

When does an iterate equal a power?

By CHE TAT NG (Waterloo) and WEINIAN ZHANG (Chengdu)

Abstract. Let f be a continuous self-map on the real line, $f^{[m]}$ denote its m -th iterate and f^n its n -th multiplicative power. In this paper we solve the functional equation $f^{[m]} = f^n$ for integers $m \geq 2$, $n \geq 2$. When $m = n$, it reveals functions whose n -th iterate and power agree.

1. Introduction

Let $f : X \rightarrow X$ be a map on a set X , $m \geq 0$ an integer. The m -th iterate $f^{[m]}$ of f is defined by

$$f^{[m]}(x) = f(f^{[m-1]}(x)), \quad f^{[0]}(x) \equiv x.$$

When f is bijective, with inverse f^{-1} , iterates with negative exponents are defined by $f^{[-m]} = (f^{-1})^{[m]}$. Sometimes the brackets around m in $f^{[m]}$ are omitted when there is no confusion [1], [4]–[6]. As pointed out in [2], in circumstances where f^m has other natural meaning, such an omission would possibly lead some readers astray. For functions defined on the real line $\mathbb{R}$, let f^m denote its m -th (multiplicative) power. We would like to ask when $f^{[m]}$ and f^m actually agree, i.e.

$$f^{[m]} = f^m. \tag{1}$$

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Corresponding author: Weinian Zhang (matzwn@sohu.com)

On various domains, this functional equation was discussed ([2], [3]) for $m = -1$. In particular, in [2] the problem leads to a discussion on the 4-th iterative roots of the identity as [4] does. In this paper we seek continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f^{[m]} = f^n \quad (2)$$

for given integers $m \geq 2, n \geq 1$.

2. Fundamental results

The following theorem, as a fundamental result, links our problem to iterative roots of special functions.

Theorem 1. *A continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution of equation (2) if and only if there exists an interval I in $\mathbb{R}$ such that*

- (i) *I is non-empty, closed relative to $\mathbb{R}$, and is invariant under the power function $F(x) = x^n$,*
- (ii) *I is also f -invariant, $f^{[m-1]} = F$ on I , and*
- (iii) *$\text{ran}(f) \subset I$, where $\text{ran}(f)$ denotes the range of f .*

PROOF. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous solution of equation (2):

$$f^{[m]}(x) = (f(x))^n, \quad \forall x \in \mathbb{R}. \quad (3)$$

Putting $y = f(x)$ in it we get

$$f^{[m-1]}(y) = y^n, \quad \forall y \in \text{ran}(f). \quad (4)$$

Since f is continuous, $\text{ran}(f)$ is a non-empty interval. Let I be its closure in $\mathbb{R}$. Then I is also an interval and (iii) holds. Clearly, (iii) implies that I is f -invariant. By continuity, equation (4) can be extended to I and we have (ii):

$$f^{[m-1]}(y) = y^n, \quad \forall y \in I. \quad (5)$$

Being f -invariant, I is also invariant under $f^{[m-1]}$. In view of (5), we get (i) – that I is invariant under the power function F . The converse is easy to check. With (iii), $\text{ran}(f) \subset I$, (ii) implies (4) which is equivalent to (3). $\square$

While there is no need for I to be a closed interval in order to get the converse in the above theorem, its imposition eliminates the need to discuss a broader class of I . Its closure assures that continuous self-maps on I have continuous extension to the larger domain $\mathbb{R}$ without an expansion on its codomain. Although we could impose the condition that $\text{ran}(f)$ is dense in I , we do not do this because it would be inconvenient and unnecessary for the converse.

Referring to condition (i), let

$$a := \inf I, \quad b := \sup I.$$

Thus a is possibly $-\infty$, b is possibly ∞ , and $I = [a, b] \cap \mathbb{R}$. The determination of F -invariant I is straight forward. In the next proposition we state the result.

Proposition 1. *For even n , $n \geq 2$, I is invariant under $F(x) = x^n$ if and only if its ends a, b are in one of the following four combinations: (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$. For odd n , $n \geq 3$, I is invariant under $F(x) = x^n$ if and only if its ends a, b are in one of the following five combinations: (O_1) $a = -\infty$ and $b \in]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or (O_4) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$. For $n = 1$, a and b can be chosen arbitrarily with $a \leq b$.*

We now turn our attention to condition (ii). Letting $g = f|_I : I \rightarrow I$, we shall solve the equation

$$g^{[m-1]} = F|_I \tag{6}$$

in the next two sections. Together with the above fundamental results, (2) is solved fully. Solving (6) for $m = 2$ is a trivial mission, so in the coming sections we will assume $m > 2$.

