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On some global and local geometric properties of Musielak–Orlicz spaces

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Abstract. It is well known that any SU-point is an exposed point and any LUR-point is a strongly exposed point. Conditions which complete exposed points and strongly exposed points to SU-points and LUR points (respectively) are found. Criteria for SU-points in Musielak–Orlicz spaces with the Luxemburg and the Orlicz norms are given. Moreover, criteria for compact local uniform rotundity of Musielak–Orlicz sequence spaces are established.

1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively. Let $(X, \| \ \|)$ be a real Banach space and X^* be its dual space. By $S(X)$ we denote the unit sphere of X.

A point $x \in S(X)$ is said to be an *extreme point* if for every $y, z \in S(X)$ such that $x = \frac{y+z}{2}$, we have $z = y = x$. We say that $x^* \in X^*$ is a support functional at $x \in X \setminus \{0\}$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. The set of all support functionals at $x \in X \setminus \{0\}$ is denoted by $\mathrm{Grad}(x)$. A point $x \in S(X)$ is said to be an *exposed point* if there exists $x^* \in \mathrm{Grad}(x)$ such that $x^* \notin \mathrm{Grad}(y)$, whenever $y \in S(X)$ and $y \neq x$. It is obvious that every exposed point is an extreme point.

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A point $x \in S(X)$ is said to be a *strongly exposed point* if there exists $x^* \in \mathrm{Grad}(x)$ such that for every sequence $(x_n) \subset S(X)$ the condition $x^*(x_n - x) \to 0$ implies $\|x_n - x\| \to 0$.

A point $x \in S(X)$ is called a *strong U-point* (SU-point for short) if for any $y \in S(X)$ with $\|x + y\| = 2$, we have $x = y$.

A point $x \in S(X)$ is said to be a point of *compact local uniform rotundity* (*local uniform rotundity*) (CLUR-point, (LUR-point) for short) if for every $(x_n)_{n=1}^{\infty}$ in $S(X)$ such that $\|x_n + x\| \to 2$ we have that (x_n) is a relatively compact set in S(X) (resp. $\|x_n - x\| \to 0$). If every $x \in S(X)$ is a CLUR-point, then we say that X is *compactly locally uniformly rotund* ($X \in$ (CLUR) for short). If every $x \in S(X)$ is a LUR-point, then we say that X is *locally uniformly rotund* ($X \in (LUR)$ for short). The importance of the notion of SU-point follows from the fact that $x \in S(X)$ is a LUR-point if and only if x is a CLUR-point and an SU-point as well as that "SU-point" cannot be replaced in this equivalence by "extreme point" (see [3]). Moreover, a point $x \in S(X)$ is an SU-point if and only if any $x^* \in \mathrm{Grad}(x)$ exposes x, that is, if $y \in S(X)$ and $x^*(y) = 1$, then $x = y$ (see [10], Remark 2). In consequence any SU-point of $S(X)$ is an exposed point. It is also known that in Köthe function (or sequence) spaces, if $x \in S(X)$ is an SU-point, then $|x|$ is a point of upper monotonicity as well as a point of lower monotonicity (see [10], Lemma 7). For example, the point $(1,1)$ is an extreme point of $S(l_2^{\infty})$, but it is not a point of lower monotonicity because $\|(1,0)\|_{\infty} = \|(1,1)\|_{\infty} = 1$ and $(1,0) \leq (1,1)$, $(1,0) \neq (1,1)$.

A Banach space X is said to have the Kadec–Klee property (X has the H-property) or $X \in (H)$ for short if the weak convergence and the convergence in norm of sequences in $S(X)$ to a limit element from $S(X)$ coincide. It is known that the CLUR property implies the H-property.

A sequence $\Phi = (\Phi_i)_{i=1}^{\infty}$ is called *a Musielak–Orlicz function*, if for any $i \in \mathbb{N}$, the function $\Phi_i : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is convex, even, vanishing at 0, $\lim_{u\to\infty} \Phi_i(u) = \infty$ and $\Phi_i(u_i) < \infty$ for some $u_i \in (0, \infty)$. The function $\Psi = (\Psi_i)_{i=1}^{\infty}$ with $\Phi_i(v) = \sup_{u\geq 0}\{u|v| - \Phi_i(u)\}$ for any $i \in \mathbb{N}$ and $v \in \mathbb{R}$ is called the function *complementary* to Φ in the sense of Young. The function Ψ is again a Musielak–Orlicz function.

Denote by l_0 the space of all real sequences and define $e_n = \chi_{\{n\}}$ for any $n \in \mathbb{N}$. Given a Musielak–Orlicz function Φ, we define the *Musielak–Orlicz sequence space* l_Φ, by

$$l_\Phi = \{x = (x_i)_{i=1}^{\infty} \in l_0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi_i(x_i).$$

This space equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}$$

or with the equivalent one

$$\|x\|_\Phi^o = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

called the Orlicz (or the Amemiya) norm is a Banach space. The spaces $(l_\Phi, \| \ \|_\Phi)$ and $(l_\Phi, \| \ \|_\Phi^o)$ will be denoted shortly by l_Φ and l_Φ°, respectively. Moreover, we define the space

$$\Big\{x = (x_i)_{i=1}^{\infty} \in l_0 : \forall \lambda > 0 \, \exists i_\lambda \in \mathbb{N} \text{ s.t. } \sum_{i=i_\lambda}^{\infty} \Phi_i(\lambda |x(i)|) < \infty\Big\},$$

and it will be denoted shortly by h_Φ and h_Φ°, respectively. For $x \in l_\Phi$ we denote $\theta(x) = \inf\big\{\lambda > 0 : \sum_{i=i_\lambda}^{\infty} \Phi_i(\lambda|x(i)|) < \infty$ for some $i_\lambda \in \mathbb{N}$ depending on $\lambda\big\}$.

Let $p_i^-(u)$, $p_i^+(u)$ ($q_i^-(u)$, $q_i^+(u)$ respectively) denote the left and right derivatives of Φ_i (Ψ_i respectively) at u. We have the Young inequality

$$uv \leq \Phi_i(u) + \Psi_i(v)$$

for all $u, v \geq 0$; and

$$uv = \Phi_i(u) + \Psi_i(v) \Longleftrightarrow p_i^-(u) \leq v \leq p_i^+(u) \quad \text{or} \quad q_i^-(v) \leq u \leq q_i^+(v).$$

For any $x \in l_\Phi^\circ$, we define

$$k^*(x) = \inf\{k \geq 0 : I_\Psi(p^+(k|x|)) \geq 1\};$$

$$k^{**}(x) = \sup\{k \geq 0 : I_\Psi(p^+(k|x|)) \leq 1\};$$

$$K(x) = \begin{cases} [k^*(x), k^{**}(x)], & \text{if } k^{**}(x) < \infty; \\ [k^*(x), \infty) & \text{if } k^*(x) < \infty \text{ and } k^{**}(x) = \infty. \end{cases}$$

We usually will write k^*, k^{**} instead of $k^*(x)$, $k^{**}(x)$ if it is clear which x is considered. It is known that $\|x\|_\Phi^o = \frac{1}{k}(1 + I_\Phi(kx))$ if and only if $k \in K(x)$ for any $x \in l_\Phi$ (see [21]).

The dual space of l_Φ is well known. Namely, we have

$$(l_\Phi)^* = l_\Psi \oplus S,$$

that is, any $x^* \in (l_\Phi)^*$ is uniquely represented in the form

$$x^* = \xi_v + \phi,$$

where $v \in l_\Psi$ and ξ_v is the order continuous functional on l_Φ generated by v, that is,

$$\xi_v(x) = \sum_{i=1}^{\infty} v_i x_i \quad (\forall\, x \in l_\Phi)$$

and $\phi \in S$ is a linear singular functional on l_Φ, that is, $\phi(x) = 0$ for any $x \in h_\Phi$ (order continuous functionals are also called regular functionals). We denote by $R\,\mathrm{Grad}(x)$, ($S\,\mathrm{Grad}(y)$, respectively) the set of all regular (singular, respectively) support functionals at x.

Let $\mathcal{N}_0$ be a subset of $\mathbb{N}$. We say that a Musielak–Orlicz function Φ satisfies the $\delta_2(\mathcal{N}_0)$ condition ($\Phi \in \delta_2(\mathcal{N}_0)$ for short) if for any $h > 1$ (equivalently for some $h > 1$) there exists $a > 0$, $k > 1$, $i_0 \in \mathcal{N}_0$ and a positive sequence (c_i) ($i \in \mathcal{N}_0$, $i > i_0$) such that $\sum_{i \in \mathcal{N}_0,\, i > i_0} c_i < \infty$ and the inequality

$$\Phi_i(hu) \le k\Phi_i(u) + c_i$$

holds whenever $i \in \mathcal{N}_0$, $i > i_0$ and $\Phi_i(u) \le a$. If $\Phi \in \delta_2(\mathbb{N})$, then we write simply $\Phi \in \delta_2$.

