

Congruence lattices of modular lattices

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1. Introduction

In 1974 I have proved the following (see [1]):

Theorem. *Every finite distributive lattice is the congruence lattice of some modular lattice.*

In this note we give a short, new proof of this result. We will use two well-known lattice constructions.

2. Preliminaries

Let L be a lattice and let P be a partially ordered set, L^P denotes the lattice of all order preserving maps of P to L , partially ordered by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. Then L^P is a special subdirect power of L . If L is the two element lattice 2 then 2^P is a distributive lattice.

Every finite distributive lattice D can be represented in this form, where P is the dual of the poset of all nonzero join-irreducible element of D . Let R be a well-ordered chain, R^d denotes the dual of R . Let $R + 1$ denote the lattice obtained from R by adjoining a new unit element.

Lemma 1. *Let R be a well-ordered chain, then $2^{R^d} \cong R + 1$.*

If $a \in L$ then $\bar{a}$ denotes the corresponding constant mapping: $\bar{a}(x) = a$ for all $x \in P$. The elements $\bar{a}$ ($a \in L$) form a sublattice of L^P , which is obviously isomorphic to L ; we identify L with this sublattice. Let $[a, b]$ be a prime-interval of L , then the corresponding interval $[\bar{a}, \bar{b}]$ of L^P is isomorphic to 2^P .

$\mathcal{M}_3$ denotes the five-element modular nondistributive lattice. The elements of $\mathcal{M}_3$ are $o < a, b, c < i$. In [2] it was proved the following:

Lemma 2. *Every congruence relation of $\mathcal{M}_3^P$ is determined by its restriction to the ideal $(\bar{a}]$, and conversely every congruence relation of this ideal can be extended to $\mathcal{M}_3^P$.*

If P is a cain, then $\mathcal{M}_3^P$ can be easily visualized. Consider the following three sublattices of $\mathcal{M}_3$: $E = \{o, i\}$, $F = \{o, c, i\}$ and $G = \{o, a, b, i\}$. Then E^P , F^P , G^P are sublattices of $\mathcal{M}_3^P$ and $E^P = F^P \cap G^P$ holds. Moreover, it is easy to see that $E^P \cong 2^P$, $G^P \cong 2^P \times 2^P$, while F^P is isomorphic to the following lattice $\{(x, y) \in 2^P \times 2^P : x \leq y\}$. G^P is a “square” and F^P is a “half square”. F^P is called a flap. If P is three element chain that $\mathcal{M}_3^P$ is illustrated by *Figure 1*. (The elements of F^P are the black circles.) It is clear, that $\mathcal{M}_3^P = F^P \cup G^P$.

Figure 1.

Let C_1 and C_2 be two chains. The direct product $C_1 \times C_2$ we shall call the “grid”; its elements are the “grid elements”. We augment the grid as follows: let $a, b \in C_1$, $a < b$, $c, d \in C_2$, $c < d$ and assume that the intervals are isomorphic. Then we add a flap to $[a, b] \times [c, d] = [(a, c), (b, d)]$ such that this flap with the direct product $[a, b] \times [c, d]$ is a lattice isomorphic to $\mathcal{M}_3^P$ where P denotes the dual of the poset $\{x : a \leq x < b\}$. If $[a, b]$ and $[c, d]$ are prime-intervals then in the lattice $C_1 \times C_2$ the interval $[(a, c), (b, d)]$ is a prime square. In this case we add to $C_1 \times C_2$ only one new element m , and $(a, c), (a, d), (b, c), (b, d), m$ form a sublattice isomorphic to $\mathcal{M}_3$ (the “flap” contains three elements: $(a, c), (b, d)$ and m). If we have a family of disjoint squares then we can apply this augmentation simultaneously.

The second construction is the Hall-Dilworth gluing: if a nonempty filter $\mathcal{F}$ of a lattice L_0 is isomorphic to an ideal $\mathcal{I}$ of a lattice L_1 , let L be the union of L_0 and L_1 with the elements of $\mathcal{F}$ and $\mathcal{I}$ identified via the isomorphism. L can be ordered with the transitive closure of the union of the orders on L_0 and L_1 . Then under this order L is a lattice; L_0 is an ideal of L and L_1 is a filter of L . If L_0 and L_1 are both modular then L is a modular lattice. A congruence relation Θ of $\mathcal{F} = \mathcal{I}$ can be extended to L if and only if Θ can be extended to L_0 and L_1 .

The ordinary sum of the lattices K_0 and K_1 will be denoted by $K_0 \oplus K_1$, we place K_1 on the top of K_0 and identify the unit element of K_0 with the zero of K_1 . (This is obviously a special Hall-Dilworth gluing, where $\mathcal{I}$ is the zero element of K_0 and $\mathcal{F}$ is the unit element of K_1).

