

## Varieties of involution semilattices of Archimedean semigroups

By IGOR DOLINKA (Novi Sad)

**Abstract.** We give an indicator characterization of involution semigroup varieties consisting of involution semilattices of Archimedean semigroups. As a consequence, we describe identities which induce such a structure on an involution semigroup. This description is made explicit for the one-variable case.

The principal source of motivation for the present note is the paper [1] by ĆIRIĆ and BOGDANOVIĆ. It is concerned with some special aspects of a general problem of great importance in the theory of semigroup varieties: given a family of identities (with certain syntactical properties), what can be said about the structure of semigroups satisfying such identities? And conversely: given a prescribed structural feature of semigroups, which identities ‘force’ the semigroups satisfying them to have the required feature? Namely, the most significant result of [1] is the solution of Problem 7.1 posed in the survey of SHEVRIN and SUKHANOV [8], which asked for characterizations of semigroup varieties consisting entirely of semilattices of Archimedean semigroups. A sufficiently complete solution to this problem was known earlier only for periodic varieties, see SAPIR and SUKHANOV [7].

---

*Mathematics Subject Classification:* 20M07.

*Key words and phrases:* involution semigroup, variety, identity, Archimedean semigroup, semilattice decomposition.

Supported by Grant No. 1227 of the Ministry of Science, Technologies and Development of Republic of Serbia.

It is the purpose of this note to sketch the main points of [1] and [2] for varieties of involution semigroups.

Recall that by an *involution semigroup* we mean a structure  $(S, \cdot, *)$ , where  $(S, \cdot)$  is a semigroup, while the following identities:  $(x^*)^* = x$  and  $(xy)^* = y^*x^*$  are satisfied (i.e.  $*$  is an antiautomorphism of  $S$  of order 2). Since these laws suffice to transform each expression involving multiplication and star into a form in which  $*$  acts only on variables, we find it convenient to introduce the notion of an *involution semigroup word* over an alphabet  $X$ , which is just an ordinary word over the ‘double’ alphabet  $X \cup X^*$ , where  $X^* = \{x^* : x \in X\}$ . Then, clearly, the free involution semigroup on  $X$ ,  $F_X^*$ , consists of all nonempty involution semigroup words over  $X$ , the involution being defined by

$$(y_1 \dots y_n)^* = y_n^* \dots y_1^*$$

for all  $y_1, \dots, y_n \in X \cup X^*$ , where  $y_i^* = x^*$  if  $y_i = x \in X$ , while  $y_i^* = x$  for  $y_i = x^* \in X^*$ .

If  $W$  is an involution semigroup word, the following functions may be useful. First, we have  $c(W)$ , the *content* of  $W$ , which is the set of all variables from  $X$  occurring in  $W$ . However, one may consider the elements of  $X^*$  as irreducible symbols of the alphabet, so that we obtain  $c^*(W)$  the *\*-content* of  $W$  (for example,  $c^*(x^*yz^*x) = \{x, x^*, y, z^*\}$ , while  $c(x^*yz^*x) = \{x, y, z\}$ ). Finally, we define the set of *paired* variables of  $W$  as

$$\pi(W) = \{x \in X : x, x^* \in c^*(W)\}.$$

One of the easiest ways to embed any semigroup into an involutorial one is the following construction. Let  $S$  be the given semigroup, and let  $S^\partial$  stand for its dual semigroup ( $S^\partial = \{\bar{a} : a \in S\}$  and  $\bar{a} \cdot \bar{b} = \overline{ba}$ ). Construct a semigroup on the set  $S \cup S^\partial \cup \{\mathbf{0}\}$  (where  $\mathbf{0} \notin S \cup S^\partial$ ) such that the multiplication  $\circ$  is given by  $a \circ b = ab$  for  $a, b \in S$ ,  $\bar{a} \circ \bar{b} = \overline{ba}$  for  $\bar{a}, \bar{b} \in S^\partial$  and  $a \circ b = \mathbf{0}$  otherwise. The involution is defined by  $\mathbf{0}^* = \mathbf{0}$  and  $a^* = \bar{a}$ ,  $\bar{a}^* = a$  for all  $a \in S$ . In this way, it is easily checked that we obtain an involution semigroup, which we denote by  $I_0^*(S)$ . This is just the 0-direct union (or orthogonal sum) of  $S$  and its dual. In particular, if  $E$  is the trivial semigroup, then  $I_0^*(E)$  is a three-element involution semilattice, which generates the variety  $\mathcal{SL}^0$  (it is determined by the identities  $xy = yx$ ,  $xx^*y = xx^*$ , cf. [4]).