3. Cases of odd n

In the special case $n = 1$, (6) is known as a Babbage equation, i.e., $F = \text{id}$ on I , where id denotes the identity map. By Theorem 11.7.1 in [4], either $g = \text{id}$ on I or m has to be odd and g is a strictly decreasing involution. As described in [4], decreasing involutions on an interval have simple geometric interpretation: their graph has to be symmetric with respect to the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$. In the sequel, we need only discuss the cases of an odd $n \geq 3$.

Lemma 1. *Let $n \geq 3$ be odd. Let g be a continuous $(m - 1)$ -th order iterative root of $F(x) = x^n$ on I . Then (i) g is strictly monotonic, (ii) g has no periodic points other than $0, 1$ and -1 , and they are in fact periodic points of g whenever they are in I , (iii) for strictly increasing g , its periodic points can only be of order 1 (a fixed point), (iv) for strictly decreasing g , its periodic points are of order 1 and 2, while at most one is of order 1, and $(-1, 1)$ is its only possible 2-cycle in I .*

PROOF. (i) With odd n , F is injective. If $g(x_1) = g(x_2)$, then $g^{[m-1]}(x_1) = g^{[m-1]}(x_2)$. Thus $F(x_1) = F(x_2)$, implying $x_1 = x_2$. This shows that g is also injective. Being continuous on the interval, g must be strictly monotonic. (ii) If x_0 is a periodic point of g , then x_0 must be a fixed point of an iterate of F . However, for any integer $k \geq 1$ the function $F^{[k]}(x) = x^{n^k}$ has exactly three fixed points at $0, 1$ and -1 . This proves that x_0 can only be $0, 1$ or -1 . Conversely, every fixed point x_0 of F is a periodic point of g , as $g^{[m-1]}(x_0) = x_0$ where $m - 1 \geq 1$.

(iii) This is a general observation that the only order preserving (finite) cycles on a linearly ordered set are the trivial 1-cycles. (iv) An order reversing cycle on a linearly ordered set must be a 1-cycle or a 2-cycle. A map with more than one fixed point cannot be order reversing. The 2-cycles $(-1, 0)$ and $(0, 1)$ cannot occur for continuous g , because no fixed points are present in the open intervals $] -1, 0[$ and $] 0, 1[$. $\square$

According to the list in Proposition 1, we shall seek roots on intervals $I = [a, b] \cap \mathbb{R}$ where a, b are in one of the following five combinations (O_1) $a = -\infty$ and $b \in]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or (O_4) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$.

In light of Lemma 1, (i), all roots are strictly monotonic. Within the following two subsections, the shorter form “increasing g ”, for instance, will have the same effect as “strictly increasing g ”.

3.1. For increasing g on I . The solving of (6) for increasing g on intervals of type (O_3) can be further simplified by solving it “componentwise”, as stated more accurately in the following:

Proposition 2. *Let $n \geq 3$ be odd. Then g is a strictly increasing continuous $(m-1)$ -th order iterative root of $F(x) = x^n$ on I if and only if it is the union of strictly increasing roots on each of the closed connected sub-intervals of I separated by the fixed points $\{0, \pm 1\} \cap I$.*

Theorem 11.2.2 in [4] gives the results on increasing iterative roots on I . In particular it shows that F possesses increasing iterative roots g of all orders, and they can be constructed by *piecewise defining*.

Moreover, it is easy to show that $g(x) \leq x$ (resp. $\geq x$) for $x \in]-\infty, -1] \cup [0, 1]$ (resp. $x \in [-1, 0] \cup [1, +\infty[$) if g is defined at the point x . For example, if g is an increasing second order root and $g(t_0) > t_0$ for some $t_0 \in [0, 1]$ where g is defined, then $F(t_0) = g(g(t_0)) > g(t_0) > t_0$ since g is strictly increasing. This will contradict $F(t_0) = t_0^n \leq t_0$. This property of g is observed when we apply Theorem 11.2.5 in [4] during a subsequent discussion on the decreasing roots.

