

## A note on generalized inverses and a block-rank equation

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**Abstract.** In this paper we study the rank equation  $\text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A)$  and find the necessary and sufficient conditions when  $X = A^{(1,2)}$  and  $X = A^d$  are the solutions of that equation. In both cases we give an explicit form of matrices  $B$  and  $C$ .

### 1. Introduction

Let  $C^{m \times n}$  denote the set of complex  $m \times n$  matrices.  $I_n$  denotes the unit matrix of order  $n$ . By  $A^*$ ,  $R(A)$ ,  $\text{rank}(A)$  and  $N(A)$  we denote the conjugate transpose, the range, the rank and the null space of  $A \in C^{n \times m}$ . The symbol  $A^-$  stands for an arbitrary generalized inner inverse of  $A$ , i.e.  $A^-$  satisfies  $AA^-A = A$ . By  $A^\dagger$  we denote the Moore–Penrose inverse of  $A$ , i.e. the unique matrix  $A^\dagger$  satisfying

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

For  $A \in C^{m \times n}$  the smallest nonnegative integer  $k$  such that  $\text{rank}(A^{k+1}) = \text{rank}(A^k)$  is called the index of  $A$  and denoted by  $\text{ind}(A)$ . If  $A \in C^{n \times n}$ , with  $\text{ind}(A) = k$ , then the matrix  $X \in C^{n \times n}$  which satisfies the following

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conditions

$$A^k X A = A^k, \quad X A X = X, \quad A X = X A,$$

is called the Drazin inverse of  $A$  and it is denoted by  $A^d$ . When  $\text{ind}(A) = 1$  then the Drazin inverse  $A^d$  is called the group inverse and it is denoted by  $A^\#$ . Also, the matrix  $X$  which satisfies

$$A X A = A \quad \text{and} \quad X A X = X$$

is called the reflexive inverse of  $A$  and it is denoted by  $A^{(1,2)}$ . For other important properties of generalized inverses see [1] and [3].

In this paper we will consider the rank equation

$$\text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A), \quad (1)$$

for arbitrary  $A \in C^{n \times n}$ . First, we give a necessary and sufficient conditions such that  $X = A^{(1,2)}$  is the solution of equation (1) and all possible matrices  $B$  and  $C$  are described. As a corollary we obtain the result of J. GROSS [8] and N. THOME and Y. WEI [7]. Moreover, we consider when  $X = A^d$  is the solution of the equation (1), for an arbitrary matrix  $A$  with  $\text{ind}(A) = k \geq 1$  and we obtain some interesting corollaries.

## 2. Main results

We start this section with some well-known results. The following lemma was proved in [4], [5] and [6].

**Lemma 2.1.** *Let  $A \in C^{n \times n}$ ,  $B \in C^{n \times m}$ ,  $C \in C^{m \times n}$  and  $X \in C^{m \times m}$ . Then*

$$\text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A) + \text{rank}(L) + \text{rank}(M) + \text{rank}(W),$$

where  $S = I_n - A^- A$ ,  $L = CS$ ,  $M = SB$  and  $W = (I_m - LL^-)(X - CA^-B)(I_m - M^-M)$ .

The following theorem, which is proved by J. GROSS [8], gives a characterization of the existence of the solution of the equation (1) by means of geometrical conditions.

**Theorem 2.1.** *Let  $A \in C^{m \times n}$ ,  $B \in C^{m \times m}$  and  $C \in C^{n \times n}$ . Then there exists a solution  $X \in C^{m \times m}$  of the equation (1) if and only if  $R(B) \subseteq R(A)$  and  $R(C^*) \subseteq R(A^*)$ , in which case  $X = CA^\dagger B$ .*

Notice that the conditions  $R(B) \subseteq R(A)$  and  $R(C^*) \subseteq R(A^*)$  are equivalent to  $AA^\dagger B = B$  and  $CA^\dagger A = C$ . Also, the matrix product  $CA^-B$  is invariant with respect to the choice of generalized inverse  $A^-$  of  $A$  if and only if  $R(B) \subseteq R(A)$  and  $R(C^*) \subseteq R(A^*)$ .