The identities of involution semigroups of the above form were investigated in [3]. The results of [3] will be used here in the form of the following

**Lemma 1** ([3]). *Let  $U, V$  be involution semigroup words and let  $S$  be a semigroup. Then the identity  $U = V$  holds in  $I_0^*(S)$  if and only if either*

- (i)  $\pi(U) \neq \emptyset$  and  $\pi(V) \neq \emptyset$ , or
- (ii)  $\pi(U) = \pi(V) = \emptyset$ , and  $U = V$  is obtained from a homotypical semigroup identity satisfied by  $S$  whose reverse is also true in  $S$ , by replacing some of the variables by their stars.

*In particular,  $U = V$  holds in  $\mathcal{SL}^0$  if and only if it is either of type (i), or  $\pi(U) = \pi(V) = \emptyset$  and  $c^*(U) = c^*(V)$ .*

Of course, by a *reverse* of a word  $W = a_1a_2 \dots a_{m-1}a_m$  we mean  $\overline{W} = a_na_{n-1} \dots a_2a_1$ , while the reverse of the (semigroup) identity  $U = V$  is just  $\overline{U} = \overline{V}$ . The following remark will be also useful in the sequel.

**Lemma 2.** *If a semigroup  $S$  admits an involutorial antiautomorphism, then the set of its identities is closed for taking reverses.*

PROOF. Assume that  $U = V$  holds in  $S$  and let  $a_1, \dots, a_n \in S$  be arbitrary. Then we have  $U(a_1^*, \dots, a_n^*) = V(a_1^*, \dots, a_n^*)$ , implying  $(U(a_1^*, \dots, a_n^*))^* = (V(a_1^*, \dots, a_n^*))^*$ . The lemma now follows immediately by noting that from the involution axioms we have

$$\overline{W} = (W(x_1^*, \dots, x_n^*))^*$$

for any word  $W$ . □

For a semigroup  $S$  and  $a, b \in S$  we write  $a \mid b$  (*a divides b*) if  $b = xay$  for some  $x, y \in S^1$ . If there exists  $n \in \mathbb{N}$  such that  $a \mid b^n$ , we write  $a \longrightarrow b$ . A semigroup  $S$  is *Archimedean* if  $a \longrightarrow b$  for all  $a, b \in S$ . On the other hand,  $S$  is a semilattice of Archimedean semigroups if  $S$  has a congruence  $\theta$  such that  $S/\theta$  is a semilattice, and each of the  $\theta$ -classes is an Archimedean subsemigroup of  $S$ . We refer to [1] for an extensive list of papers dealing with such semigroups.

However, if  $S$  is an involution semigroup, then the congruence  $\theta$  with the above properties is easily shown to be compatible with  $*$ , so that it is

a  $*$ -congruence, and  $S/\theta$  is an involution semilattice (namely, such  $\theta$ , if it exists, is unique [6], and it is given by  $a \theta b$  if and only if  $\Sigma(a) = \Sigma(b)$ , where  $\Sigma(a) = \{x \in S : a \longrightarrow x\}$ , whence it suffices to see that we have  $\Sigma(a^*) = (\Sigma(a))^*$ ). In such a case, we use the term *involution semilattice* of Archimedean semigroups.

A classical result is that  $S$  is a semilattice of Archimedean semigroups if and only if  $a^2 \longrightarrow ab$  for all  $a, b \in S$  (see [1, Theorem 1]). The lemma below gives a somewhat modified sufficient condition for an involution semigroup to have such a decomposition.

