

On generalized h -recurrent Finsler connection with deflection and torsion

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Summary. In 1934 E. CARTAN [1] published his monograph ‘Les espaces de Finsler’ and fixed his method to determine a notion of connection in the Geometry of Finsler space. MATSUMOTO [4] determined uniquely the Cartan connection CT by the following conditions: (1) The connection is metrical; (2) the deflection tensor field vanishes; (3) the torsion tensor field T vanishes; (4) the torsion tensor field S vanishes.

HOJO [3] introduced the connections, which depend on a real parameter p and make the v -covariant derivative $\varphi_{ij||k}^{(p)}$ of $\varphi_{ij}^{(p)} (= \dot{\partial}_i \dot{\partial}_j L^P)$ zero just as $g_{ij|k} = 0$ in case of CT . The Cartan connection is really the case when p takes the value two and so the connection determined by Hojo is a generalization of CT .

Recently B.N. PRASAD and LALJI SRIVASTAVA ([7]) have investigated the generalized h -recurrent Finsler connection which is deflection and torsion free. In this paper we investigate a generalized h -recurrent Finsler connection with given deflection- and torsion-tensor fields.

1. Introduction

A Finsler manifold (F^n, L) of dimension n is a manifold F^n associated with a fundamental function $L(x, y)$, where $x = (x^i)$ denotes the positional variable of F^n and $y = (y^i)$ denote the components of a tangent vector with respect to (x^i) . Throughout the following, L is assumed to be positively homogeneous of degree one with respect to (y^i) . The metric tensor of (F^n, L) is given by $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ where $\dot{\partial}_i = \partial / \partial y^i$.

A Finsler connection of (F^n, L) is a triad $(F_{jk}^i, N_k^i, C_{jk}^i)$ of an h -connection F_{jk}^i , a non-linear connection N_k^i and a vertical connection C_{jk}^i (MATSUMOTO [5]). If a Finsler connection is given, the h - and v -covariant derivatives of any tensor field V_j^i are defined as

$$(1.1) \quad V_{j|k}^i = d_k V_j^i + V_j^m F_{mk}^i - V_m^i F_{jk}^m$$

$$(1.2) \quad V_j^i|_k = \dot{\partial}_k V_j^i + V_j^m C_{mk}^i - V_m^i C_{jk}^m,$$

where $d_k = \partial_k - N_k^m \dot{\partial}_m$, $\partial_k = \partial/\partial x^k$.

For any Finsler connection $(F_{jk}^i, N_k^i, C_{jk}^i)$ the h - v -curvature tensor P_{hjk}^i is given by ([6])

$$(1.3) \quad P_{hjk}^i = \dot{\partial}_k F_{hj}^i - C_{hk|j}^i + C_{hm}^i P_{jk}^m$$

2. Generalized h -recurrent Finsler connection

Let $p \neq 1$ be a real number. We define $\phi^{(p)}(x, y)$ as

$$(2.1) \quad \phi^{(p)} = \frac{1}{p} L^p \quad (p \neq 0), \quad \phi^{(0)} = \log L.$$

We denote $\dot{\partial}_i \phi^{(p)}$ and $\dot{\partial}_i \dot{\partial}_j \phi^{(p)}$ as $\phi_i^{(p)}$ and $\phi_{ij}^{(p)}$ and so on. Thus

$$(2.2) \quad \phi_i^{(p)} = L^{(p-1)} \ell_i, \quad \phi_{ij}^{(p)} = L^{(p-2)} (g_{ij} + (p-2) \ell_i \ell_j).$$

In the following, we restrict our considerations to a domain, where the matrix $\|\phi_{ij}^{(p)}\|$ is regular and then its inverse $\phi^{(p)ij}$ is given by

$$(2.3) \quad \phi^{(p)ij} = L^{-(p-2)} \left[g^{ij} - \frac{(p-2)}{(p-1)} \ell^i \ell^j \right].$$

Differentiating (2.2) by y^k , we have

$$(2.4) \quad \begin{aligned} \phi_{ijk}^{(p)} = L^{(p-2)} & \left[2C_{ijk} + (p-2)L^{-1} \{ h_{ij} \ell_k + h_{jk} \ell_i \right. \\ & \left. + h_{ki} \ell_j + (p-1) \ell_i \ell_j \ell_k \} \right]. \end{aligned}$$

To avoid confusion, we denote h - and v -covariant derivatives with respect to Cartan's connection by $|_k$ and $|_k$, while these covariant derivatives with respect to a generalized h -recurrent Finsler connection will be denoted by $\|_k$ and $\|_k$ respectively. The quantities corresponding to a generalized h -recurrent Finsler connection will be denoted by putting p on the top of the quantity while the quantities corresponding to Cartan's connection will be denoted as usual.