First we consider a necessary and sufficient conditions such that  $X = A^{(1,2)}$  is the solution of the equation (1) and in this case we find the explicit form for  $B$  and  $C$ .

Matrix  $A \in C^{m \times n}$  such that  $\text{rank}(A) = r$  can be decomposed by

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q, \quad (2)$$

where  $P \in C^{m \times m}$ ,  $Q \in C^{n \times n}$  and  $D \in C^{r \times r}$  are invertible matrices. Given that decomposition arbitrary reflexive generalized inverse of  $A$  has the following form

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & U \\ V & VDU \end{bmatrix} P^{-1} \quad (3)$$

where  $U$  and  $V$  are arbitrary matrices of suitable size (see [2]).

The following theorem gives a sufficient and necessary conditions such that  $X = A^{(1,2)}$  is the solution of the equation (1).

**Theorem 2.2.** *Let  $A \in C^{m \times n}$ ,  $B \in C^{m \times m}$ ,  $C \in C^{n \times n}$  and  $X \in C^{n \times m}$  and let the matrix  $A$  and its reflexive generalized inverse be given by (2) and (3) respectively. Then  $X = A^{(1,2)}$  is the solution of the equation (1) if and only if*

$$B = P \begin{bmatrix} DL & (DLD)U \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ VL^{-1} & 0 \end{bmatrix} Q \quad (4)$$

for some nonsingular matrix  $L \in C^{r \times r}$ .

PROOF. Suppose that  $X = A^{(1,2)}$  is the solution of the equation (1). Then there exist matrices  $G \in C^{m \times m}$  and  $F \in C^{n \times m}$  such that  $B = AG$ ,

$C = FA$  and  $CA^-B = A^{(1,2)}$ . Let

$$QGP = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad QFP = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

Hence,

$$B = AG = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P^{-1} = P \begin{bmatrix} DG_1 & DG_2 \\ 0 & 0 \end{bmatrix} P^{-1} \quad (5)$$

and

$$C = FA = Q^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q = Q^{-1} \begin{bmatrix} F_1D & 0 \\ F_3D & 0 \end{bmatrix} Q. \quad (6)$$

Also,

$$A^{(1,2)} = FAG = Q^{-1} \begin{bmatrix} F_1DG_1 & F_1DG_2 \\ F_3DG_1 & F_3DG_2 \end{bmatrix} P^{-1}.$$

Now, from (3) we have that

$$F_1DG_1 = D^{-1}, \quad F_1DG_2 = U, \quad F_3DG_1 = V.$$

From the first equation we obtain that  $F_1, G_1$  are invertible matrices and  $F_1D = D^{-1}G_1^{-1}$ . Now,  $DG_2 = F_1^{-1}U = DG_1DU$  and  $F_3D = VG_1^{-1}$ . If we replace that in (5) and (6) and put  $G_1 = L$ , we obtain (4).

Now, suppose that (4) holds. Then  $AA^-B = B$  and  $C = CA^-A$ , for generalized inner inverse  $A^-$  of  $A$ , which is given by

$$A^- = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

So by Theorem 2.1 there exists a solution  $X = CA^-B$  of the equation (1). By (4) we can easily check that  $X = CA^-B = A^{(1,2)}$ .  $\square$

Remark that when we consider the special reflexive inverse of  $A$ ,

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad (7)$$

for  $U = V = 0$ , we obtain the ([7], Theorem 3).

**Corollary 2.1.** *Let  $A \in C^{m \times n}$ ,  $B \in C^{m \times m}$ ,  $C \in C^{n \times n}$  and  $X \in C^{n \times m}$ . Let the matrix  $A$  and one its reflexive generalized inverse be given by (2) and (7) respectively. Then  $X = A^{(1,2)}$  is the solution of the equation (1) if and only if*

$$B = P \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q, \quad (8)$$

for some nonsingular matrix  $L \in C^{r \times r}$ .

Now, we consider the singular value decomposition of  $A \in C^{m \times n}$  such that  $\text{rank}(A) = r$

$$A = M \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} N^*, \quad (9)$$

where  $M \in C^{m \times m}$  and  $N \in C^{n \times n}$  are unitary and  $D \in C^{r \times r}$  is a real positive definite diagonal matrix. By Theorem 2.2 we obtain ([8], Theorem 2).