For the theories of Orlicz and Musielak–Orlicz spaces and the notions concerning l_Φ defined above we refer to [2], [18], [17], [21], [15], [19] and [16].

We say that a function $\Phi : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is *upper (lower) affine* at $w \in \mathbb{R}$ if there exists $v \in \mathbb{R}$ with $|v| > |w|$ ($|v| < |w|$) such that Φ is affine on the intervals $\big[|w|, |v|\big]$ and $\big[-|v|, -|w|\big]$ (respectively on the intervals $\big[|v|, |w|\big]$ and $\big[-|w|, -|v|\big]$). We introduce the following notations:

$$A_u^i = \{z \in \mathbb{R} : \Phi_i \text{ is upper affine at } z\},$$

$$A_l^i = \{z \in \mathbb{R} : \Phi_i \text{ is lower affine at } z\}$$

$$AS_u^i = \{z \in A_u^i : p_i^-(z) = p_i^+(z)\}, \quad AS_l^i = \{z \in A_l^i : p_i^-(z) = p_i^+(z)\}$$

$$ANS_u^i = A_u^i \setminus AS_u^i, \qquad ANS_l^i = A_l^i \setminus AS_l^i$$

$$A_u(x) = \{i \in \mathbb{N} : x_i \in A_u^i\}, \qquad A_l(x) = \{i \in \mathbb{N} : x_i \in A_l^i\},$$

$$A_u^s(x) = \{i \in A_u(x) : x_i \in AS_u^i\}, \qquad A_l^s(x) = \{i \in A_l(x) : x_i \in AS_l^i\},$$

$$A_u^{ns}(x) = A_u(x) \setminus A_u^s(x), \qquad A_l^{ns}(x) = A_l(x) \setminus A_l^s(x).$$

Moreover, for every $i \in \mathbb{N}$ we denote $\mathrm{Smooth}(\Phi_i) = \{u \in \mathbb{R} : p_i^+(u) = p_i^-(u)\}$, $a(\Phi_i) = \sup\{u \geq 0 : \Phi_i(u) = 0\}$, $b(\Phi_i) = \sup\{u \geq 0 : \Phi_i(u) < \infty\}$, $\mathrm{Ext}(\Phi_i) = \{u \in \mathbb{R} : \Phi_i \text{ is strictly convex at } u\}$, $\partial\Phi_i(u) = [p_i^-(u),\ p_i^+(u)]$. If $a(\Phi_i) = 0$ ($b(\Phi_i) = +\infty$, respectively) for all $i \in \mathbb{N}$, then we will write $\Phi > 0$ ($\Phi < \infty$, respectively).

We will use a few known results that we present below.

Theorem 1.1 ([3]). *Let X be a Banach space and $x \in S(X)$. If x is an SU-point, then x is an exposed point.*

Theorem 1.2 ([1]). *A point $x \in S(l_\Phi)$ is exposed if and only if:*

(i) $I_\Phi(x) = 1$,

(ii) $\mathrm{card}(\{i \in \mathbb{N} : a(\Phi_i) > 0,\ x_i \in \mathrm{Smooth}(\Phi_i)\}) = 0$,

(iii) $\mathrm{card}(\{i \in \mathbb{N} : x_i \notin \mathrm{Ext}(\Phi_i)\}) \leq 1$,

(iv) $(p_i^-(|x_i|))_{i\in\mathbb{N}} \in l_\Psi$,

(v) *there are no pair $i, j \in \mathbb{N}$ with $i \neq j$ such that $x_i \in AS_u^i$ and $x_j \in AS_l^j$.*

Theorem 1.3 ([6]). *A Musielak–Orlicz space l_Φ° has property H if and only if $\Phi \in \delta_2$ or $\sum_{i=1}^\infty \Psi_i(c_i) \leq 1$,where*

$$c_i = \begin{cases} b(\Phi_i), & \textit{if } \Psi_i(b(\Phi_i)) < 1; \\ \Psi_i^{-1}(1), & \textit{otherwise.} \end{cases}$$

Theorem 1.4 ([4]). *Let X be a reflexive Banach space. Then both X and X^* have the CLUR-property if and only if the both X and X^* have the H-property.*

Lemma 1.1 (see [20], [14]). *Let $\Phi < \infty$, $x \in S(l_\Phi)$ and $I_\Phi(x) = 1$. Then $x^* \in R\,\mathrm{Grad}(x)$ if and only if it is of the form*

$$x^*(y) = \sum_{i=1}^{\infty} \eta_i y_i \Big/ \sum_{i=1}^{\infty} \eta_i x_i \quad (y \in l_\Phi),$$

where $\eta = (\eta_i) \in l_\Psi$ is such that $\eta_i \in \partial\Phi_i(x_i) = [p_i^-(|x_i|), p_i^+(|x_i|)]$ for any $i \in \mathbb{N}$.

Lemma 1.2 ([20]). *If $\Psi \notin \delta_2$, then there exists a sequence $(I_n) \subset \mathbb{N}$ $(0 = I_0 < I_1 < I_2 < \dots)$ and a family of sequences (u_i^n) in $\mathbb{R}_+$ $(n = 1, 2 \dots;$ $i = I_{n-1} + 1, \dots, I_n)$ such that*

$$\sum_{i=I_{n-1}+1}^{I_n} \Phi_i(u_i^n) > 1 \quad \text{for any } n \in \mathbb{N},$$

$$\Phi_i(u_i^n) \le \frac{1}{n}, \quad \Phi_i\left(\frac{u_i^n}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{\Phi_i(u_i^n)}{2}$$

for any $n \in \mathbb{N}$ and any $i \in \{I_{n-1} + 1, \dots, I_n\}$.

Theorem 1.5 ([12]). *Let $x \in S(l_\Phi^\circ)$ and $k \in K(x)$. Then*

(i) *if $k^* < k^{**}$, then there exists exactly one support functional at x and it is generated by $w\,\mathrm{sgn}(x)$, where $w_i = p_i^+(k^* x_i) = p_i^-(k^{**} x_i)$ for any $i \in \mathbb{N}$;*

(ii) *if $K(x) = \{k\}$, then*

  (a) *$I_\Psi(p^-(kx)) \le 1$ and $\theta(kx) \le 1$;*

  (b) *if $\theta(kx) < 1$, then $I_\Psi(p^+(kx)) \ge 1$. Moreover, each support functional at x belongs to l_Ψ and it is of the form : $w\,\mathrm{sgn}(x)$, where $p_i^-(kx_i) \le w_i \le p_i^+(kx_i)$ for any $i \in \mathbb{N}$ and $I_\Psi(w) = 1$;*

  (c) *if $\theta(kx) = 1$, then each support functional at x is of the form $w\,\mathrm{sgn}(x) + \phi$, where $p_i^-(k|x_i|) \le w_i \le p_i^+(k|x_i|)$ for any $i \in \mathbb{N}$, $I_\Psi(w) + a = 1$, ϕ is a singular functional, $\phi(x) = a/k$ and $a \in [0, 1 - I_\Psi(p^-(kx))]$.*

2. Results

We start with two auxiliary lemmas and two remarks.

Lemma 2.1. *In the Musielak–Orlicz space h_Φ, the condition $I_\Phi(x)=1$ is equivalent to the condition $\|x\|_\Phi = 1$ if and only if $\Phi_i(b(\Phi_i)) \geq 1$ for any $i \in \mathbb{N}$.*

Proof. *Necessity.* Suppose that there exists $i_0 \in \mathbb{N}$ such that $\Phi_{i_0}(b(\Phi_{i_0})) < 1$. Defining $x = b(\Phi_{i_0})e_{i_0}$ we have $I_\Phi(x) < 1$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$, which implies $\|x\|_\Phi = 1$.