**Lemma 3.** *Let  $S$  be an involution semigroup such that we have  $a^2 \longrightarrow aa^* \longrightarrow ab$  ( $a^2 \longrightarrow a^*a \longrightarrow ab$ ) for all  $a, b \in S$ . Then  $S$  is an involution semilattice of Archimedean semigroups.*

PROOF. Let us consider only the first case, the other one being analogous. We have that for each  $a, b \in S$  there are  $n, k \in \mathbb{N}$  such that  $a^2 \mid (aa^*)^n$  and  $aa^* \mid (ab)^k$ , i.e.  $(aa^*)^n = xa^2y$  and  $(ab)^k = uaa^*v$  for some  $x, y, u, v \in S^1$ . We claim that for arbitrary  $a, b \in S$  and any  $\ell \in \mathbb{N}$  we have  $(aa^*)^\ell \longrightarrow ab$ . For this, it suffices to show that  $(aa^*)^{2^r} \longrightarrow ab$  for all  $r \geq 0$ . For  $r = 0$  the claim is true by assumption. So, assume that  $(aa^*)^{2^r} \longrightarrow ab$  for some  $r$ . Then

$$(ab)^{k_1} = x_1(aa^*)^{2^r}y_1$$

for some  $k_1 \in \mathbb{N}$  and  $x_1, y_1 \in S^1$ . But we also have that

$$((aa^*)^{2^r}y_1x_1)^{k_2} = x_2(aa^*)^{2^{r+1}}y_2$$

for some  $k_2 \in \mathbb{N}$  and  $x_2, y_2 \in S_1$ . Hence,

$$\begin{aligned} (ab)^{k_1(k_2+1)} &= (x_1(aa^*)^{2^r}y_1)^{k_2+1} = x_1((aa^*)^{2^r}y_1x_1)^{k_2}(aa^*)^{2^r}y_1 \\ &= (x_1x_2)(aa^*)^{2^{r+1}}(y_2(aa^*)^{2^r}y_1), \end{aligned}$$

proving that  $(aa^*)^{2^{r+1}} \longrightarrow ab$ . Thus, our claim follows by induction. Finally, by setting  $\ell = n$ , we obtain  $a^2 \longrightarrow ab$ , and  $S$  is a semilattice of Archimedean semigroups.  $\square$

As it is usual, we denote by  $B_2$  the five-element combinatorial Brandt semigroup, given by the presentation  $\langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle$ .

However, we shall be much more interested in considering  $B_2$  as an inverse semigroup, which is presented in the variety of involution semigroups by

$$\langle a \mid a^2 = 0, aa^*a = a \rangle.$$

So,  $B_2$  consists of elements  $0, a, a^*, aa^*$  and  $a^*a$ .

**Lemma 4.** *Let  $\mathcal{V}$  be a variety of involution semigroups not containing  $B_2$ . Then for any  $S \in \mathcal{V}$  and  $a \in S$  we have  $a^2 \longrightarrow aa^*$  and  $a^2 \longrightarrow a^*a$ .*

PROOF. First of all, consider the following partition of the set of all 1-letter involution semigroup words:

$$\begin{aligned} W_a &= \{(xx^*)^n x : n \geq 0\}, & W_{a^*} &= \{(x^*x)^n x^* : n \geq 0\}, \\ W_{aa^*} &= \{(xx^*)^n : n \geq 0\}, & W_{a^*a} &= \{(x^*x)^n : n \geq 0\}, \\ W_0 &= F_{\{x\}}^* \setminus (W_a \cup W_{a^*} \cup W_{aa^*} \cup W_{a^*a}). \end{aligned}$$

Clearly,  $W_0$  consists of all words from  $F_{\{x\}}^*$  which contain  $x^2$  or  $(x^*)^2$  as a subword. The above partition defines an equivalence  $\theta$  on  $F_{\{x\}}^*$ , which is easily seen to be a congruence, and  $F_{\{x\}}^*/\theta \cong B_2$ .