In what follows we select a typical interval under the case $-1 < a < 0$ and $b = 0$ to illustrate the general construction of an increasing root.

Let g be a strictly increasing continuous self-map on $[a, 0]$ and

$$g^{[m-1]}(x) = F(x) = x^n \quad (x \in [a, 0]). \quad (7)$$

According to Lemma 1, g has 0 as its unique fixed point. Letting

$$c_j = g^{[j]}(a), \quad j = 0, \dots, m-1 \quad (8)$$

we first get $-1 < a = c_0 \leq c_1 \leq c_2 \leq \dots \leq c_{m-2} \leq c_{m-1} = F(a) = a^n < 0$ from the range condition $\text{ran}(F|_I) = \text{ran}(g^{[m-1]}) \subset \text{ran}(g^{[m-2]}) \subset \dots \subset \text{ran}(g) \subset [a, 0]$ while using $g(0) = 0$. Because g has 0 as its unique fixed point, the above inequalities must be strict:

$$-1 < a = c_0 < c_1 < c_2 < \dots < c_{m-2} < c_{m-1} = a^n < 0.$$

Recall that $m > 2$ has been assumed. A *fundamental region* for g is $[c_0, c_{m-2}]$, in the sense that the restriction

$$g_0 := g|_{[c_0, c_{m-2}]}$$

can be initiated reasonably freely, and it determines g on the full $[a, 0]$. The *initial* g_0 is an order preserving homeomorphism, mapping $[c_0, c_{m-2}]$ onto $[c_1, F(a)]$, and $g_0(c_j) = c_{j+1}$ for each $j = 0, \dots, m-2$. It implies

$$g_0^{[m-2]}([c_0, c_1]) = [c_{m-2}, c_{m-1}] \quad (9)$$

in particular, and g on $[a, 0]$ is uniquely determined by g_0 via:

Step 1. For each $\ell \geq 1$ and $x \in [F^{[\ell]}(c_0), F^{[\ell]}(c_{m-2})]$, there exists a unique $y \in [c_0, c_{m-2}]$ such that $F^{[\ell]}(y) = x$. We have $g(x) = g(F^{[\ell]}(y)) = F^{[\ell]}(g(y))$. Thus $g(x) = F^{[\ell]}(g_0(y))$.

It corresponds to the observation that for each $\ell \geq 0$, g maps $[F^{[\ell]}(c_{m-3}), F^{[\ell]}(c_{m-2})]$ homeomorphically onto $[F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$, order preserving.

Step 2. For each $\ell \geq 0$ and $x \in [F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$, there exists a unique $y \in [c_{m-2}, F(a)]$ such that $F^{[\ell]}(y) = x$. Further, by (9), there exists a unique $z \in [c_0, c_1]$ such that $g_0^{[m-2]}(z) = y$. We have $g(x) = g(F^{[\ell]}(y)) = F^{[\ell]}(g(g_0^{[m-2]}(z))) = F^{[\ell+1]}(z)$.

It reflects that g maps $[F^{[\ell]}(c_{m-2}), F^{[\ell+1]}(a)]$ onto $[F^{[\ell+1]}(a), F^{[\ell+1]}(c_1)]$ homeomorphically, order preserving.

Step 3. $g(0) = 0$.

Conversely, let $(c_j)_{j=0}^{m-1}$ be a strictly increasing sequence with $c_0 = a$ and $c_{m-1} = F(a)$, and let g_0 be an order preserving homeomorphism from $[c_0, c_{m-2}]$ onto $[c_1, c_{m-1}]$ satisfying (8), we can verify that the above three steps well define an extension of g_0 to a continuous increasing g satisfying (7).

3.2. For decreasing g on I .

Proposition 3. *Let $n \geq 3$ be odd. Let g be a strictly decreasing continuous $(m-1)$ -th order iterative root of $F(x) = x^n$ on I . Then the five interval types listed in Proposition 1 are confined further to (C_1) I is one of the degenerated singletons $\{-1\}$, $\{0\}$ and $\{1\}$, (C_2) $I = \mathbb{R}$ or $[-1, 1]$, or (C_3) $I = [a, b]$ where $-1 < a < 0$, $0 < b < 1$.*