Sufficiency. It suffices to show that $\|x\|_\Phi = 1$ implies $I_\Phi(x) = 1$ when $\Phi_i(b(\Phi_i)) \geq 1$ for any $i \in \mathbb{N}$. Assume that there exists $x \in h_\Phi$ such that $I_\Phi(x) < 1$. The function $f(\lambda) = I_\Phi(\lambda x)$ is convex. It follows from the definition of the space h_Φ that there exists $i_0 \in \mathbb{N}$ such that $\sum_{i=i_0}^{\infty} \Phi_i(2x_i) < \infty$. Moreover, there exists $\lambda_0 \in (1, 2)$ such that $\sum_{i=1}^{i_0-1} \Phi_i(\lambda_0 x_i) < 1$, because of $I_\Phi(x) < 1$. Therefore, the function f is real-valued in the interval $(0, \lambda_0)$, so it is continuous on this interval, by its convexity. Since $f(1) = I_\Phi(x) < 1$ and f is continuous in a right-neighbourhood of 1, there exists $\lambda_1 > 1$ such that $I_\Phi(\lambda_1 x) < 1$, whence $\|\lambda_1 x\|_\Phi \leq 1$, which gives $\|x\|_\Phi \leq \frac{1}{\lambda_1} < 1$. This contradiction finishes the proof. □

Using similar techniques as in [11], we get

Lemma 2.2. *Assume that $\Phi = (\Phi_i)$ is a Musielak–Orlicz function such that for any $i \in \mathbb{N}$ there exists $u_i \in \mathbb{R}$ satisfying $\Phi_i(u_i) = 1$. Let $\Phi \in \delta_2$, $x \in l_\Phi$, $(x_n) \subset l_\Phi$, $x_n \to x$ coordinatewise and $\|x_n\|_\Phi \to \|x\|_\Phi$. Then $\|x_n - x\|_\Phi \to 0$.*

Remark 2.1. *For a Banach space X and $x \in S(X)$ define the following condition*

$$(A) \qquad \text{If } \ y \in S(X) \text{ and } \|x + y\| = 2, \text{ then } \operatorname{Grad}(x) = \operatorname{Grad}(y).$$

Then the point x is an SU-point if and only if x is an exposed point and condition (A) holds.

Proof. $\Leftarrow$) Let $x, y \in S(X)$ and $\|x + y\| = 2$. Then condition (A) gives $\operatorname{Grad}(y) = \operatorname{Grad}(x)$. Using the fact that x is an exposed point we get $x = y$.

$\Rightarrow$) First we will show that any SU-point in $S(X)$ is an exposed point. Let $y \in S(X)$ and $x^*(x) = x^*(y) = 1$ for $x^* \in S(X^*)$. Then $x^*(x+y) = 2$. Therefore

$$2 = \|x^*\| \|x+y\| \geq |x^*(x+y)| = 2,$$

whence $\|x+y\| = 2$. By the assumption that x is an SU-point we get that $x = y$, which means x is an exposed point. Now assume that $x \in S(X)$ is an SU-point, $y \in S(X)$, $\|x+y\| = 2$. Then $x = y$ and so $\mathrm{Grad}(y) = \mathrm{Grad}(x)$. Therefore we have obtained condition (A). $\square$

Remark 2.2. *For a Banach space X and $x \in S(X)$ define the following condition*

(B) *If $(x_n) \subset S(X)$, $x^* \in \mathrm{Grad}(x)$ and $\|x + x_n\| = 2$, then $x^*(x_n) \to 1$.*

Then x is a LUR-point if and only if x is a strongly exposed point and condition (B) holds.

Proof. $\Rightarrow$) It is well known that a LUR-point is a strongly exposed point. Let x be a LUR-point and $\|x + x_n\| \to 2$. Then $\|x - x_n\| \to 0$, which implies that $x_n \xrightarrow{w} x$, i.e. $x^*(x_n) \to x^*(x) = 1$. Therefore we have obtained condition (B).

$\Leftarrow$) Let $\|x_n + x\| \to 2$ and $x^* \in S(X^*)$ be a functional which exposes x strongly. Then $x^* \in \mathrm{Grad}(x)$ and by condition (B) we get $x^*(x_n) \to 1 = x^*(x)$, i.e. $x^*(x_n - x) \to 0$ for any $x^* \in \mathrm{Grad}(x)$. Since x^* strongly exposes x, so $\|x_n - x\| \to 0$. $\square$

Theorem 2.1. *Let $\Phi < \infty$ and $x \in S(l_\Phi)$. Then x is an SU-point if and only if:*

(i) $I_\Phi(x) = 1$,

(ii) $\mathrm{card}(\{i \in \mathbb{N} : a(\Phi_i) > 0,\, x_i \in \mathrm{Smooth}(\Phi_i)\}) = 0$,

(iii) $\mathrm{card}(\{i \in \mathbb{N} : x_i \notin \mathrm{Ext}(\Phi_i)\}) \leq 1$,

(iv) *there are no $i, j \in \mathbb{N}$, $i \neq j$, such that $x_i \in A_u^i$ and $x_j \in A_l^j$,*

(v) $\theta(x) < 1$,

Proof. *Necessity.* Assume, without loss of generality that $x \geq 0$ and x is an SU-point of $S(l_\Phi)$. By Theorem 1.1 and Theorem 1.2 we only need to show the necessity of conditions (v) and (iv). Suppose $\theta(x) \geq 1$. Since

$x \in S(l_\Phi)$ and $\theta(x) \leq \|x\|_\Phi$, we have $\theta(x) = 1$. Then $I_\Phi(\lambda x) = +\infty$ for any $\lambda > 1$. Let $i_0 \in \mathbb{N}$ be such that $x_{i_0} \neq 0$ and let $y = \{y_i\}$, where

$$y_i = \begin{cases} x_i, & i \neq i_0; \\ 0, & i = i_0. \end{cases}$$

Then $I_\Phi(y) \leq 1$, whence $\|y\|_\Phi \leq 1$, and $I_\Phi(\lambda y) = +\infty$ for any $\lambda > 1$, which implies $\|y\|_\Phi \geq 1$. Consequently $\|y\|_\Phi = 1$. Therefore $\|x+y\|_\Phi \leq 2$. Moreover, $|y| \leq |x|$ and $|y| \neq |x|$, which implies $2 = 2\|y\|_\Phi \leq \|x+y\|_\Phi$, whence we get $\|x+y\|_\Phi = 2$. This contradicts the assumption that x is an SU-point and gives the necessity of condition (v). Now we will show that condition (iv) is necessary. By Theorem 1.2 we need to consider only two cases: I) there exist $i, j \in \mathbb{N}$, $i \neq j$, such that $x_i \in AS_u^i$ and $x_j \in ANS_l^j$; II) there exist $i, j \in \mathbb{N}$, $i \neq j$, such that $x_i \in ANS_u^i$ and $x_j \in ANS_l^j$. We will prove only case I), because the proof of the second one is nearly the same. Assume that there exist $i, j \in \mathbb{N}$, $i \neq j$, such that $x_i \in AS_u^i$ and $x_j \in ANS_l^j$. We choose $a, b \in \mathbb{R}_+$ such that

$$\Phi_i(x_i) + \Phi_j(x_j) = \Phi_i(x_i + a) + \Phi_j(x_j - b)$$

and define

$$y = x\chi_{\mathbb{N}\setminus\{i,j\}} + (x_i + a)e_i + (x_j - b)e_j.$$

Then $I_\Phi(y) = I_\Phi(x) = 1$, by condition (i). Let $\eta_i = p_i^+(x_i) = p_i(x_i)$, $\eta_j = p_j^-(x_j)$ and $\eta_n = p_n^-(x_n)$ for $n \notin \{i, j\}$. Then, by Lemma 1.1, the support functional x^* generated by η belongs to $\mathrm{Grad}(x) \cap \mathrm{Grad}(y)$. So

$$2 = x^*(x+y) \leq \|x+y\|_\Phi \leq 2,$$

whence $\|x+y\|_\Phi = 2$. This contradicts the assumption that x is an SU-point. In the same way we consider the case: $x_i \in ANS_u^i$ and $x_j \in ANS_l^j$ for some natural $i \neq j$.

Sufficiency. Assume that conditions (i)–(v) hold and $\|x\|_\Phi = \|y\|_\Phi = \|\frac{x+y}{2}\|_\Phi = 1$. In the same way as in the proof of Theorem 5 in [3] we show that $I_\Phi(y) = I_\Phi(\frac{x+y}{2}) = 1$ and the coordinates x_i, y_i belong to the same interval of the affinity of the functions Φ_i, for any $i \in \mathbb{N}$. If there exists $i_0 \notin \mathrm{Ext}(\Phi_i)$ (condition (iii)), then condition (ii) yields $\Phi_{i_0}(x_{i_0}) \neq 0$. Since x_{i_0} and y_{i_0} belong to the same interval of affinity of the function Φ_{i_0} and

$I_\Phi(x) = I_\Phi(y) = 1$, so we have $x_{i_0} = y_{i_0}$. If $x_i \in \mathrm{Ext}(\Phi_i) \cap A_u^i$ for some $i \in \mathbb{N}$, then, by condition (iv), $x_j \in A_u^j$ for any natural $j \neq i$. Therefore $x_i \leq y_i$ for any $i \in \mathbb{N}$. Now condition (i) implies that $x_i = y_i$ for any $i \in \mathbb{N}$, because of $I_\Phi(y) = 1$. Similarly we consider the case $x_i \in \mathrm{Ext}(\Phi_i) \cap A_l^i$ for some $i \in \mathbb{N}$. □

Theorem 2.2. *Let $x \in S(L_\Phi)$ and $b(\Phi(t,\cdot)) = +\infty$ for μ-almost every $t \in T$. Then x is SU-point if and only if:*

(i) $I_\Phi(x) = 1$,

(ii) $\mu(\{t \in T : a(\Phi(t,\cdot)) > 0,\ x(t) \in \mathrm{Smooth}(\Phi(t,\cdot))\}) = 0$,

(iii) $\mu(\{t \in T : x(t) \notin \mathrm{Ext}(\Phi(t,\cdot))\}) = 0$,

(iv) $\mu(A_l(x))\mu(A_u(x)) = 0$,

(v) $\theta(x) < 1$.