Recently PRASAD and L. SRIVASTAVA [7] have introduced a generalized h -recurrent Finsler connection $\{F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i}\}$ which is determined uniquely by the following axioms:

- (C_1) The connection is h -recurrent with respect to the vector field a_k i.e. $g_{ij\|k} = a_k g_{ij}$

(C₂) the v -covariant derivative of $\phi_{ij}^{(p)}$ vanishes i.e. $\phi_{ij}^{(p)} \parallel_k = 0$

(C₃) the deflection tensor field $D_k^{(p)i}$ vanishes i.e.

$$D_k^{(p)i} = F_{jk}^{(p)i} y^j - N_k^{(p)i} = 0$$

(C₄) the torsion tensor field $T_{jk}^{(p)i}$ vanishes i.e.

$$T_{jk}^{(p)i} = F_{jk}^{(p)i} - F_{kj}^{(p)i} = 0$$

(C₅) the torsion tensor field $S_{jk}^{(p)i}$ vanishes i.e.

$$S_{jk}^{(p)i} = C_{jk}^{(p)i} - C_{kj}^{(p)i} = 0$$

In this paper we omit conditions (C₃), (C₄) and investigate a generalized h -recurrent Finsler connection with given deflection- and torsion-tensor fields.

3. Generalized h -recurrent Finsler connections with deflection and torsion

1. First we investigate connections where the nonlinear connection and the $(h)h$ -torsion are prefixed.

Theorem 3.1. *Given in a Finsler space, a nonlinear connection $N_k^{(p)i}$, a skew symmetric (1,2) tensor field $T_{jk}^{(p)i}$ and a covariant vector field a_k , there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying axioms (C₁), (C₂), (C₅) and the new axioms (C_{3'}): the nonlinear connection is the given $N_k^{(p)i}$; (C_{4'}): the (h) h -torsion tensor field is the given $T_{jk}^{(p)i}$.*

PROOF. From (C₂) it follows that

$$\phi_{ij}^{(p)} \parallel_k = \phi_{ijk}^{(p)} - \tilde{C}_{ijk}^{(p)} - \tilde{C}_{jik}^{(p)} = 0,$$

where

$$\tilde{C}_{ijk}^{(p)} = \phi_{rj}^{(p)} C_{ik}^{(p)r}.$$

By cyclic permutation of the indices i, j and k , we get

$$\tilde{C}_{ijk}^{(p)} = (1/2)[\phi_{ijk}^{(p)} + \phi_{jki}^{(p)} - \phi_{kij}^{(p)}] = (1/2)\phi_{ijk}^{(p)},$$

which implies

$$(3.1) \quad C_{ik}^{(p)r} = (1/2)\phi^{(p)rj} \phi_{ijk}^{(p)} = C_{ik}^r + \sigma_{ik}^{(p)r},$$

where $\sigma_{ik}^{(p)r}$ are given as below by (2.3) and (2.4),

$$(3.2) \quad \sigma_{ik}^{(p)r} = \{(p-2)/2L\}[\delta_i^r \ell_k + \delta_k^r \ell_i + h_{ik} \ell^r / (p-1) - \ell_i \ell_k \ell^r].$$

From the axioms (C1) and (C3') we have

$$\partial_k g_{ij} - N_k^{(p)m} \dot{\partial}_m g_{ij} - g_{mj} F_{ik}^{(p)m} - g_{im} F_{jk}^{(p)m} = a_k g_{ij}.$$

Applying the Christoffel process to the above equation and using axiom (C4'), we get

$$(3.3) \quad \begin{aligned} F_{jk}^{(p)i} = & \gamma_{jk}^i - (C_{km}^i N_j^{(p)m} + C_{jm}^i N_k^{(p)m} - g^{hi} C_{jkm} N_h^{(p)m}) \\ & - \frac{1}{2}(a_j \delta_k^i + a_k \delta_j^i - a^i g_{jk}) + A_{jk}^{(p)i}, \quad \text{where} \end{aligned}$$

$$(3.4) \quad \gamma_{jk}^i = \frac{1}{2} g^{ih} (\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}),$$

$$(3.5) \quad \begin{aligned} A_{jk}^{(p)i} = & \frac{1}{2} (T_{kjh}^{(p)} g^{hi} + T_{jkh}^{(p)} g^{hi} + T_{jk}^{(p)i}) \\ & a^i = g^{ij} a_j \quad \text{and} \quad T_{kjh}^{(p)} = g_{jr} T_{kh}^{(p)r}. \end{aligned}$$

In view of (3.1), (3.3) and axiom (C3') it is clear that the Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ is uniquely determined from the metric function L and from the given vector fields $a_k, T_{jk}^{(p)i}$.