PROOF. Amongst the five interval types (O_1) $a = -\infty$ and $b \in]-\infty, -1]$, or (O_2) $a = -1$ and $b = -1$, or (O_3) $a \in \{-\infty\} \cup [-1, 0]$ and $b \in [0, 1] \cup \{\infty\}$, or (O_4) $a = 1$ and $b = 1$, or (O_5) $a \in [1, \infty[$ and $b = \infty$, we shall rule some out quickly. When $a = -\infty$, b cannot be finite because the condition $\text{ran}(F|_I) \subset \text{ran}(g)$ cannot be met by finite b . For the same reason, a finite a cannot be paired with $b = \infty$. By Lemma 1, I cannot contain two points in $\{-1, 0, 1\}$ without having the third. This rules out both $a = -1$ and $0 \leq b < 1$, and $-1 < a \leq 0$ and $b = 1$. The case $a = 0$ and $0 < b < 1$ is not admissible because the order reversing g cannot map $[0, b]$ into $[0, b]$ while keeping 0 fixed. For the same reason the case $-1 < a < 0$ and $b = 0$ is not admissible. $\square$

In the following we give the construction of g on intervals I listed in Proposition 3. On a degenerated singleton I , the answer for g is trivial. For the rest of this subsection, we shall assume that I is not a singleton. Because F is strictly increasing, $m - 1$ must be even, say $m - 1 = 2k$.

Let

$$\phi := g^{[2]}.$$

Then ϕ is continuous, strictly increasing and satisfies

$$\phi^{[k]}(x) = x^n. \quad (10)$$

As illustrated in the previous subsection we can solve for all increasing ϕ from (10). By Lemma 1, ϕ has no periodic points other than 0, 1 and -1 and they are in fact fixed points of ϕ whenever they are in I .

For each solved ϕ , which is perhaps not a power function, we continue to solve for continuous and strictly decreasing g from

$$g^{[2]} = \phi. \quad (11)$$

CASE $a = -1$ and $b = 1$, or $a = -\infty$ and $b = \infty$.

In this case, the fixed points of ϕ are -1 , 0 and 1. Notice that F is a bijection on I and so is ϕ . Thus ϕ is an increasing homeomorphism of I onto I .

Theorem 11.2.5 in [4] is applicable, giving the construction of g . A fundamental region for g is $[x_0, \phi(x_0)]$ on $[-1, 1]$, or $[x_0, \phi(x_0)] \cup [\phi(y_0), y_0]$ on $\mathbb{R}$, where $-1 < x_0 < 0$ and $y_0 < -1$ are arbitrarily chosen.

CASE $-1 < a < 0$, $0 < b < 1$.

From $\text{ran}(\phi) = \text{ran}(g^{[2]}) \subset \text{ran}(g) \subset [a, b]$ we get $[\phi(a), \phi(b)] \subset [g(b), g(a)] \subset [a, b]$. Letting

$$[c, d] = \text{ran}(g), \quad \text{where } c = g(b), \quad d = g(a),$$

we note their relative positions in $\mathbb{R}$:

$$-1 < a \leq c \leq \phi(a) < 0 < \phi(b) \leq d \leq b < 1.$$

Consider the sequences

$$a, \phi(a), \phi^{[2]}(a), \dots, \phi^{[\ell]}(a), \dots$$

and

$$d, \phi(d), \phi^{[2]}(d), \dots, \phi^{[\ell]}(d), \dots$$

which are strictly increasing and strictly decreasing, respectively. They tend to 0, the unique fixed point of ϕ . Observe that g maps each term in the first sequence to a corresponding term in the second sequence, i.e.,

$$g(\phi^{[\ell]}(a)) = \phi^{[\ell]}(d) \quad \ell = 0, 1, 2, \dots \quad (12)$$

as g and ϕ commute. Moreover, each term in the second sequence is mapped by g to a shifted term in the first sequence, i.e.,

$$g(\phi^{[\ell]}(d)) = \phi^{[\ell+1]}(a) \quad \ell = 0, 1, 2, \dots \quad (13)$$

The interval $[a, \phi(a)]$ is a fundamental region for g as we shall explain.

(1) The initial g_0 maps $[a, \phi(a)]$ homeomorphically onto $[\phi(d), d]$, $g_0(a) = d$, $g_0(\phi(a)) = \phi(d)$ and $d \in [\phi(b), b]$.