Proof. The proof proceeds in the same way as the proof of Theorem 2.1, using Theorem 3 in [1] (which characterizes exposed points in L_Φ) instead of Theorem 1.2. □

The next theorem is an analogue of Theorem 2.1, but it concernes the Orlicz norm.

Theorem 2.3. *A point $x \in S(l_\Phi^\circ)$ is an SU-point if and only if the following conditions hold:*

(i) *$K(x) = \{k\}$ for some $0 < k < \infty$,*

(ii) *$A_l^s(kx) = A_u^s(kx) = \emptyset$,*

(iii) *if $A_l^{ns}(kx) \neq \emptyset$, then $\theta(kx) < 1$ and $I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i\}}) + \Psi_i(p_i^-(kx_i)) < 1$ for any $i \in A_l^{ns}(kx)$.*

(iv) *if $A_u^{ns}(kx) \neq \emptyset$, then $\theta(kx) < 1$ and $I_\Psi(p^-(kx)\chi_{\mathbb{N}\setminus\{i\}}) + \Psi_i(p_i^+(kx_i)) > 1$ for any $i \in A_u^{ns}(kx)$.*

Proof. *Necessity.* Condition (i) is necessary in order that x is an extreme point (see [13] and [5]). Therefore, it is also necessary in order that x is an SU-point.

(ii) Assume for simplicity (but without loss of generality) that $x \geq 0$. Since condition (i) is necessary, we can assume in the remaining part of the proof that $K(x) = \{k\}$. Suppose that there exists $i_0 \in A_l^s(kx)$. Then the function Φ_{i_0} is affine in the interval $[kx_{i_0} - b, kx_{i_0}]$ for some $b > 0$. We will consider two cases.

1° Let $\theta(kx) < 1$. Define

$$y = x\chi_{\mathbb{N}\setminus\{i_0\}} + \left(x_{i_0} - \operatorname{sgn}(x_{i_0})\frac{b}{k}\right)e_{i_0}. \tag{1}$$

Then $p_i^+(ky_i) = p_i^+(kx_i)$ for any $i \in \mathbb{N}$. Hence $I_\Psi(p^+(ky)) = I_\Psi(p^+(kx))$. Moreover, for $\alpha \neq k$, we have $I_\Psi(p^+(\alpha y)) \le I_\Psi(p^+(\alpha x))$, which implies $k = k^*(x) \le k^*(y)$. Since $0 \le y \le x$, we have $\theta(ky) \le \theta(kx) < 1$. If $I_\Psi(p^+(kx)) < 1$, then by Theorem 1.5(ii)(b), we have that $\operatorname{Grad}(x) = \emptyset$, which contradicts the Hahn–Banach theorem. Hence $I_\Psi(p^+(kx)) \ge 1$.

If $I_\Psi(p^+(kx)) > 1$, then $I_\Psi(p^+(ky)) > 1$, whence $k^*(y) \le k$ and we get $k^*(y) = k$, by $k^*(y) \ge k$. Consequently $K(y) = \{k\}$, because in the opposite case $I_\Psi(p^+(kx)) = 1$, by Theorem 1.5(i) and the Young inequality. Since $k \in K(y)$ if and only if $k\|y\|_\Phi^o \in K\left(y/\|y\|_\Phi^o\right)$, so $K\left(y/\|y\|_\Phi^o\right) = \{k\|y\|_\Phi^o\}$. Moreover

$$p_i^-\left(k\|y\|_\Phi^o \frac{y_i}{\|y\|_\Phi^o}\right) = p_i^-(ky_i) = p_i^-(kx_i),$$

$$p_i^+\left(k\|y\|_\Phi^o \frac{y_i}{\|y\|_\Phi^o}\right) = p_i^+(ky_i) = p_i^+(kx_i)$$

for any $i \in \mathbb{N}$, so by Theorem 1.5(ii)(b), $\operatorname{Grad}\left(y/\|y\|_\Phi^o\right) = \operatorname{Grad}(x)$.

If $I_\Psi(p^+(kx)) = 1$, then $I_\Psi(p^+(ky)) = 1$ and $k \in K(y)$, so $k^*(y) = k$, by $k^*(y) \ge k$, which has been already proved. If $k = k^*(y) < k^{**}(y)$ then, by Theorem 1.5(i), $p^+(kx)$ generates a support functional x^* belonging to $\operatorname{Grad}(x) \cap \operatorname{Grad}\left(\frac{y}{\|y\|_\Phi^o}\right)$. If $k = k^*(y) = k^{**}(y)$ then, by Theorem 1.5(ii)(b), $p^+(kx)$ generates a support functional x^* that belongs to $\operatorname{Grad}(x) \cap \operatorname{Grad}(y/\|y\|_\Phi^o)$. For such a functional we have

$$2 = x^*\left(x + \frac{y}{\|y\|_\Phi^o}\right) \le \left\|x + \frac{y}{\|y\|_\Phi^o}\right\|_\Phi^\circ \le 2,$$

whence $\|x + \frac{y}{\|y\|_\Phi^o}\|_\Phi^o = 2$, which means that x is not an SU-point. This proves the necessity of the condition $A_l^s(kx) = \emptyset$ for k satisfying $K(k) = \{k\}$ whenever $\theta(kx) < 1$.

2° Let $\theta(kx) = 1$. Define y as in (1). Then $\theta(ky) = 1$, because in the opposite case there exists $l > k$ such that $I_\Phi(ly) < \infty$. On the other hand, by $I_\Phi(lx) = \infty$, we have

$$I_\Phi(ly) = I_\Phi(lx\chi_{\mathbb{N}\setminus\{i_0\}}) + \Phi_{i_0}(ly_{i_0}) = I_\Phi(lx) + \Phi_{i_0}(ly_{i_0}) - \Phi_{i_0}(lx_{i_0}) = +\infty,$$

which gives a contradiction. By Theorem 1.5(ii)(c), the case $I_\Psi(p^+(kx))\geq 1$ can be considered as the case 1°. So let $I_\Psi(p^+(kx))<1$. Then $I_\Psi(p^+(ky))<1$. Since $\theta(ky) = 1$, so $I_\Phi(\alpha y) = \infty$ for any $\alpha > k$, whence $k^{**}(y) = k$ and $K(y) = \{k\}$, by $k^*(y) = k$. It follows from Theorem 1.5(ii)(c) that any $y^* \in \operatorname{Grad}\left(\frac{y}{\|y\|^o_\Phi}\right)$ is generated by the triple (v, ϕ, a), where $v \in l_\Psi$, ϕ is a singular functional, $a \geq 0$, $I_\Psi(v) + a = 1$, $\phi\left(y/\|y\|^o_\Phi\right) = a/(k\|y\|^o_\Phi)$, $a \in [0, 1 - I_\Psi(p^-(ky))]$ and

$$p_i^-(kx_i) = p_i^-\left(k\|y\|^o_\Phi \frac{y_i}{\|y\|^o_\Phi}\right) \leq v_i \leq p_i^+\left(k\|y\|^o_\Phi \frac{y_i}{\|y\|^o_\Phi}\right) = p_i^+(kx_i)$$

for any $i \in \mathbb{N}$. Moreover

$$\sum_{i=1}^\infty \frac{y_i}{\|y\|^o_\Phi} \operatorname{sgn}(y_i) v_i = 1 - \frac{a}{k\|y\|^o_\Phi},$$

because $I_\Psi(v) = 1 - a$, $I_\Phi(ky) = k\|y\|^o_\Phi - 1$ and

$$k\|y\|^o_\Phi \sum_{i=1}^\infty \frac{y}{\|y\|^o_\Phi} \operatorname{sgn}(y_i) v_i = I_\Phi(ky) + I_\Psi(v).$$

Similarly

$$k \sum_{i=1}^\infty x_i \operatorname{sgn}(x_i) v_i = I_\Phi(kx) + I_\Psi(v) = k - 1 + 1 - a = k - a.$$

Hence $\sum_{i=1}^\infty x_i \operatorname{sgn}(x_i) v_i = 1 - \frac{a}{k}$. Moreover $x - y \in h^\Phi$, so $\phi(y) = \phi(x) = \frac{a}{k}$. It means that

$$\operatorname{Grad}\left(\frac{y}{\|y\|^o_\Phi}\right) \subset \operatorname{Grad}(x).$$

Now we can proceed as in the case 1° of the proof, obtaining that x is not an SU-point.