2. For the above connection the deflection tensor field $D_k^{(p)i}$ defined in (C3) is obtained by contracting (3.3) by y^j

$$(3.6) \quad \begin{aligned} D_k^{(p)i} = & G_k^i + 2C_{km}^i G^m - C_{km}^i N_o^{(p)m} - N_k^{(p)i} \\ & - \frac{1}{2}(a_o \delta_k^i + a_k y^i - a^i y_k) + A_{ok}^{(p)i}, \quad \text{where} \end{aligned}$$

$$(3.7) \quad G_k^i = \dot{\partial}_k G^i = \gamma_{ok}^i - 2C_{km}^i G^m,$$

$$(3.8) \quad G^i = \frac{1}{2} \gamma_{oo}^i.$$

The Suffix 'o' denotes contraction with respect to the element of support y^i .

Contracting (3.6) with y^k , we get

$$(3.9) \quad N_o^{(p)i} = 2G^i - D_o^{(p)i} - a_o y^i + \frac{1}{2} a^i L^2 + A_{oo}^{(p)i}.$$

Substituting the value of $N_o^{(p)i}$ in (3.6) and using $C_{jk}^i y^j = 0$, $C_{jhk} y^j = 0$, we get

$$\begin{aligned} N_k^{(p)i} &= G_k^i - C_{km}^i (A_{oo}^{(p)m} - D_o^{(p)m} + \frac{1}{2} a^m L^2) \\ &\quad + (A_{ok}^{(p)i} - D_k^{(p)i}) - \frac{1}{2} (a_o \delta_k^i + a_k y^i - a^i y_k). \end{aligned}$$

Hence we have the following

Theorem 3.2. *Given in a Finsler space a (1,1) tensor field $D_k^{(p)i}$, a covariant vector field a_k and a skew-symmetric (1,2) tensor field $T_{jk}^{(p)i}$ there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying the axioms (C1), (C2), (C4'), (C5) and the new axiom (C3''): the deflection tensor field is the given $D_k^{(p)i}$.*

3. The v -connection $F_{jk}^{(p)i}$ is given by (3.3) in which the nonlinear connection is given by

$$(3.10) \quad N_k^{(p)i} = G_k^i - C_{km}^i B_o^{(p)m} + B_k^{(p)i}, \quad \text{where}$$

$$(3.11) \quad B_k^{(p)i} = A_{ok}^{(p)i} - D_k^{(p)i} - \frac{1}{2} (a_o \delta_k^i + a_k y^i - a^i y_k).$$

The vertical connection is given by (3.1).

As a special case of the above theorem, if we impose the axiom (C3) instead of (C3''), the $B_k^{(p)i}$ in (3.11) becomes

$$(3.12) \quad B_k^{(p)i} = A_{ok}^{(p)i} - \frac{1}{2} (a_o \delta_k^i + a_k y^i - a^i y_k),$$

and we have the following:

Theorem 3.3. *Given in a Finsler space a skew-symmetric (1,2) tensor field $T_{jk}^{(p)i}$ and a covariant vector field a_k there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying the axioms (C1), (C2), (C3), (C4') and (C5).*

These coefficients are given by (3.3), (3.1) and

$$(3.13) \quad \begin{aligned} N_k^{(p)i} &= G_k^i - C_{km}^i (A_{oo}^{(p)m} - \frac{1}{2} a^m L^2) + A_{ok}^{(p)i} \\ &\quad - \frac{1}{2} (a_o \delta_k^i + a_k y^i - a^i y_k). \end{aligned}$$

4. If we assume that $B_k^{(p)i} = 0$, equation (3.10) reduces to $N_k^{(p)i} = G_k^i$, and we have the following results which gives the Finsler connection with deflection and torsion:

Theorem 3.4. *Given in a Finsler space a skew-symmetric (1,2) tensor field $T_{jk}^{(p)i}$ and a covariant vector field a_k , there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying the axioms (C1), (C2), (C4'), (C5) and the new axiom (C3''): the nonlinear connection is the one given by E. CARTAN.*

The coefficients $F_{jk}^{(p)i}$ are given in this case by

$$(3.14) \quad \begin{aligned} F_{jk}^{(p)i} = & \gamma_{jk}^i - (C_{km}^i G_j^m + C_{jm}^i G_k^m - g^{hi} C_{jkm} G_h^m) \\ & - \frac{1}{2}(a_j \delta_k^i + a_k \delta_j^i - a^i g_{jk}) + A_{jk}^{(p)i} \end{aligned}$$