(2) g_0 determines g elsewhere as follows:

Step 1. For each $\ell \geq 1$ and $x \in [\phi^{[\ell]}(a), \phi^{[\ell+1]}(a)]$, there exists a unique $y \in [a, \phi(a)]$ such that $\phi^{[\ell]}(y) = x$. We have $g(x) = g(\phi^{[\ell]}(y)) = \phi^{[\ell]}(g(y))$. Thus $g(x) = \phi^{[\ell]}(g_0(y))$.

Step 2. For each $\ell \geq 0$ and $x \in [\phi^{[\ell+1]}(d), \phi^{[\ell]}(d)]$, there exists a unique $y \in [\phi^{[\ell]}(a), \phi^{[\ell+1]}(a)]$ such that $g(y) = x$. We have $g(x) = g(g(y)) = \phi(y)$.

Step 3. For each $x \in [d, b]$, we see $\phi(x) \in [\phi(d), \phi(b)] \subset [\phi(d), d]$. Since g_0 maps $[a, \phi(a)]$ homeomorphically onto $[\phi(d), d]$, there exists a unique

$y \in [a, \phi(a)]$ such that $g(y) = \phi(x)$. Because $\phi(x) = g(g(x))$ and g is injective, we have $g(x) = y$.

So g maps $[d, b]$ homeomorphically onto $[c, \phi(a)]$, order reversing.

Step 4. $g(0) = 0$ according to Lemma 1.

Conversely, it is straight forward to check that if g_0 is a continuous order reversing map of the interval $[a, \phi(a)]$ onto the interval $[\phi(d), d]$ for some $d \in [\phi(b), b]$, then the above four steps well define a continuous extension g on $[a, b]$ which satisfies $g^{[2]} = \phi$.

4. Cases of even n

In this final section, we determine the continuous $(m-1)$ -th order iterative roots of $F(x) = x^n$ on I when n is an even positive integer. According to Proposition 1, there are four types of $I = [a, b] \cap \mathbb{R}$ to cover: Case (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$.

Lemma 2. *Let $n \geq 2$ be even. Let g be a continuous $(m-1)$ -th order iterative root of $F(x) = x^n$ on I . Then (i) g has no periodic points other than 0 and 1, and they are in fact fixed points of g whenever they are in I , (ii) g is strictly increasing on $I_+ := I \cap [0, +\infty[$, and is strictly decreasing on $I_- := I \cap]-\infty, 0]$.*

PROOF. (i) If x_0 is a periodic point of g , then x_0 must be a fixed point of an iterate of F . For any integer $k \geq 1$ the function $F^{[k]}(x) = x^{n^k}$ has exactly two fixed points at 0 and 1. This proves that x_0 can only be 0 or 1. Conversely, every fixed point x_0 of F , if it is in I , is a periodic point of g . So 0 and 1 are indeed periodic points of g whenever they are in I . Having no third fixed point between 0 and 1, they cannot form a 2-cycle under the continuous g . This shows that 0 and 1 are in fact fixed points of g whenever they are in I .

(ii) With even $n \geq 2$, F is injective on $]-\infty, 0]$ and on $[0, +\infty[$. Repeating the arguments in the proof of Lemma 1, we see that g must be strictly monotonic on each of the two intervals $I \cap]-\infty, 0]$ and $I \cap [0, +\infty[$. Consider the four types of I .

Case (E_4) : $a \in [1, \infty[$ and $b = \infty$. Subcase 1. Suppose $a > 1$. By (i), $g(a) \neq a$. The interval I being g -invariant, we must have $g(a) > a$. If g were strictly decreasing, we would have $a \leq g^{(2)}(a) < g(a)$. The compact interval $[a, g(a)]$ will then be g -invariant and must contain a fixed point of g . This is a contradiction to (i), that g has no fixed point in I when $a > 1$. So g is strictly increasing. Subcase 2. $a = 1$. By (i), $g(a) = a$. As I is g -invariant, g cannot be strictly decreasing.