In the same way we can show that $A^s_u(kx) = \emptyset$.

(iii) By the previous part of the proof we can assume that conditions (i)–(ii) hold. Consider first the case $\theta(kx) = 1$. Assume that there exists $i_0 \in A_l^{ns}(kx)$. Then the function Φ_{i_0} is affine on the interval $[kx_{i_0} - b, kx_{i_0}]$ for some $b > 0$. Define y as in (1). Then, similarly as in the proof of condition the necessity of (ii), we have $\theta(ky) = 1$. First we will show that

$$I_\Psi(p^-(kx)) < 1. \tag{2}$$

Suppose for the contrary that $I_\Psi(p^-(kx)) \geq 1$. Since $K(x) = \{k\}$, it follows from Theorem 1.5(ii)(a) that $I_\Psi(p^-(kx)) \leq 1$, whence

$$I_\Psi(p^-(kx)) = 1. \tag{3}$$

Consequently, since $i_0 \in A_l^{ns}(kx)$, we get

$$I_\Psi(p^+(kx)) > 1. \tag{4}$$

Since $K(x) = \{k\}$, we get $I_\Psi(p^+(\alpha x)) < 1$ for any $\alpha < k$, because in the opposite case $k^*(x) \leq \alpha < k$. So by $i_0 \in A_l^{ns}(kx)$ and $y \leq x$, we have

$$I_\Psi(p^+(\alpha y)) < 1 \quad \text{for any } \alpha < k. \tag{5}$$

If $I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i_0\}}) + \Psi_{i_0}(p_-^{i_0}(kx_{i_0})) < 1$, then $I_\Psi(p^+(ky)) < 1$, which contradicts the equality $I_\Psi(p^-(ky)) = 1$, being a consequence of (3). So we should consider only two subcases.

3° Let$I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i_0\}}) + \Psi_{i_0}(p_{i_0}^-(kx_{i_0})) > 1$. Then $I_\Psi(p^+(ky)) > 1$, whence $k^*(y) \leq k$ and, by (5), $k^*(y) = k$. If $k = k^* < k^{**}$, then it follows from Theorem 1.5(i) and from the Young inequality that $I_\Psi(p^+(ky)) = 1$, which contradicts the fact that $I_\Psi(p^+(ky)) > 1$, whence $K(y) = \{k\}$. Therefore $K\left(y/\|y\|_\Phi^o\right) = \{k\|y\|_\Phi^o\}$. Moreover $I_\Psi(p^-(ky)) = I_\Psi(p^-(kx)) = 1$, by (3). By Theorem 1.5(ii)(c) we have that the support functional x^*, generated by $p^-(kx)$ belongs to $\mathrm{Grad}(x) = \mathrm{Grad}\left(y/\|y\|_\Phi^o\right)$. Proceeding as in the case 1° of the proof, we get that x is not an SU-point.

4° Let $I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i_0\}}) + \Psi_{i_0}(p_-^{i_0}(kx_{i_0})) = 1$. Note that in this case $\mathrm{supp}(x) \neq A_l^{ns}(kx)$, because in the opposite case we have $I_\Psi(p^-(kx)) < 1$, which contradicts condition (3). So, in particular, we have $\mathrm{supp}(x) \neq \{i_0\}$. Moreover $I_\Psi(p^+(ky)) = 1$, whence $k \in K(y)$ and $I_\Psi(p^+(ky)) =$

$I_\Psi(p^-(ky)) = 1$, by (3) and the fact that $I_\Psi(p^-(ky)) = I_\Psi(p^-(kx))$. By the assumption that $\Phi_i(u)/u \to 0$ as $u \to 0$ for any $i \in \mathbb{N}$, we get

$$p_i^-(ky_i) = p_i^+(ky_i) \quad \text{for any } i \neq i_0,\ i \in \mathbb{N}. \tag{6}$$

Condition (5) and $I_\Psi(p^+(ky)) = 1$ imply that $k^*(y) = k$. By (6) we have $I_\Psi(p^+(\alpha y)) > I_\Psi(p^+(ky)) = 1$ for any $\alpha > k$. Hence $I_\Psi(p^+(ky)) = 1$ implies that $k^{**}(y) = k$. Finally $K(y) = \{k\}$. Proceeding as in the case 3° of the proof, we get that x is not an SU-point.

Therefore, we have proved that if $\theta(kx) = 1$, then condition (2) is necessary in order that x is an SU-point and we can assume that this condition holds.

Now once again consider y as in (1). If $I_\Psi(p^+(ky)) \geq 1$, then, as above (cases 3° and 4° of the proof), we have $k^*(y) = k$. If $k^{**}(y) = k^* = k$, then, by condition (2) and Theorem 1.5(ii)(c), the support functional $x^* \in \operatorname{Grad}(x)$, which is generated by the triple (w, ϕ_1, a_1), where $w = (p_i^-(kx_i))_1^\infty = (p_i^-(ky_i))_1^\infty$ and a_1, ϕ_1 satisfy conditions from Theorem 1.5(ii)(c), also belongs to $\operatorname{Grad}(y/\|y\|_\Phi^o)$. Proceeding with x^* as in the case 1° of the proof, we get $\|x + y\|_\Phi^o = 2$. It means that x is not an SU-point. If $k^{**}(y) > k^* = k$, then, by Theorem 1.5(i) and the Young inequality, we get $I_\Psi(p^+(ky)) = 1$, whence $k^{**}(y) \geq k$. Since $\theta(ky) = 1$, so $I_\Phi(ly) = \infty$ for any $l > k$, whence $k^{**}(y) = k$ and finally $K(y) = k$. This contradiction shows that the case $k^{**}(y) > k^*$ is impossible.

If $I_\Psi(p^+(ky)) < 1$, then $k^{**}(y) \geq k$ and as above, using $\theta(ky) = 1$, we get $k^{**}(y) = k$. From Theorem 1.5(i) and the Young inequality, we get that $K(y) = \{k\}$, because in the opposite case $I_\Psi(p^-(ky)) = 1$. Now we proceed as above, obtaining a contradiction.

In order to prove condition (2) in the case $\theta(kx) < 1$ together with the assumption that there exists $i_0 \in A_l^{ns}(kx)$ such that $I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i_0\}}) + \Psi_{i_0}(p_{i_0}^-(kx_{i_0})) \geq 1$, we proceed exactly in the same way as in the cases 3° and 4° of the proof, using only Theorem 1.5(ii)(b) instead of Theorem 1.5(ii)(c). Then we have $I_\Psi(p^-(kx)) < 1$ and $I_\Psi(p^+(kx)) > 1$. Defining y as in (1), we get $I_\Psi(p^+(ky)) \geq 1$, whence $k^*(y) \leq k$ and, by (5), $k^*(y) = k$. We will consider two cases.

If $I_\Psi(p^+(ky)) = 1$, then the support functional x^*, generated by $p^+(ky)$, belongs to the $\operatorname{Grad}(x) \cap \operatorname{Grad}(y/\|y\|_\Phi^o)$, by Theorem 1.5(ii)(b) and the fact that $p^-(kx) = p^-(ky) \leq p^+(ky) \leq p^+(kx)$. Therefore, as in the case 1° of the proof, we have $\|x + y/\|y\|_\Phi^o\|_\Phi^o = 2$. So, we get that x is not an SU-point.

If $I_\Psi(p^+(ky)) > 1$, then there exists $A^{ns}(kx) \ni i_1 \neq i_0$, because in the opposite case $\operatorname{Grad}(x) = \emptyset$, by Theorem 1.5(ii)(b). If

$$\sum_{\substack{i=1 \\ i \notin \{i_0, i_1\}}} \Psi_i(p_i^+(kx_i)) + \Psi_{i_0}(p_{i_0}^-(kx_{i_0})) + \Psi_{i_1}(p_{i_1}^-(kx_{i_1})) = 1,$$

then the support functional x^*, generated by $v = (v_i)_1^\infty$, where $v_i = p_i^+(kx_i)$ for $i \notin \{i_0, i_1\}$, $v_i = p_i^-(kx_i)$ for $i \in \{i_0, i_1\}$, belongs to the $\operatorname{Grad}(x) \cap \operatorname{Grad}(y\|y\|_\Phi^o)$. So, as above, x is not an SU-point.

