1}{p^{n-1}}, \quad a_1 = \frac{1}{p^{n-1}}(p^2 - 1), \quad a_2 = \frac{1}{p^{n-3}}(p^2 - 1)$$

and

$$a_3 = (1 - a_{n-3})\frac{1}{p^{n-7}} - \frac{1}{p^{n-5}}.$$

The next two propositions will be used to prove Theorem 6.

Proposition 4. *Let $f(x) = \sum_{k=0}^{n} a_k z^k$ be a polynomial whose zeros are z_j, $1 \le j \le n$. Suppose that $a_n = 1$, $a_{n-1} = 0$ and*

$$|z_1| = |z_2| = \cdots = |z_u| = \frac{1}{s}, \quad |z_{u+1}| = |z_{u+2}| = \cdots = |z_n| = \frac{1}{t},$$

where $s > t > 0$. Then we have

$$|a_1| \le \left(1 - \left(\frac{t}{s}\right)^2\right) \frac{u}{s^{u-1} t^{n-u}}.$$

Proof. Since $a_{n-1} = 0$, we have $z_1 + z_2 + \cdots + z_n = 0$. So

$$(-1)^{n+1} a_1 = \sum_{k=1}^{n} \prod_{\substack{1 \le j \le n \\ j \neq k}} z_j = \sum_{k=1}^{n} \left(\prod_{\substack{1 \le j \le n \\ j \neq k}} z_j - \bar{z}_k \prod_{j=1}^{n} z_j \right) = \sum_{k=1}^{n} (1 - |z_k|^2) \prod_{\substack{1 \le j \le n \\ j \neq k}} z_j$$

$$= \sum_{k=1}^{u}\left(1-\frac{1}{s^2}\right)\prod_{\substack{1\le j\le n\\ j\ne k}} z_j + \sum_{k=u+1}^{n}\left(1-\frac{1}{t^2}\right)\prod_{\substack{1\le j\le n\\ j\ne k}} z_j$$

$$= \left(1-\frac{1}{s^2}\right)\sum_{k=1}^{u}\prod_{\substack{1\le j\le n\\ j\ne k}} z_j + \left(1-\frac{1}{t^2}\right)\left(\sum_{k=1}^{n}\prod_{\substack{1\le j\le n\\ j\ne k}} z_j - \sum_{k=1}^{u}\prod_{\substack{1\le j\le n\\ j\ne k}} z_j\right)$$

$$= \left(\frac{1}{t^2}-\frac{1}{s^2}\right)\sum_{k=1}^{u}\prod_{\substack{1\le j\le n\\ j\ne k}} z_j + \left(1-\frac{1}{t^2}\right)(-1)^{n+1}a_1.$$

Hence

$$\frac{1}{t^2}(-1)^{n+1}a_1 = \left(\frac{1}{t^2}-\frac{1}{s^2}\right)\sum_{k=1}^{u}\prod_{\substack{1\le j\le n\\ j\ne k}} z_j.$$

The desired result follows from triangle inequality that

$$|a_1| \le \left(1-\left(\frac{t}{s}\right)^2\right)\frac{u}{s^{u-1}t^{n-u}}. \qquad \square$$

Using same idea of the above proof, we have

Proposition 5. *Let $f(x)=\sum_{k=0}^{n} a_k z^k$ be a polynomial whose zeros are z_j, $1\le j\le n$. Suppose that $a_n=1$, $a_{n-1}=a_{n-2}=0$ and*

$$|z_1|=|z_2|=\cdots=|z_u|=\frac{1}{s}, \quad |z_{u+1}|=|z_{u+2}|=\cdots=|z_n|=\frac{1}{t},$$

where $s>t>0$. Then we have

$$|a_2| \le \left(1-\left(\frac{t}{s}\right)^4\right)\frac{u(u-1)}{2(s^{u-2}t^{n-u})} + \left(1-\left(\frac{t}{s}\right)^2\right)\frac{u(n-u)}{s^{u-1}t^{n-u-1}}.$$

Proof. Since $a_{n-1}=a_{n-2}=0$, we have

$$z_1+z_2+\cdots+z_n = \sum_{k=1}^{n-1}\sum_{l=k+1}^{n} z_k z_l = 0.$$

So

$$(-1)^n a_2 = \sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j = \sum_{k=1}^{n-1}\sum_{l=k+1}^{n}\left(\prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j - \bar{z}_k\bar{z}_l\prod_{j=1}^{n} z_j\right)$$

$$= \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \left(1 - |z_k|^2 |z_l|^2\right) \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j$$

$$= \left(1 - \frac{1}{s^4}\right) \sum_{k=1}^{u-1} \sum_{l=k+1}^{u} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j + \left(1 - \frac{1}{s^2t^2}\right) \sum_{k=1}^{u-1} \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j$$

$$+ \left(1 - \frac{1}{s^2t^2}\right) \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne u,l}} z_j + \left(1 - \frac{1}{t^4}\right) \sum_{k=u+1}^{n-1} \sum_{l=k+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j.$$