Case (E_1) : $a \in [-1, 0]$ and $b \in [a^n, 1]$. Subcase 1. Suppose $a < 0$. By (i), $g(0) = 0$. First, we observe that if g were strictly increasing on both I_+ and I_- , then it is strictly increasing on I . This would imply all its iterates, including F , are strictly increasing on I . But F is not strictly increasing. This contradiction shows that g cannot be strictly increasing on both I_+ and I_- . For similar reasons it cannot be strictly decreasing on both I_+ and I_- . Next, if g were strictly increasing on I_- and strictly decreasing on I_+ , then g and therefore all its iterates will have their range included in I_- . This is a contradiction as F does not comply with this property. So we reached the conclusion that g is strictly increasing on I_+ and strictly decreasing on I_- . Subcase 2. Suppose $a = 0$. The argument for g strictly increasing is the same as that of Subcase 2 in Case (E_4) .

Case (E_2) : $a \in [-\infty, 0]$ and $b = \infty$. Subcase 1. Suppose $a < 0$. In this case, 0 is an interior point of I and the proof given above for Subcase 1 in Case (E_1) is also valid. Subcase 2. Suppose $a = 0$. The argument for an increasing g is the same as that of Subcase 2 in Case (E_4) .

Case (E_3) : $a = 1$ and $b = 1$. The statements are trivial. $\square$

Lemma 3. *Let $n \geq 2$ be even. Let g be a continuous $(m-1)$ -th order iterative root of $F(x) = x^n$ on $I = [a, b] \cap \mathbb{R}$ which contains 0 as an interior point. Then g is even on the maximal subinterval of I symmetric about 0. Furthermore, it can be extended uniquely to a continuous $(m-1)$ -th order iterative root $\bar{g}$ of F on the balanced interval $\bar{I} := [-c, c] \cap \mathbb{R}$ where $c = \max(|a|, b)$, with $\text{ran}(\bar{g}) = \text{ran}(g) \subset I$.*

PROOF. Let $x \in I$ and assume $-x \in I$. By Lemma 2, g is minimized at the fixed point 0. So $y_1 := g(x)$ and $y_2 := g(-x)$ are both in I_+ . We also have $g^{[m-2]}(y_1) = g^{[m-1]}(x) = F(x)$ and $g^{[m-2]}(y_2) = g^{[m-1]}(-x) = F(-x)$. Because F is even, it follows that $g^{[m-2]}(y_1) = g^{[m-2]}(y_2)$. The strict monotonicity of g on I_+ which is g -invariant yields $y_1 = y_2$. This

proves the evenness of g , that $g(x) = g(-x)$ whenever both x and $-x$ are in I . The function $\bar{g} : \bar{I} \rightarrow \bar{I}$ given by

$$\bar{g}(x) = \begin{cases} g(x), & \forall x \in I \\ g(-x), & \forall x \in -I \end{cases}$$

is thus well defined. It is straight forward to check that on the interval $\bar{I} = I \cup (-I)$, $\bar{g}$ is again a continuous $(m-1)$ -th iterative root of F . It is immediate from the definition of $\bar{g}$ that $\text{ran}(\bar{g}) = \text{ran}(g)$. The function $\bar{g}$ is clearly the unique even extension of g from I to $\bar{I}$. $\square$

Combining the above two lemmas we arrive at the following conclusion for the Section. The convention $-(-\infty) = \infty$ will be in use.

Proposition 4. *Let $F(x) = x^n$, $n \geq 2$ even, $I = [a, b] \cap \mathbb{R}$ F -invariant, and let $g : I \rightarrow I$ denote a continuous $(m-1)$ -th order iterative root of F . (i) When $a \geq 0$, g exists (for every $m > 2$) and its general construction is given in the same manner as in the discussions in Section 3.1. (ii) When $a < 0$ and $-a \leq b$, g exists. Its restriction to I_+ , denoted by g_* , is a root of $F|_{I_+}$ and g is the even extension of g_* to I . Conversely, for every root g_* of $F|_{I_+}$ whose general construction is covered in (i), the even extension $g(x) := g_*(|x|)$ to I (which is not necessarily the balanced $\bar{I}$) is a root on I . (iii) When $a < 0$ and $-a > b > a^n$, g exists. It is the restriction, to I , of some root $\bar{g} : [a, -a] \rightarrow [a, -a]$ of F on $[a, -a]$ satisfying the extra range condition $\text{ran}(\bar{g}) \subset [0, b]$. (iv) When $a < 0$ and $-a > b = a^n$, g does not exist (for every $m > 2$).*

PROOF. Recall that I belongs to one of the following types: (E_1) $a \in [-1, 0]$ and $b \in [a^n, 1]$, or (E_2) $a \in [-\infty, 0]$ and $b = \infty$, or (E_3) $a = 1$ and $b = 1$, or (E_4) $a \in [1, \infty[$ and $b = \infty$.