It is easy to see that the augmented grid can be defined as repeated gluing of lattices which are either isomorphic to the direct product of two chains or they are isomorphic to $\mathcal{M}_3^R$ for some chain R .

3. Proof

For every finite poset P we have to construct a modular lattice L_P such that $\text{Con } L_P \cong 2^P$. We use induction on the size of P . If $|P| = 1$, i.e. $2^P \cong 2$, then L_P is the two element chain, 2. We construct L_P having the following two properties:

- (1) L_P has an element a_P with a complement a'_P , and the filter $[a_P]$ is a well-ordered chain.
- (2) L_P contains a subchain $a_P = b_0 < b_1 < \dots < b_n = 1_P$, where $n = |P|$ and the irreducible congruences of L_P are exactly the congruences in the form $\Theta(b_{i-1}, b_i)$ ($i = 1, 2, \dots, n$).

It is clear, that for $|P| = 1$ the lattice $L_1 \cong 2$ satisfies these properties. Let p be a minimal element of P , where $|P| = n > 1$. Then by our assumption for the poset $Q = P \setminus \{p\}$ there exists a modular lattice L_Q satisfying (1), (2) and $\text{Con } L_Q \cong 2^Q$. The element a_Q is given in (1). By condition (2) L_Q contains a chain $a_Q = b_0 < b_1 < \dots < b_{n-1} = 1_Q$, and the join-irreducible congruences of L_Q are the principal congruences $\Theta(b_{k-1}, b_k)$ ($k = 1, 2, \dots, n-1$), consequently we have a bijection $\varphi([b_{k-1}, b_k]) = p_k \in Q$, the map φ is called a coloring, p_k is the color of the interval $[b_{k-1}, b_k]$. Assume that $p_{k_1}, p_{k_2}, \dots, p_{k_r}$, are the covers of p in the poset P , i.e. $p \prec p_{k_j}$ ($j = 1, 2, \dots, r$). Let C be the chain $[a_Q] \subseteq L_Q$. Then $a_Q = b_0 < b_1 < \dots < b_{n-1} = 1_Q$ is a subchain of C . For every natural number i let C_i be a chain isomorphic to $[b_{k_1-1}, b_{k_1}] \oplus \dots \oplus [b_{k_r-1}, b_{k_r}]$ and b_k^i denotes the image of b_k under this isomorphism. Finally, we consider the ordinary sum of these chains with a new unit element 1^* adjoined, i.e. $\mathcal{C} = \{C_0 \oplus C_1 \oplus C_2 \oplus \dots\} \cup \{1^*\}$.

We extend φ to $\mathcal{C}$ as follows: $\varphi([b_{k-1}^i, b_k^i]) = p_k \in Q$ for $i = 0, 1, 2, \dots$.

Consider the grid $\mathcal{C} \times \mathcal{C}$ and for every i and k the square $[(b_{k-1}^i, b_{k-1}), (b_k^i, b_k)] = [b_{k-1}^i, b_k^i] \times [b_{k-1}, b_k]$. By the definition of φ , $\varphi([b_{k-1}^i, b_k^i]) = \varphi([b_{k-1}, b_k]) = p_k$, i.e. this is a monochromatic square. By Lemma 1 there exists a poset P_k such that $[b_{k-1}, b_k] \cong 2^{P_k}$ (indeed P_k is the dual of the chain $\{x \in [b_{k-1}, b_k] : x < b_k\}$). We extend all monochromatic squares to $\mathcal{M}_3^{P_k}$ as described in paragraph 2 for all $k \in \{k_1, k_2, \dots, k_r\}$ (don't forget that $p_{k_1}, p_{k_2}, \dots, p_{k_r}$ are the covers of p), the resulting lattice is C_w .

In the lattice C_w the interval $[(b_{k_0}^0, b_0), (b_{k_0}^0, b_{n-1})]$ is isomorphic to C . This interval is an ideal of C_w . On the other hand C is a filter of L_Q . Now, we apply the Hall-Dilworth gluing for the lattices L_Q and C_w , we obtain the lattice T . Then $(b_{k_0}^0, b_k)$ is identified with b_k . (See *Figure 2*.)

Figure 2.

Lemma 3. $\text{Con } T \cong 2^P$.

PROOF. We determine the irreducible congruence relations of T . By Lemma 2 every congruence relation of C_w is determined by its projections to $\mathcal{C}$ and C . On the other hand every congruence relation of L_Q is determined by its restriction to filter $C = [a_Q]$. Then we have:

- (*) Every congruence relation of T is determined by its restriction to the subchains $\mathcal{C}$ and C .