And the sum of the last summand, i.e.,

$$\sum_{k=u+1}^{n-1} \sum_{l=k+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j$$

equals

$$\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j - \sum_{k=1}^{u-1} \sum_{l=k+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j - \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne u,l}} z_j$$

$$= (-1)^n a_2 - \left(\sum_{k=1}^{u-1} \sum_{l=k+1}^{u} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j + \sum_{k=1}^{u-1} \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j\right) - \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne u,l}} z_j.$$

Hence, in all,

$$(-1)^n a_2 = \left(\frac{1}{t^4} - \frac{1}{s^4}\right) \sum_{k=1}^{u-1} \sum_{l=k+1}^{u} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j + \left(\frac{1}{t^4} - \frac{1}{s^2t^2}\right) \sum_{k=1}^{u-1} \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne k,l}} z_j$$

$$+ \left(\frac{1}{t^4} - \frac{1}{s^2t^2}\right) \sum_{l=u+1}^{n} \prod_{\substack{1\le j\le n\\ j\ne u,l}} z_j + \left(1 - \frac{1}{t^4}\right) (-1)^n a_2.$$

Now, by triangle inequality, we get

$$\frac{|a_2|}{t^4} \le \left(\frac{1}{t^4} - \frac{1}{s^4}\right) \frac{u(u-1)}{2(s^{u-2}t^{n-u})} + \left(\frac{1}{t^4} - \frac{1}{s^2t^2}\right) \frac{(u-1)(n-u)}{s^{u-1}t^{n-u-1}}$$

$$+\left(\frac{1}{t^4}-\frac{1}{s^2t^2}\right)\frac{n-u}{s^{u-1}t^{n-u-1}},$$

which follows the result by simple calculation. □

Now we are ready to prove the following theorem.

Theorem 6. (1) *If $p > n-1$, then there does not exist $C(1/p)$-polynomials whose the degree of weighted sums is $n-2$.*

(2) *If $2p^4-(n-1)(n-2)p^2-2(n-1)p-(n-1)(n-2) > 0$, then there does not exist $C(1/p)$-polynomials whose the degree of weighted sums is $n-3$.*

Proof. Applying $u = n-1$, $s = p$, $t = 1$ to Proposition 4 and Proposition 5, respectively, we get

$$\begin{aligned}|a_1| &\le \left(1-\frac{1}{p^2}\right)\frac{n-1}{p^{n-2}},\\ |a_2| &\le \left(1-\frac{1}{p^4}\right)\frac{(n-1)(n-2)}{2p^{n-3}}+\left(1-\frac{1}{p^2}\right)\frac{n-1}{p^{n-2}}.\end{aligned} \tag{7}$$

But, by Remark 3,

$$a_1=\frac{1}{p^{n-1}}(p^2-1), \quad a_2=\frac{1}{p^{n-3}}(p^2-1).$$

Substituting these into (7) easily proves the theorem. □

Remark 7. (1) An example of an identity

$$z^n-\frac{1}{2^{n-1}}\left(Q_n(z)+z\right)=\frac{1}{2^{n-1}}(z-1)(2z+1)Q_n(z),$$

where

$$Q_n(z)=\frac{(2z)^{n-1}-1}{2z-1}$$

and the polynomial $Q_n(z)+z$ has degree $n-2$ shows that, for all $n \ge 3$, there exist $C(1/2)$-polynomials with the degree of the weighted sum $n-2$. And, for $n = 3$, the first result of Theorem 6 asserts nonexistence of $C(1/p)$-polynomials whose the degree of weighted sum is 1, where $p > 1/2$. Hence our sufficient condition in case of the degree of weighted sum $n-2$ is best possible in this sense.

(2) For each n, by computer algebra, we can check the hypothesis in second result of Theorem 6. Here, in Table 1, we give ranges of p satisfying the hypothesis for each $n = 3, 4, 5, 6, 7$.

n	P
3	$p > 1.6180$
4	$p > 2.2257$
5	$p > 2.8529$
6	$p > 3.4994$
7	$p > 4.1604$
$\vdots$	$\vdots$

Table 1

Acknowledgements. The author wishes to thank Professor Kenneth B. Stolarsky for helpful discussions, and also to thank the referee for suggestions and interest.

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(Received March 14, 2003; revised September 30, 2003)