(iv) This case can be proved similarly to the case (iii), so we omit the proof. $\square$

Sufficiency. Assume that $y \in S(l_\Phi^\circ)$, $\|x + y\|_\Phi^o = 2$. Let $l \in K(y)$ and conditions (i)–(iv) hold. Then $\frac{kl}{k+l} \in K(x + y)$ (see [4]) and $\|x + y\|_\Phi^o = \|x\|_\Phi^o + \|y\|_\Phi^o$, whence

$$\frac{(k+l)\left(1 + I_\Phi\left(\frac{kl}{k+l}(x+y)\right)\right)}{kl} = \frac{l(1 + I_\Phi(kx)) + k(1 + I_\Phi(ly))}{kl},$$

$$(l+k)I_\Phi\left(\frac{kl}{k+l}(x+y)\right) = lI_\Phi(kx)) + kI_\Phi(ly),$$

$$\sum_{i=1}^\infty \Phi_i\left(\frac{kl}{k+l}(x_i+y_i)\right) = \frac{l}{k+l}\sum_{i=1}^\infty \Phi_i(kx_i) + \frac{k}{k+l}\sum_{i=1}^\infty \Phi_i(ly_i),$$

$$\Phi_i\left(\frac{l}{k+l}kx_i + \frac{k}{k+l}ly_i\right) = \frac{l}{k+l}\Phi_i(kx_i) + \frac{k}{k+l}\Phi_i(ly_i)$$

for all $i \in \mathbb{N}$. It means that all functions Φ_i are affine on the intervals $[\min\{kx_i, ly_i\}, \max\{kx_i, ly_i\}]$. Suppose that there exist $i \neq j$ such that $i \in A_l^{ns}(kx)$ and $j \in A_u^{ns}(kx)$. Then, by conditions (iii) and (iv), we get

$$\begin{aligned}
1 &> I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i,j\}}) + \Psi_i(p_i^-(kx_i)) + \Psi_j(p_j^+(kx_j)) \\
&\geq I_\Psi(p^-(kx)\chi_{\mathbb{N}\setminus\{i,j\}}) + \Psi_i(p_i^-(kx_i)) + \Psi_j(p_j^+(kx_j)) \\
&= I_\Psi(p^-(kx)\chi_{\mathbb{N}\setminus\{j\}}) + \Psi_j(p_j^+(kx_j)) > 1,
\end{aligned}$$

a contradiction, whence either $\mathbb{N} = A_l^{ns}(kx)$ or $\mathbb{N} = A_u^{ns}(kx)$. Assume that $\mathbb{N} = A_l^{ns}(kx)$. Then $ly_i \le kx_i$ for any $i \in \mathbb{N}$ and since $x, y \in S(l_\Phi^\circ)$, so $l \le k$. By (iii) we get $\theta(kx) < 1$. Suppose that there exists $i_0 \in \mathbb{N}$ such that $ly_{i_0} < kx_{i_0}$. By condition (iii), we have

$$I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus\{i_0\}}) + \Psi_{i_0}(p_-^{i_0}(kx_{i_0})) < 1. \tag{7}$$

Without loss of generality we can assume that $ly_i = kx_i$ for any natural $i \ne i_0$, because condition (7) holds for any $i \in \mathbb{N}$, not only for fixed i_0. Moreover $p_+^{i_0}(ly_{i_0}) = p_-^{i_0}(kx_{i_0})$ and $p_+^i(ly_i) \le p_+^i(kx_i)$ for any $i \ne i_0$. Then, by (7), we get $I_\Psi(p^+(ly)) < 1$. Moreover, $ly \le kx$, whence $\theta(ly) \le \theta(kx) < 1$. Therefore, it follows from Theorem 1.5(ii)(b) that $\mathrm{Grad}(y) = \emptyset$, which contradicts Hahn–Banach Theorem. Therefore $ly_{i_0} = kx_{i_0}$. It implies that $ly = kx$, and since $x, y \in S(l_\Phi^\circ)$, so $k = l$ and finally $x = y$.

If $A_u^{ns}(kx) = \mathbb{N}$, then we proceed in the same way as above, using condition (iv) instead of condition (iii).

In a similar way we can prove the following

Theorem 2.4. *A point $x \in S(L_\Phi^\circ)$ is an SU-point if and only if the following conditions hold:*

(i) $K(x) = \{k\}$ *for some* $0 < k < \infty$,

(ii) $\mu(A_l^s(kx)) = \mu(A_u^s(kx)) = 0$,

(iii) *if* $\mu(A_l^{ns}(kx)) > 0$, *then* $\theta(kx) < 1$ *and* $I_\Psi(p^+(kx)\chi_{T\setminus E}) + I_\Psi(p^-(kx)\chi_E) < 1$ *for any* $E \subset A_l^{ns}(kx)$ *such that* $\mu(E) > 0$,

(iv) *if* $\mu(A_u^{ns}(kx)) > 0$, *then* $\theta(kx) < 1$ *and* $I_\Psi(p^-(kx)\chi_{T\setminus E}) + I_\Psi(p^+(kx)\chi_E) > 1$ *for any* $E \subset A_u^{ns}(kx)$ *such that* $\mu(E) > 0$.

Theorem 2.5. *If Φ is a Musielak–Orlicz function such that $\Phi > 0$ and $\Phi_i(1) = 1$ for any $i \in \mathbb{N}$, then the following conditions are equivalent:*

(i) $l_\Phi \in$ (CLUR),

(ii) (a) $\Phi \in \delta_2$

(b) $\Psi \in \delta_2$ *or* Φ_i *is strictly convex on* $[0, \Phi_i^{-1}(1)]$ *for every* $i \in \mathbb{N}$.

Proof. (i)$\Rightarrow$ (ii)(a) The implication follows from the facts that $X \in$ (CLUR) $\Rightarrow X \in (H)$ for any Banach space X and $\Phi \notin \delta_2 \Rightarrow l_\Phi \notin (H)$, because Banach function lattices with property H are order continuous.

(i)$\Rightarrow$ (ii)(b) Assume that condition (ii)(b) does not hold, that is, $\Psi \notin \delta_2$ and there is $j \in \mathbb{N}$ and an interval $[u,v] \cup [0, \Phi_j^{-1}(1)]$ such that Φ_j is affine on $[u,v]$. We may assume without loss of generality that $j=1$ and $u>0$. Denote $w=(u+v)/2$ and take $x \in S(l_\Phi)$ with $I_\Phi(x)=1$ and $x(1)=v$. Let $(I_n)_{n=1}^\infty$ in $\mathbb{N}$ and (u_i^n) in $\mathbb{R}$ ($n \in \mathbb{N}$, $i \in \{I_{n-1}+1,\dots,I_n\}$) be sequences from Lemma 1.2. Denote $a=\Phi_1(v)-\Phi_1(w)$. Let $k \in \mathbb{N}$ be the smallest number that satisfies $\frac{1}{k} \leq a$. Since $\Phi_i(u_i^n) \leq \frac{1}{n}$ for any $i \in \{I_{n-1}+1,\dots,I_n\}$, $n \in \mathbb{N}$ being arbitrary, there exists $m_1 \in \{I_{k-1}+1,\dots,I_k\}$ such that

$$\sum_{i=I_{k-1}+1}^{m_1} \Phi_i(u_i^k) \leq a \quad \text{and} \quad \sum_{i=I_{k-1}+1}^{m_1+1} \Phi_i(u_i^k) > a.$$

Continuing this procedure we can find a sequence $(m_n)_{n=1}^\infty$ with $I_{k+n-2}+1 \leq m_n < I_{k+n-1}$ for any $n \in \mathbb{N}$ such that

$$\sum_{i=I_{k+n-2}+1}^{m_n} \Phi_i(u_i^{k+n-1}) \leq a \quad \text{and} \quad \sum_{i=I_{k+n-2}+1}^{m_n+1} \Phi_i(u_i^{k+n-1}) > a.$$

Let $e_i = \chi_{\{i\}}$ be the basis vector for the space c_0 and define

$$x_n = we_1 + \sum_{i=2}^{I_{k+n-2}} x(i)e_i + \sum_{i=I_{k+n-2}+1}^{m_n} u_i^{k+n-1} e_i.$$