The deflection tensor field $D_k^{(p)i}$ is expressed as

$$(3.15) \quad D_k^{(p)i} = A_{ok}^{(p)i} - \frac{1}{2}(a_o \delta_k^i + a_k y^i - a^i y_k).$$

5. Now we investigate a connection which bears resemblance to the Wagner connection.

Theorem 3.5. *Given in a Finsler space the covariant vector field $s_j \neq 0$ and the recurrence vector $a_j \neq 0$, there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying the axioms (C1), (C2), (C3), (C5) and (C4''): the (h)h-torsion field is the given $T_{jk}^{(p)i} = \delta_j^i s_k - \delta_k^i s_j$.*

PROOF. From the axiom (C2) it follows that the vertical connection $C_{jk}^{(p)i}$ is given by (3.1).

From axiom (C1) we have

$$\partial_k g_{ij} - N_k^{(p)m} \dot{\partial}_m g_{ij} - g_{mj} F_{ik}^{(p)m} - g_{im} F_{jk}^{(p)m} = a_k g_{ij}.$$

Applying the Christoffel process to the above equation and using axiom (C4'') we get

$$(3.16) \quad \begin{aligned} F_{jk}^{(p)i} = & \gamma_{jk}^i - (C_{km}^i N_j^{(p)m} + C_{jm}^i N_k^{(p)m} - g^{hi} C_{jkm} N_h^{(p)m}) \\ & - \frac{1}{2}(a_j \delta_k^i + a_k \delta_j^i - a^i g_{jk}) + g_{jk} s^i - \delta_k^i s_j. \end{aligned}$$

Contracting (3.16) with y^j , using axiom (C3) and the fact that C_{jk}^i is the indicatory tensor, we get

$$(3.17) \quad \begin{aligned} N_k^{(p)i} = & \gamma_{ok}^i - C_{km}^i N_o^{(p)m} - \frac{1}{2}(a_k y^i + a_o \delta_k^i - a^i y_k) \\ & + y_k s^i - \delta_k^i s_o. \end{aligned}$$

Again contracting (3.17) with y^k , we get

$$(3.18) \quad N_o^{(p)i} = \gamma_{oo}^i - a_o y^i + \frac{1}{2} L^2 a^i + L^2 s^i - y^i s_o.$$

Substituting (3.18) in (3.17) and using (3.7), we get

$$(3.19) \quad N_k^{(P)i} = G_k^i + B_k^{ir} (s_r + \frac{1}{2} a_r) + s_k y^i, \text{ where}$$

$$(3.20) \quad B_k^{ir} = (y_k g^{ir} - \delta_k^i y^r - \delta_k^r y^i - L^2 C_k^{ir}) \text{ and}$$

$$(3.21) \quad C_k^{ir} = C_{kh}^i g^{hr}.$$

Substituting (3.19) in (3.16), we get

$$(3.22) \quad F_{jk}^{(p)i} = \Gamma_{jk}^{*i} + U_{jk}^{ir} (s_r + \frac{1}{2} a_r) + \delta_j^i s_k, \text{ where}$$

$$(3.23) \quad \Gamma_{jk}^{*i} = \frac{1}{2} g^{ih} [d_k g_{jh} + d_j g_{kh} - d_h g_{jk}] \text{ and}$$

$$(3.24) \quad U_{jk}^{ir} = g_{jk} g^{ir} - \delta_j^i \delta_k^r - C_j^{ir} y_k - C_k^{ir} y_j \\ + C_{jk}^r y^i + C_{jk}^i y^r - \delta_k^i \delta_j^r \\ + L^2 (C_j^{mr} C_{mk}^i + C_j^{im} C_{mk}^r - C_m^{ir} C_{jk}^m).$$

From (3.22), (3.19) and (3.1) it is clear that the connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ is uniquely determined from the metric function L and from the given vector fields s_j and a_j .

The connection defined in the above theorem will be called generalized h -recurrent Wagner connection with respect to the vector field s_j and the recurrence vector a_j .

Theorem 3.6. *Given the covariant vector field s_j and the recurrence vector a_j in a Finsler space, there exists a unique Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ satisfying the axioms (C1), (C2), (C4''), (C5) and (C3''): the nonlinear connection $N_k^{(p)i}$ is the one given by CARTAN.*

PROOF. Putting $N_k^{(p)i} = G_k^i$ in (3.16) and using (3.23) we get

$$(3.25) \quad F_{jk}^{(p)i} = \Gamma_{jk}^{*i} - \frac{1}{2} (a_j \delta_k^i + a_k \delta_j^i - a^i g_{jk}) + g_{jk} s^i - \delta_k^i s_j.$$

Thus $C_{jk}^{(p)i}$ is determined uniquely from axiom (C2), $N_k^{(p)i}$ is determined from axiom (C3'') and $F_{jk}^{(p)i}$ is determined from axioms (C1) and (C3'').