Now, if  $B_2 \notin \mathcal{V}$ , then  $\mathcal{V}$  satisfies an identity  $U = V$  which fails in  $B_2$ . In fact, there are then  $a_1, \dots, a_n \in B_2$  such that  $U(a_1, \dots, a_n)$  and  $V(a_1, \dots, a_n)$  represent different elements of  $B_2$  (considered as an involution semigroup generated by  $a$ ). Define a sequence of 1-letter involution semigroup words  $W_1, \dots, W_n$  such that for all  $1 \leq i \leq n$  we have

$$W_i = \begin{cases} x & \text{if } a_i = a, \\ x^* & \text{if } a_i = a^*, \\ xx^* & \text{if } a_i = aa^*, \\ x^*x & \text{if } a_i = a^*a, \\ x^2 & \text{if } a_i = 0. \end{cases}$$

Then  $U_1 = U(W_1, \dots, W_n)$  and  $V_1 = V(W_1, \dots, W_n)$  are 1-letter involution semigroup words, and the identity  $U_1(x) = V_1(x)$  holds in  $\mathcal{V}$ , but it still fails in  $B_2$ , namely,  $U_1(a) \neq V_1(a)$ .

Our goal is now to show that one of  $U_1, V_1$  can be assumed to be from  $W_{aa^*}$ , while the other belongs to  $W_0$ . Indeed, one of these words (say,  $U_1$ )

does not belong to  $W_0$ . But then, depending on the  $\theta$ -class of  $U_1$ , transform  $U_1$  to  $U'_1 \in W_{aa^*}$  by defining

$$U'_1 = \begin{cases} U_1 x^* & \text{if } U_1 \in W_a, \\ x U_1 & \text{if } U_1 \in W_{a^*}, \\ U_1 & \text{if } U_1 \in W_{aa^*}, \\ x U_1 x^* & \text{if } U_1 \in W_{a^* a}. \end{cases}$$

By applying this transformation to  $V_1$ , we obtain the word  $V'_1 \notin W_{aa^*}$ . If  $V'_1 \notin W_0$  then depending on the  $\theta$ -class of  $V_1$ , we can similarly transform the identity  $U'_1 = V'_1$  to the identity  $U''_1 = V''_1$ , where  $V''_1 \in W_{aa^*}$ . As  $U'_1 \in W_{aa^*}$ , it is a matter of a direct verification that  $U''_1 \in W_0$ .

In any case,  $\mathcal{V}$  satisfies an identity  $U = V$  such that  $U \in W_{aa^*}$  and  $V \in W_0$ , or, in more detail, an identity which is either of the form

$$(xx^*)^{n_1} = Px^2Q,$$

or of the form

$$(xx^*)^{n_2} = P'(x^*)^2Q',$$

for some involution semigroup words  $P(x, x^*), Q(x, x^*), P'(x, x^*), Q'(x, x^*)$  and  $n_1, n_2 \in \mathbb{N}$ . By applying  $*$  to each of the equations above, we see that  $\mathcal{V}$  must satisfy identities of both kinds just described. But the second one implies  $(x^*x)^{n_2} = P''x^2Q''$ , where  $P'' = P'(x^*, x)$  and  $Q'' = Q'(x^*, x)$ , so that for every  $S \in \mathcal{V}$  and  $a \in S$  we have that  $a^2$  divides both  $(aa^*)^{n_1}$  and  $(a^*a)^{n_2}$ , as required.  $\square$

**Theorem 5.** *Let  $\mathcal{V}$  be an involution semigroup variety. Then the following conditions are equivalent:*

- (i) *any member of  $\mathcal{V}$  can be decomposed (as a semigroup) into a semilattice of Archimedean semigroups;*
- (ii)  *$\mathcal{V}$  does not contain  $B_2$  and  $I_0^*(B_2)$ .*

PROOF. (i) $\Rightarrow$ (ii) This is immediate, as the semigroup reducts of both  $B_2$  and  $I_0^*(B_2)$  are not semilattices of Archimedean semigroups (see, for example, [1, Theorem 1]).

(ii) $\Rightarrow$ (i) *Case 1:*  $\mathcal{V}$  contains  $\mathcal{SL}^0$ . Therefore, for every identity  $U = V$  which holds in  $\mathcal{V}$  we have (by Lemma 1) either  $\pi(U) \neq \emptyset$  and  $\pi(V) \neq \emptyset$ ,

or that  $U = V$  is equivalent to a semigroup identity (some of the variables are replaced by their stars). Now,  $I_0^*(B_2) \notin \mathcal{V}$ , so  $\mathcal{V}$  satisfies an identity which is false in  $I_0^*(B_2)$ . Such an identity clearly cannot be of the first type above. It follows that  $\mathcal{V}$  satisfies a semigroup identity which is not true in  $I_0^*(B_2)$ . Since  $B_2$  has a zero and admits an involution, this semigroup identity is false in  $B_2$  (since  $I_0^*(S)$  satisfies precisely those semigroup identities which are homotypical and, together with their reverses, hold in  $S$ ). Thus, the semigroup reducts of members of  $\mathcal{V}$  generate a (semigroup) variety which cannot contain  $B_2$ . By [1, Corollary 1], this variety (and, *a fortiori*, the class of reducts above) consists entirely of semilattices of Archimedean semigroups.