Since $\Phi_i(u_i^l) \leq \frac{1}{l}$ for any $i,l \in \mathbb{N}$, we have

$$a - \frac{1}{k+n-1} < \sum_{i=I_{k+n-2}+1}^{m_n} \Phi_i(u_i^{k+n-1}) \leq a.$$

Consequently

$$1 - \frac{1}{k+n-1} < I_\Phi(x_n) \leq 1$$

and so

$$1 - \frac{1}{k+n-1} < \|x_n\|_\Phi \leq 1$$

for any $n \in \mathbb{N}$. Moreover

$$1 \geq I_\Phi\left(\frac{x+x_n}{2}\right) = \Phi_1\left(\frac{w+v}{2}\right) + \sum_{i=2}^{I_{k+n-2}} \Phi_i\big(x(i)\big)$$

$$+\sum_{i=I_{k+n-2}+1}^{m_n}\Phi_i\left(\frac{u_i^{k+n-1}}{2}\right)\geq\frac{1}{2}\Phi_1(w)+\frac{1}{2}\Phi_1(v)+\sum_{i=2}^{I_{k+n-2}}\Phi_i\big(x(i)\big)$$

$$+\frac{1-\frac{1}{k+n-1}}{2}\left(\sum_{i=I_{k+n-2}+1}^{m_n+1}\Phi_i(u_i^{k+n-1})-\frac{1}{k+n-1}\right)$$

$$=\sum_{i=2}^{I_{k+n-2}}\Phi_i(x(i))-\frac{1}{2}\{\Phi_1(v)-\Phi_1(w)\}$$

$$+\frac{1-\frac{1}{k+n-1}}{2}\left(\Phi_1(v)-\Phi_1(w)-\frac{1}{k+n-1}\right)$$

$$=\sum_{i=2}^{I_{k+n-2}}\Phi_i(x(i))-\frac{1}{2(k+n-1)}\{\Phi_1(v)-\Phi_1(w)\}$$

$$-\frac{1-\frac{1}{k+n-1}}{2(k+n-1)}\to I_\phi(x)=1.$$

Consequently $\left\|\frac{x+x_n}{2}\right\|_\Phi\to 1$ as $n\to\infty$. Moreover

$$I_\Phi(x_m-x_n)\geq\sum_{i=I_{k+n-2}+1}^{m_n}\Phi_i(u_i^{k+n-1})$$

$$=\sum_{i=I_{k+n-2}+1}^{m_n+1}\Phi_i(u_i^{k+n-1})-\Phi_{m_n+1}(u_i^{k+m_n})\geq\Phi_1(v)-\Phi_1(w)-\frac{1}{k+m_n}$$

for all $m,n\in\mathbb{N}$. Consequently

$$I_\Phi(x_m-x_n)\geq\frac{1}{2}\{\Phi_1(v)-\Phi_1(w)\}$$

for $m,n\in\mathbb{N}$ large enough, which yields

$$\|x_m-x_n\|_\Phi\geq\min\left(1,\frac{1}{2}\{\Phi_1-\Phi_1(a)\}\right)>0$$

for $m,n\in\mathbb{N}$ large enough. This means that sequence (x_n) has no Cauchy subsequences, that is, $l_\Phi\notin$ (CLUR).

(ii) $\Rightarrow$ (i) If $\Phi\in\delta_2$ and $\Psi\in\delta_2$, then the proof proceeds in the same way as the proof of the analogous fact in the case of the Orlicz spaces (see [3]), so we omit it. Now assume that $\Phi \in \delta_2$, $\|x\|_\Phi = \|x_n\|_\Phi = \|\frac{x+x_n}{2}\|_\Phi = 1$ and Φ_i is strictly convex on $[0, \Phi_i^{-1}(1)]$ for any $i \in \mathbb{N}$. The last condition implies that $\mathrm{card}(A_l(x)) = 0$. We define the sets

$$A_{i,\varepsilon} = \{n \in \mathbb{N} : |x_n(i)| \le |x(i)| \text{ and } |x(i) - x_n(i)| \ge \varepsilon\},$$

for any $\varepsilon > 0$ and any $i \in \mathbb{N}$. First we will show that

$$\mathrm{card}(A_{i,\varepsilon}) < \infty \quad \text{for any } \varepsilon > 0 \quad \text{and } i \in \mathbb{N}. \tag{8}$$

Otherwise, there exists $\varepsilon_0 > 0$ and $i_0 \in \mathbb{N}$ such that $\mathrm{card}(A_{i_0,\varepsilon_0}) = \infty$, that is, there exists a subsequence (x_{n_k}) of (x_n) which satisfies the conditions: $|x_{n_k}(i_0)| \le |x(i_0)|$ and $|x_{n_k}(i_0) - x(i_0)| \ge \varepsilon_0$. Since $A_l(x) = \emptyset$, there exists $\delta > 0$ such that

$$\Phi_{i_0}\left(\frac{x(i_0) + x_{n_k}(i_0)}{2}\right) \le \frac{1-\delta}{2}\left[\Phi_{i_0}(x(i_0)) + \Phi_{i_0}(x_{n_k}(i_0))\right],$$

for any $k \in A_{i_0,\varepsilon_0}$. By Lemma 2.1 we have $I_\Phi(x) = I_\Phi(x_1) = 1$. Moreover, $I_\Phi(\frac{x+x_n}{2}) \to 1$ as $n \to \infty$. Then we get

$$\begin{aligned}
0 \leftarrow & \sum_{i=1}^{\infty}\left[\frac{\Phi_i(x_{n_k}(i)) + \Phi_i(x(i))}{2} - \Phi_i\left(\frac{x_{n_k}(i) + x(i)}{2}\right)\right] \\
\ge & \frac{\Phi_{i_0}(x_{n_k}(i_0)) + \Phi_{i_0}(x(i_0))}{2} - \Phi_{i_0}\left(\frac{x_{n_k}(i_0) + x(i_0)}{2}\right) \\
\ge & \delta\frac{\Phi_{i_0}(x_{n_k}(i_0)) + \Phi_{i_0}(x(i_0))}{2} \ge \delta\Phi_{i_0}\left(\frac{x_{n_k}(i_0) - x(i_0)}{2}\right) \ge \delta\Phi_{i_0}\left(\frac{\varepsilon_0}{2}\right),
\end{aligned}$$

a contradiction, which shows that (8) is true. We will show now that $x_n \to x$ coordinatewise. Suppose the opposite. Then there exist $\varepsilon_0 > 0$, $i_0 \in \mathbb{N}$ and a subsequence (x_{n_k}) of (x_n) satisfying

$$|x_{n_k}(i_0) - x(i_0)| > \varepsilon_0 \quad \text{for any } k \in \mathbb{N}. \tag{9}$$

By the previous part of the proof we have $\mathrm{card}(A_{i_0,\varepsilon_0}) < \infty$. By condition (9) and the definition of the set A_{i_0,ε_0} we conclude that there exists $k_0 \in \mathbb{N}$ such that $|x_{n_k}(i_0)| > |x(i_0)|$ for all $k > k_0$, whence we conclude from (9) the existence of $\varepsilon_1 > 0$ such that $\Phi_{i_0}(x_{n_k}(i_0)) - \Phi_{i_0}(x(i_0)) > \varepsilon_1$ for any $k > k_0$. Since $I_\Phi(x) < \infty$, so there exists a natural number $i_1 > i_0$

such that $\sum_{i>i_1}\Phi_i(x(i)) < \frac{\varepsilon_1}{2}$. Moreover, we can assume, without loss of generality, that $x_n(i) \to x(i)$ for any $i \le i_1, i \ne i_0$. Hence for any $i < i_1, i \ne i_0$, there exists $l_i \in \mathbb{N}$ such that $|\Phi_i(x_n(i)) - \Phi_i(x(i))| < \frac{\varepsilon_1}{3i_1}$ for any $n > l_i$. Therefore, in the worst case we have

$$\begin{aligned}
1 = I_\Phi(x_{n_k}) &\ge \Phi_{i_0}(|x_{n_k}(i_0)|) + \sum_{\substack{i\le i_1\\ i\ne i_0}} \Phi_i(|x_{n_k}(i)|)\\
&\ge \Phi_{i_0}(|x(i_0)|) + \varepsilon_1 + \sum_{\substack{i\le i_1\\ i\ne i_0}} \Phi_i(|x(i)|) + \sum_{\substack{i\le i_1\\ i\ne i_0}} [\Phi_i(|x_{n_k}(i)|) - \Phi_i(|x(i)|)]\\
&> 1 + \frac{\varepsilon_1}{2} + \sum_{\substack{i\le i_1\\ i\ne i_0}} [\Phi_i(|x_{n_k}(i)|) - \Phi_i(|x(i)|)] > 1 + \frac{\varepsilon_1}{2} - \frac{\varepsilon_1}{3} = 1 + \frac{\varepsilon_1}{6}
\end{aligned}$$

for any $k > \max\{k_0, l_1, l_2, \ldots, l_{i_1}\}$. This contradiction proves that $x_n \to x$ coordinatewise. Using Lemma 2.2 we can finish the proof. □

Remark 2.3. *Let $B = \{i \in \mathbb{N} : b(\Phi_i) < \infty\}$ and $I_\Phi(b(\Phi)\chi_B)) < 1$. Then $l_\Phi \notin$ (CLUR).*

Proof. Assume first additionally that $\operatorname{card}(B) = \infty$. Then we may assume, without loss of generality, that $b(\Phi_i) < \infty$ for any $i \in \mathbb{N}$. Define

$$x = \sum_{i\in\mathbb{N}} b(\Phi_i)e_i, \quad x_n = \sum_{i=1}^{n} b(\Phi_i)e_i.$$

It is clear that $\|x\|_\Phi = \|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $n > m$, we get $x_n - x_m = \sum_{i=m+1}^{n} b(\Phi_i)e_i$, so $\|x_n - x_m\|_\Phi = 1$. Moreover

$$\frac{x_n + x}{2} = \sum_{i=1}^{n} b(\Phi_i)e_i + \sum_{i=n+1}^{\infty} \frac{b(\Phi_i)}{2}e_i.$$

Therefore $I_\Phi(\frac{x+x_n}{2}) \le I_\Phi(x) \le 1$ and consequently, using the triangle inequality, we get $\|\frac{x+x_n}{2}\|_\Phi = 1$. This shows that $l_\Phi \notin$ (CLUR).