Let Θ be an irreducible congruence relation of T . We distinguish two cases:

Case 1. $(1^*, b_0) \not\equiv (c, b_0)(\Theta)$ for all $c \in \mathcal{C}$. If $b_{k-1} \leq x < y \leq b_k$ for some $k \in \{k_1, k_2, \dots, k_r\}$ in the chain C and $(x, b_0) \equiv (y, b_0)(\Theta)$ then by Lemma 2 $(b_{k_0}^0, x) \equiv (b_{k_0}^0, y)(\Theta)$ holds, i.e. Θ is determined by its restriction to C . This proves that Θ is the extension of a congruence relation in the form $\Theta(b_{k-1}, b_k)$ of L_Q , i.e. $\Theta = \bar{\Theta}(b_{k-1}, b_k)$.

It is easy to see that every $\Theta = \Theta(b_{k-1}, b_k)$ can be extended to T . We describe the Θ classes.

If $k \in \{k_1, k_2, \dots, k_r\}$ i.e. p_k is a cover of p in the poset P then the nontrivial Θ -classes on C_w are the intervals: $[(b_{k-1}^i, x), (b_k^i, x)]$, $[(y, b_{k-1}), (y, b_k)]$ and the monochromatic squares $[(b_{k-1}^i, b_{k-1}), (b_k^i, b_k)]$, where $x \in C$, $y \in \mathcal{C}$.

If $k \notin \{k_1, k_2, \dots, k_r\}$ then the nontrivial Θ -classes on C_w are the intervals $[(y, b_{k-1}), (y, b_k)]$.

It is easy to prove that these relations are indeed congruences.

Case 2. $(1^*, b_0) \equiv (c, b_0)(\Theta)$ for some $c \in \mathcal{C}$, $c < 1^*$. Then by the definition of $\mathcal{C}$ there exists a natural number i , with the property: $c \in C_{i-1} \subseteq \mathcal{C}$. In this case $(b_k^i, b_0) \equiv (b_{k-1}^i, b_0)(\Theta)$ for $k \in \{k_1, \dots, k_r\}$, i.e. $(1^*, b_0) \equiv (b_{k_0}^0, b_0) = a_Q(\Theta)$ and $(b_{k_0}^0, b_k) \equiv (b_{k_0}^0, b_{k-1})(\Theta)$. This proves that $\Theta \geq \bar{\Theta}(b_{k-1}, b_k)$ for all $k \in \{k_1, \dots, k_r\}$, and $\Theta_p = \Theta((b_{k_0}^0, b_0), (1^*, b_0))$. Denote this congruence relation by Θ_p then we have

$$\Theta \geq \bar{\Theta}(b_{k-1}, b_k) \text{ if and only if } k \in \{k_1, k_2, \dots, k_r\}.$$

φ can be extended on the following way: $\varphi([(b_{k_0}^0, b_0), (1^*, b_0)]) = p$. Now φ is a bijection between $J(\text{Con } T)$ and P^d , i.e. $\text{Con } T \cong 2^P$.

The lattice T does not satisfy condition (2), therefore we define L_P as an extension of T .

The chain $\mathcal{C}$ can be represented in the form 2^{R^d} with a well-ordered chain R . Let 0_T be the zero element of T . First we consider the direct product $S_0 \cong [0_T, a_Q] \times 2^{R^d}$. This lattice has the elements $(a_Q, 0)$, $(0_T, 1)$, $(a_Q, 1)$, where 0 is the zero of 2^{R^d} and 1 is the unit element of 2^{R^d} . $\mathcal{M}_3$ has the canonical embedding into $\mathcal{M}_3^{R^d}$, i.e. $\bar{o} < \bar{a}$, $\bar{b}$, $\bar{c} < \bar{i}$ is a sublattice of $\mathcal{M}_3^{R^d}$ isomorphic to $\mathcal{M}_3$. The ideal $[\bar{o}, \bar{b}]$ of $\mathcal{M}_3^{R^d}$ is isomorphic to 2^{R^d} and the filter $[(a_Q, 0)]$ of S is isomorphic to 2^{R^d} . Then we apply the Hall-Dilworth gluing construction for $\mathcal{M}_3^{R^d}$ and S_0 . The resulting lattice is S . Finally we apply the Hall-Dilworth gluing construction for the lattice S and T as follows: the ideal $[(1^*, b_0)]$ of T and the filter $[(0_T, 1)]$ of S are isomorphic. Then we identify by the gluing the elements $\bar{b} \in S$ and $a_Q \in T$. (see Figure 3.). This lattice is L_P . By Lemma 2. $\bar{b} \equiv \bar{1}(\Theta)$ iff $\bar{a} \equiv \bar{1}(\Theta)$, i.e.

$\Theta(\bar{o}, \bar{1}) = \Theta_p$. Let a_p be the element $\bar{a}$ then the chain required in condition (2) is the following: $a_P = b'_0$, $b'_1 = a_P \vee b_0$, $b'_2 = a_P \vee b_1, \dots, b'_n = a_P \vee b_{n-1}$, i.e. we have a lattice L_P such that $\text{Con } L_P \cong 2^P$ and conditions (1), (2) are satisfied.

Figure 3.

References

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