*Case 2:*  $\mathcal{SL}^0$  is not contained in  $\mathcal{V}$ . Since any identity  $U = V$  such that  $\pi(U) = \pi(V) = \emptyset$  and  $c^*(U) \neq c^*(V)$  implies an identity  $U' = V'$  such that  $\pi(U') = \emptyset$ ,  $\pi(V') \neq \emptyset$  (if, for example,  $x \in c^*(U)$ ,  $x^* \in c^*(V)$ , it suffices to take  $U' = xU$ ,  $V' = xV$ ), by Lemma 1 we have that  $\mathcal{V}$  satisfies an identity of the latter type. Of course (by a suitable substitution), we may further assume that  $U'$  is a semigroup word. By identifying all variables in  $U' = V'$  we obtain an identity which is either of the form  $x^k = W(x, x^*)xx^*Z(x, x^*)$ , or of the form  $x^k = W(x, x^*)x^*xZ(x, x^*)$ . In the first case, for each  $S \in \mathcal{V}$  and any  $a, b \in S$ , we have

$$(ab)^{k+1} = a(ba)^kb = (aW(ba, (ba)^*)b)aa^*(b^*Z(ba, (ba)^*)b),$$

that is,  $aa^* \rightarrow ab$ . In the second case, we similarly obtain  $a^*a \rightarrow ab$ . By Lemmas 3 and 4, these facts imply that every  $S \in \mathcal{V}$  is a semilattice of Archimedean semigroups.  $\square$

Let  $U = V$  be an involution semigroup identity. We call it an  $\mathcal{SA}^*$ -identity if the satisfaction of  $U = V$  in an involution semigroup  $S$  forces  $S$  to be an involution semilattice of Archimedean semigroups (note that semigroup identities with an analogous property for ordinary semigroups were described in [1, Theorem 2]). As a consequence of the above results, we obtain the following

**Corollary 6.** *The following conditions are equivalent for an involution semigroup identity  $U = V$ :*

- (i)  $U = V$  is a  $\mathcal{SA}^*$ -identity;

- (ii)  $U = V$  is not satisfied in  $B_2$  and  $I_0^*(B_2)$ ;
- (iii)  $U = V$  is not satisfied in  $I_0^*(B_2)$ , and there exist one-letter involution semigroup words  $W_1, \dots, W_n$  such that

$$U' = U(W_1, \dots, W_n) = V(W_1, \dots, W_n) = V'$$

fails in  $B_2$  (or equivalently,  $U', V'$  belong to different classes of words defined in the proof of Lemma 4);

- (iv)  $U = V$  is not satisfied in  $I_0^*(B_2)$  and has consequences of the form  $(xx^*)^n = W$ , where  $W$  contains either  $x^2$ , or  $(x^*)^2$  as a subword.

We omit the proof, as it can be easily reconstructed from the material already presented. Namely, (i) $\Rightarrow$ (ii) is immediate; the proof of (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is contained in the proof of Lemma 3, while (iv) $\Rightarrow$ (i) follows from Theorem 5 and the fact that an identity of the form  $(xx^*)^n = W$ , with  $W$  as described above, fails in  $B_2$ .

As an application, we finish the note by explicitly listing all one-letter  $\mathcal{SA}^*$ -identities. Recall that all two-letter semigroup identities implying a semilattice decomposition into Archimedean components were described in Theorem 1 of [2], and the following result is basically its involutorial counterpart.