Now let $\operatorname{card}(B) < \infty$, $x = \sum_{i\in B} b(\Phi_i)e_i$ and $a = 1 - I_\Phi(x) > 0$. Define $x_n = x$ for $n \in B$ and

$$x_n(i) = \begin{cases} x(i), & \text{for } i \in B;\\ \Phi_n^{-1}(a), & i = n;\\ 0, & \text{otherwise} \end{cases}$$

for $n \notin B$. Therefore $I_\Phi(x_n) = I_\Phi(x) < 1$ for $n \in B$ and $I_\Phi(x_n) = I_\Phi(x) + a = 1$ for $n \notin B$. Hence $\|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. Moreover, $\frac{x_n+x}{2} = x$ for $n \in B$, and

$$\frac{x_n(i)+x(i)}{2} = \begin{cases} x(i), & \text{for } i \in B; \\ \frac{1}{2}\Phi_n^{-1}(a), & i = n; \\ 0, & \text{otherwise} \end{cases}$$

for $n \notin B$. Since $I_\Phi(x) \leq 1$ and $I_\Phi(x_n) \leq 1$ for any $n \in \mathbb{N}$, by convexity of I_Φ, we get $I_\Phi(\frac{x_n+x}{2}) \leq 1$ and consequently $\|\frac{x_n+x}{2}\|_\Phi \leq 1$ for any $n \in \mathbb{N}$. Moreover, $I_\Phi(\lambda\frac{x_n+x}{2}) \geq I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$, whence $\|\frac{x_n+x}{2}\|_\Phi \geq 1$. This proves that $\|\frac{x+x_n}{2}\|_\Phi = 1$ for any $n \in \mathbb{N}$. We also have for $n > m$ and $n, m \notin B$:

$$x_n(i) - x_m(i) = \begin{cases} -\Phi_m^{-1}(a), & \text{for } i = m; \\ \Phi_n^{-1}(a), & \text{for } i = n; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $I_\Phi(x_n - x_m) = 2a$ for $m, n \notin B$, $m \neq n$, so $\|x_n - x_m\|_\Phi \geq \min\{1, 2a\} > 0$ for $m, n \notin B$, $m \neq n$. This shows that l_Φ is not CLUR. $\square$

Theorem 2.6. *Let Φ be a Musielak–Orlicz function such that $\Phi < \infty$ and $\lim_{u\to 0}\frac{\Phi_i(u)}{u} = 0$ for any $i \in \mathbb{N}$. Then l_Φ° has property CLUR if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.*

Proof. *Sufficiency.* By Theorem 1.3, $l_\Phi^\circ \in (H)$. By $\lim_{u\to 0}\frac{\Phi_i(u)}{u} = 0$ for any $i \in \mathbb{N}$, we get $\Psi < \infty$. Moreover, condition $\Psi \in \delta_2$ is equivalent to $l_\Psi \in (H)$ (see [8]). Since our assumptions imply reflexivity of l_Φ°, so by Theorem 1.4 we get $l_\Phi^\circ \in$ (CLUR).

Necessity. Since property CLUR implies property H and property H implies order continuity (see [7]), we have $\Phi \in \delta_2$. Now suppose $\Psi \notin \delta_2$. Then (see [8]) there exists a sequence $(u_n) \subset \mathbb{R}$, $u_n \geq 0$, and sequence of sets (E_n): $E_n \in 2^N$, $E_n \cap E_m = \emptyset$ for $m \neq n$ such that:

(a) $\frac{1}{2^{n+1}} < \sum_{i\in E_n}\Psi_i(u_n(i)) \leq \frac{1}{2^n}$,

(b) $\Psi_i((1+\frac{1}{2^n})u_n(i)) > 2^{n+1}\Psi_i(u_n(i))$ for any $i \in E_n$.

Define the sequence (z_n) in l_Ψ by

$$z_n(i) = \begin{cases} u_n(i), & \text{if } i \in E_n; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$I_\Psi(z_n) = \sum_{i=1}^{\infty} \Psi_i(z_n(i)) = \sum_{i \in E_n} \Psi_i(z_n(i)) \le \frac{1}{2^n}.$$

Hence $\|z_n\|_\Psi \le 1$. If $\lambda = \frac{2^n}{2^n+1}$, then

$$I_\Psi\left(\frac{z_n}{\lambda}\right) = \sum_{i \in E_n} \Psi_i\left(\left(1 + \frac{1}{2^n}\right) u_n(i)\right) > \sum_{i \in E_n} 2^{n+1} \Psi_i(u_n(i)) > 1.$$

Therefore $\frac{2^n}{2^n+1} < \|z_n\|_\Psi \le 1$. Since $z_n \in h_\Psi$ and $(h_\Psi)^* = l_\Phi^\circ$, so for any $n \in \mathbb{N}$ there exists $x_n \in S(l_\Phi^\circ)$ such that

$$\frac{2^n}{2^n+1} < \|z_n\|_\Psi = \langle z_n, x_n \rangle = \sum_{i \in E_n} x_n(i) u_n(i). \tag{10}$$

Let $x \in S(l_\Phi^\circ)$ and $x \ge 0$. Since $\Phi \in \delta_2$, so $(l_\Phi^\circ)^* = l_\Psi$ and there exists $y \in S(l_\Psi)$ such that $\langle x, y \rangle = \sum_{i \in N} x_i y_i = \|x\|_\Phi^o$. Define

$$y_n = \frac{2^n}{2^n+1} (u_n \chi_{E_n} + y \chi_{N \setminus E_n}).$$

Then

$$I_\Psi(y_n) \le \frac{2^n}{2^n+1} (I_\Psi(y) + I_\Psi(z_n)) \le \frac{2^n}{2^n+1} \left(1 + \frac{1}{2^n}\right) = 1. \tag{11}$$

Moreover

$$\langle x_n, y_n \rangle \ge \frac{2^n}{2^n+1} \sum_{i \in E_n} x_n(i) z_n(i) = \frac{2^n}{2^n+1} \langle x_n, z_n \rangle \ge \left(\frac{2^n}{2^n+1}\right)^2. \tag{12}$$

By the definition of y_n and the fact that $x \ge 0$ and $z_n \ge 0$, we get

$$\langle x, y_n \rangle = \frac{2^n}{2^n+1} \left[\sum_{i \in E_n} x(i) u_n(i) + \sum_{i \notin E_n} x(i) y(i) \right]$$

$$= \frac{2^n}{2^n+1}\left[\sum_{i\in N} x(i)y(i) + \sum_{i\in E_n} x(i)u_n(i) - \sum_{i\in E_n} x(i)y(i)\right]$$
$$\geq \frac{2^n}{2^n+1}\left[\langle x,y\rangle - \sum_{i\in E_n} x(i)y(i)\right]. \tag{13}$$

Moreover, the Young inequality yields

$$\sum_{i\in E_n} x(i)y(i) \leq I_\Phi(x\chi_{E_n}) + I_\Psi(y\chi_{E_n}) \to 0. \tag{14}$$

By (12), (13), (14) and by the definition of the Orlicz norm, we have

$$2 \geq \|x+x_n\|_\Phi^o \geq \langle x+x_n, y_n\rangle$$
$$\geq \left(\frac{2^n}{2^n+1}\right)^2 + \frac{2^n}{2^n+1}\left[1 - \sum_{i\in E_n} x(i)y(i)\right] \to 2.$$

Hence $\|x_n+x\|_\Phi^o \to 2$. On the other hand, since supports of the elements x_n are separated, we get $\|x_n - x_m\|_\Phi^o \geq \|x_n\|_\Phi^o = 1$ for any $m,n \in \mathbb{N}$, $m \neq n$. This contradicts the CLUR-property for l_Φ°. □

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