**Corollary 2.2.** *Let  $A \in C^{m \times n}$ ,  $B \in C^{m \times m}$ ,  $C \in C^{n \times n}$  and  $X \in C^{n \times m}$ . Let the matrix  $A$  be given by (9). Then  $X = A^\dagger$  is the solution of the equation (1) if and only if*

$$B = M \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} M^* \quad \text{and} \quad C = N \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} N^*, \quad (10)$$

for some nonsingular matrix  $L \in C^{r \times r}$ .

PROOF. Taking  $P = M$  and  $Q = N^*$  in (2), we obtain that the matrix  $A$  has the representation (9) and in that case  $A^\dagger = N \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} M^*$ , which has the form (7). Hence, the result follows from Corollary 2.1.  $\square$

In the rest of the paper, we consider the following question: When  $X = A^d$  is the solution of the equation (1)?

First, let  $A \in C^{m \times n}$  and  $\text{ind}(A) = 1$ . Using the Jordan canonical form of  $A$ , there exist nonsingular matrices  $P \in C^{n \times n}$  and  $D \in C^{r \times r}$  such that

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{-1}. \quad (11)$$

We obtain the result of N. THOME and Y. WEI ([7], Theorem 2).

**Theorem 2.3.** *Let  $A \in C^{n \times n}$  with  $\text{ind}(A) = 1$  and  $\text{rank}(A) = r$  be given by (11) and  $B, C, X \in C^{n \times n}$ . Then  $X = A^\#$  is the solution of the equation (1) if and only if*

$$B = P \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = P \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad (12)$$

for some nonsingular matrix  $L \in C^{r \times r}$ .

PROOF. If the matrix  $A$  is given by (11), then  $A^\# = P \begin{bmatrix} D_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$ . Hence, the result follows from Corollary 2.1 taking  $Q = P^{-1}$  and noticing that  $A^\#$  is given by (7).  $\square$

Now, we consider a more general case when  $A \in C^{n \times n}$  is such that  $\text{ind}(A) = k \geq 1$  and  $\text{rank}(A) = r$ . Then the matrix  $A$  can be written as

$$A = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P, \quad (13)$$

where  $P \in C^{n \times n}$ ,  $M \in C^{r \times r}$  are nonsingular matrices and  $N \in C^{(n-r) \times (n-r)}$  is nilpotent, that is  $N^k = 0$ . In this case

$$A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P.$$

**Theorem 2.4.** *Let  $A \in C^{n \times n}$ , with  $\text{index}(A) = k$ , be represented by (13) and  $B, C, X \in C^{n \times n}$ . Then  $X = A^d$  is the solution of the equation (1) if and only if there exist  $G_1, F_1 \in C^{r \times r}$ ,  $G_2, F_2 \in C^{r \times (n-r)}$ ,  $G_3, F_3 \in C^{(n-r) \times r}$  and  $G_4, F_4 \in C^{(n-r) \times (n-r)}$  such that*

$$B = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \quad \text{and} \quad C = P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P \quad (14)$$

and

$$\begin{aligned} F_1MG_1 + F_2NG_3 &= M^{-1}, \\ F_1MG_2 + F_2NG_4 &= 0, \\ F_3MG_1 + F_4NG_3 &= 0, \\ F_3MG_2 + F_4NG_4 &= 0. \end{aligned} \quad (15)$$

PROOF. Suppose that  $X = A^d$  is the solution of the equation (1). From Theorem 2.1 we have that  $R(B) \subseteq R(A)$  and  $R(C^*) \subseteq R(A^*)$ , so there exist matrices  $G$  and  $F$  such that  $B = AG$  and  $C = FA$  and  $A^d = CA^\dagger B$ . Let

$$PGP^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad PFP^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

It follows that

$$B = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P$$

and

$$C = P^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P = P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P.$$

Since the matrix  $A$  has the form (13), it follows that

$$A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \quad \text{and} \quad A^\dagger = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} P.$$

Hence,

$$\begin{aligned} A^d &= P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} F_1MG_1 + F_2NG_3 & F_1MG_2 + F_2NG_4 \\ F_3MG_1 + F_4NG_3 & F_3MG_2 + F_4NG_4 \end{bmatrix} P. \end{aligned}$$