**Theorem 7.** *An involution semigroup identity in one variable is an  $\mathcal{SA}^*$ -identity if and only if it has one of the following forms:*

- (1)  $U = V$ , where  $U \notin W_0 \cup \{x, x^*\}$  (cf. the proof of Lemma 4) and  $V \in \{x^n : n \geq 1\} \cup \{(x^*)^n : n \geq 1\}$ ;
- (2)  $x = V$ , where  $V$  is not of the form  $(xx^*)^n x$  for any  $n \geq 0$ ;
- (3)  $x^* = V$ , where  $V$  is not of the form  $x^*(xx^*)^n$  for any  $n \geq 0$ .

PROOF. Let  $U(x) = V(x)$  be a one-letter  $\mathcal{SA}^*$ -identity. Then, by the above corollary, there is a one-letter involution semigroup word  $W$  such that  $U' = U(W)$  and  $V' = V(W)$  belong to different classes of  $\theta$  (cf. Lemma 4). Without any loss of generality, let  $U' \notin W_0$ , so that  $U'$  contains neither  $x^2$ , nor  $(x^*)^2$  as a subword. Then  $U$  has the same property. Indeed, if  $U \equiv U_1 x^2 U_2$ , then we must have that  $W$  belongs either to  $W_{aa^*}$ , or to  $W_{a^*a}$ , as  $U' \notin W_0$ . But then either  $U', V' \in W_{aa^*}$ , or  $U', V' \in W_{a^*a}$ , contradicting the above assumption on the words  $U', V'$  (a similar conclusion follows if  $U \equiv U_1 (x^*)^2 U_2$ ). Hence,  $U \notin W_0$ .



Now, if  $U \notin \{x, x^*\}$ , then  $U$  contains occurrences of both  $x$  and  $x^*$ . Thus,  $V$  cannot contain both of  $x, x^*$ , for  $U = V$  must fail in  $I_0^*(B_2)$  (see Lemma 1). It follows that  $U = V$  is just of the form (1). On the other hand, assume  $U \in \{x, x^*\}$ . If  $U \equiv x$ , then, clearly,  $V$  is not of the form  $(xx^*)^n x$ ,  $n \geq 0$ , for otherwise  $U = V$  holds in  $B_2$ . Analogously, if  $U \equiv x^*$ , then  $V$  is not of the form  $x^*(xx^*)^n$ ,  $n \geq 0$ . So,  $U = V$  has one of the forms (2), (3).

Finally, it remains to perform the routine check that all the identities listed in (1)–(3) are indeed false both in  $B_2$  and  $I_0^*(B_2)$ , which confirms them as  $\mathcal{SA}^*$ -identities.  $\square$

ACKNOWLEDGEMENTS. The author is grateful to the anonymous referee for a number of valuable suggestions.

## References

- [1] M. ĆIRIĆ and S. BOGDANOVIĆ, Decompositions of semigroups induced by identities, *Semigroup Forum* **46** (1993), 329–346.
- [2] M. ĆIRIĆ and S. BOGDANOVIĆ, Identities over the two-element alphabet, *Semigroup Forum* **52** (1996), 365–379.
- [3] S. CRVENKOVIĆ, I. DOLINKA and M. VINČIĆ, Equational bases for some 0-direct unions of semigroups, *Studia Sci. Math. Hungarica* **36** (2000), 423–431.
- [4] I. DOLINKA, Remarks on varieties of involution bands, *Comm. Algebra* **28** (2000), 2837–2852.
- [5] M. PETRICH, Introduction to Semigroups, *Merrill, Columbus*, 1973.
- [6] M. S. PUTCHA, Semilattice decompositions of semigroups, *Semigroup Forum* **6** (1973), 12–34.
- [7] M. V. SAPIR and E. V. SUKHANOV, On varieties of periodic semigroups, *Izv. Vysš. Uč. Zav. Mat.* **4** (227) (1981), 48–55 (in Russian).
- [8] L. N. SHEVRIN and E. V. SUKHANOV, Structural aspects of theory of semigroup varieties, *Izv. Vysš. Uč. Zav. Mat.* **6**(325) (1989), 3–39 (in Russian).

IGOR DOLINKA  
DEPARTMENT OF MATHEMATICS AND INFORMATICS  
UNIVERSITY OF NOVI SAD  
TRG DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD  
SERBIA AND MONTENEGRO  
E-mail: dockie@im.ns.ac.yu

(Received September 8, 2003, revised May 27, 2004)