CASE 1. Suppose $a \geq 0$.

This is the case where $I = I_+$. According to Lemma 2, g is strictly increasing having 0 and 1 as its only fixed points provided they are in I . Just like the discussions in section 3.1, its existence as well as its general construction are given by Theorem 11.2.2 in [4].

CASE 2. Suppose $a < 0$.

This is the case where 0 is an interior point of I . According to Lemma 3, g is even and has an extension $\bar{g} : \bar{I} \rightarrow \bar{I}$, $\bar{I} = I \cup (-I)$.

Furthermore, the extension satisfies the range condition

$$\text{ran}(\bar{g}) \subset I. \quad (14)$$

Subcase 1. Suppose $-a \leq b$.

In this subcase, $x \in I_-$ implies $-x \in I_+$. The consideration of the extension $\bar{g}$ is not crucial. By Lemma 2, $g_* := g|_{I_+}$ is an increasing root of F on I_+ . The existence and general construction of g_* are attended in Case 1. On the interval I_- , g is determined by its evenness $g(x) = g_*(-x)$.

Subcase 2. Suppose $-a > b$.

This can only occur within Case (E_1) , with $a \in]-1, 0[$ and $b \in [a^n, -a[$. We first attend the general construction of $\bar{g}_* := \bar{g}|_{\bar{I}_+}$ which is strictly increasing. The interval $\bar{I}_+$ is $[0, c]$ where $c = -a$. While Theorem 11.2.2 in [4] gives the general construction of all roots $\bar{g}_*$ mapping $[0, c]$ into $[0, c]$, we must confine ourselves to those meeting the stronger range condition (14) which requires that $\bar{g}_*$ maps $[0, c]$ into $[0, b]$. Its initialization on $[0, c]$ is, in part, based on the following:

The sequence

$$c_j := \bar{g}_*^{[j]}(c), \quad j = 0, \dots, m-1 \quad (15)$$

satisfies

$$1 > c = c_0 > c_1 > c_2 > \dots > c_{m-2} > c_{m-1} = c^n > 0 \quad (16)$$

and

$$c_1 \leq b. \quad (17)$$

Subcase 2.1. Suppose $a \in]-1, 0[$ and $b = a^n$.

For $m > 2$ there is no sequence satisfying (16)–(17). So F has no root.

Subcase 2.2. Suppose $a \in]-1, 0[$ and $b > a^n$. Then the conditions (16)–(17) can be met by some sequence, $\bar{g}_*$ exists, and a fundamental region for its construction is $[c_{m-2}, c_0]$. It determines g by

$$g(x) = \bar{g}_*(|x|) \quad \forall x \in I. \quad (18)$$

□

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References

- [1] K. BARON and W. JARCZYK, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* **61** (2001), 1–48.
- [2] R. CHENG, A. DASGUPTA, B. R. EBANKS, L. F. KINCH, L. M. LARSON and R. B. MCFADDEN, When does $f^{-1} = 1/f$?, *Amer. Math. Monthly* **105** (1998), 8: 704–717.
- [3] R. EULER and J. FORAN, On functions whose inverse is their reciprocal, *Math. Mag.* **54** (1981), 185–189.
- [4] M. KUCZMA, B. CHOCZEWSKI and R. GER, Iterative functional equations, Encyclopedia of Math. & Its Applications Vol. 32, *Cambridge University Press, Cambridge*, 1990.
- [5] A. MUKHERJEA and J. S. RATTI, On a functional equation involving iterates of a bijection on the unit interval, *Nonlinear Anal.* **7** (1983), 899–908.
- [6] W. ZHANG, Solutions of equivariance for a polynomial-like iterative equation, *Proc. Roy. Soc. Edinburgh* **130A** (2000), 1153–1163.

CHE TAT NG
 DEPARTMENT OF PURE MATHEMATICS
 UNIVERSITY OF WATERLOO
 WATERLOO, ONTARIO N2L 3G1
 CANADA

WEINIAN ZHANG
 DEPARTMENT OF MATHEMATICS
 SICHUAN UNIVERSITY
 CHENGDU, SICHUAN 610064
 CHINA

E-mail: matzwn@sohu.com

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