We obtain the following system

$$\begin{aligned} F_1MG_1 + F_2NG_3 &= M^{-1}, \\ F_1MG_2 + F_2NG_4 &= 0, \\ F_3MG_1 + F_4NG_3 &= 0, \\ F_3MG_2 + F_4NG_4 &= 0. \end{aligned}$$

Conversely, suppose that the matrices  $B$  and  $C$  satisfied (14). Then we see that  $AA^\dagger B = B$  and  $C = CA^\dagger A$ . From Theorem 2.1 we have that there exists a solution  $X = CA^\dagger B$  of the equation (1). Now, from the system (15) it follows that  $CA^\dagger B = A^d$ , so  $X = A^d$  is the solution of the equation (1).  $\square$

Notice that Theorem 2.4 is a generalization of Theorem 2.3.

Now, we state some interesting results.

**Theorem 2.5.** *Let  $A \in C^{n \times n}$ , with  $\text{ind}(A) = k$  has the form (13), let  $p, m, n$  be positive integers and  $m, n \geq k$ . Then  $X = A^d$  is the solution of the equation*

$$\text{rank} \begin{bmatrix} A^p & A^n \\ A^m & X \end{bmatrix} = \text{rank}(A^p), \quad (16)$$

*if and only if  $M^{m+n-p} = M^{-1}$ .*

PROOF. Suppose that  $X = A^d$  is the solution of the equation (16). Then  $A^d = A^m(A^p)^- A^n$ . Hence,

$$\begin{aligned} & P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} M^m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M^{-p} & 0 \\ 0 & (N^p)^- \end{bmatrix} \begin{bmatrix} M^n & 0 \\ 0 & 0 \end{bmatrix} P \\ &= P^{-1} \begin{bmatrix} M^{(m+n-p)} & 0 \\ 0 & 0 \end{bmatrix} P, \end{aligned}$$

i.e.  $M^{m+n-p} = M^{-1}$ .

On the contrary, suppose that  $M^{m+n-p} = M^{-1}$ . First, we show that there exists a solution  $X$  of the equation (16), i.e. that  $R(A^n) \subseteq R(A^p)$  and  $N(A^p) \subseteq N(A^m)$ .

If  $y \in R(A^n)$ , then there exists  $x$  such that  $y = A^n x$ , i.e.

$$y = P^{-1} \begin{bmatrix} M^n z_1 \\ 0 \end{bmatrix}, \quad \text{where} \quad Px = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Now,

$$y = P^{-1} \begin{bmatrix} M^p & 0 \\ 0 & N^p \end{bmatrix} \begin{bmatrix} M^{(p-n)} z_1 \\ 0 \end{bmatrix},$$

implying that  $y = A^p x'$ , where  $x' = P^{-1} \begin{bmatrix} M^{(p-n)} z_1 \\ 0 \end{bmatrix}$ . Hence,  $R(A^n) \subseteq R(A^p)$  and analogously  $N(A^p) \subseteq N(A^n)$ . Using the same computation as in the first part, we obtain that  $X = A^d$  is the solution of the equation (16).  $\square$

*Remark 1.* Notice that Theorem 2.5 is also valid if we put  $f(n)$  and  $g(m)$  instead of  $n, m$ , where  $f, g$  are arbitrary positive functions.

**Corollary 2.3.** Let  $A \in C^{n \times n}$ , with  $\text{ind}(A) = k$  has the form (13), let  $p, m, n$  be positive integers such that  $m, n \geq k$  and  $m + n = p - 1$ . Then  $X = A^d$  is the solution of the equation (16).

**Corollary 2.4.** Let  $A \in C^{n \times n}$ , then

$$\text{rank} \begin{bmatrix} A^{(2l+1)} & A^l \\ A^l & A^d \end{bmatrix} = \text{rank } A^{(2l+1)},$$

for arbitrary integer  $l \geq \text{ind}(A)$ .

**Corollary 2.5.** Let  $A \in C^{n \times n}$  and  $\text{ind}(A) = 1$ , then

$$\text{rank} \begin{bmatrix} A^3 & A \\ A & A^\# \end{bmatrix} = \text{rank}(A^3).$$

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