6. For simplicity we shall use the following terminology. A generalized h -recurrent Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ means, if nothing else is said, such a connection with vanishing deflection and $(h)h$ -torsion tensor fields. — Omitting of the term “ h -recurrent” means that $g_{ij||k} = 0$.

Definition 3.1. A Finsler space is said to be a generalized h -recurrent Berwald space resp. such a space with torsion if it is possible to introduce a generalized h -recurrent Finsler connection without torsion (resp. with torsion) in such a way that the connection coefficient $F_{jk}^{(p)i}$ depends on position only.

Definition 3.2. A Finsler space is called a generalized h -recurrent Wagner space if it is possible to introduce a generalized h -recurrent Wagner connection in such a way that the connection coefficient $F_{jk}^{(p)i}$ depends on the position alone.

Theorem 3.7. *If the generalized h -recurrent Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ with torsion satisfies the condition $\dot{\partial}_\ell F_{jk}^{(p)i} = 0$ then $\dot{\partial}_\ell a_k = 0$.*

PROOF. From (1.3) it follows that the condition $\dot{\partial}_\ell F_{jk}^{(p)i} = 0$ is equivalent to

$$(3.26) \quad P_{jk\ell}^{(p)i} = -C_{j\ell||k}^{(p)i} + C_{jm}^{(p)i} P_{k\ell}^{(p)m}.$$

Applying the Ricci identity ([6]) for the metric tensor g_{ij} we get

$$\begin{aligned} g_{ij||\ell} - g_{ij||k||\ell} &= g_{ij||h} C_{k\ell}^{(p)h} + g_{ij||h} P_{k\ell}^{(p)h} \\ &\quad + g_{hj} P_{ik\ell}^{(p)h} + g_{ih} P_{jk\ell}^{(p)h} \end{aligned}$$

which in view of $g_{ij||k} = a_k g_{ij}$, $g_{ij||\ell} = -g_{im} \sigma_{j\ell}^{(p)m} - g_{mj} \sigma_{i\ell}^{(p)m}$ and (3.26) gives

$$(3.27) \quad (\dot{\partial}_\ell a_k) g_{ij} + 2C_{jim} P_{k\ell}^{(p)m} + 2a_k C_{ij\ell} - 2C_{ij\ell||k} = 0.$$

Contracting this equation with y^i , we get

$$(\dot{\partial}_\ell a_k) y_j + 2C_{ij\ell} D_k^{(p)i} = 0.$$

Again contracting with y^j and using $C_{ij\ell} y^j = 0$, we get

$$(\dot{\partial}_\ell a_k) L^2 = 0 \quad \text{which implies that } \dot{\partial}_\ell a_k = 0.$$

4. Conformal transformations of generalized h -recurrent Wagner spaces

1. Let L be the metric function of a Berwald space and let us consider whether this Berwald space may become a generalized h -recurrent Wagner space by a conformal transformation σ :

$$(4.1) \quad \bar{L} = e^\sigma L$$

In the Finsler space with metric $\bar{L}$, a generalized h -recurrent Wagner connection $(\bar{F}_{jk}^{(p)i}, \bar{N}_k^{(p)i}, \bar{C}_{jk}^{(p)i})$ is given by

$$(4.2) \quad \bar{F}_{jk}^{(p)i} = \bar{\Gamma}_{jk}^{*i} + \bar{U}_{jk}^{ir}(s_r + \frac{1}{2}a_r) + \delta_j^i s_k$$

$$(4.3) \quad \bar{N}_k^{(p)i} = \bar{G}_k^i + \bar{B}_k^{ir}(s_r + \frac{1}{2}a_r) + y^i s_k$$

$$(4.4) \quad \bar{C}_{jk}^{(p)i} = \bar{C}_{jk}^i + \bar{\sigma}_{jk}^{(p)i}$$

Since U_{jk}^{ir} , B_k^{ir} and C_{jk}^i are conformally invariant we can express these in terms of L .

We know that

$$(4.5) \quad \bar{\Gamma}_{jk}^{*i} = \Gamma_{jk}^{*i} - U_{jk}^{ir}\sigma_r,$$

$$(4.6) \quad \bar{G}_k^i = G_k^i - B_k^{ir}\sigma_r,$$

$$(4.7) \quad \bar{C}_{jk}^i = C_{jk}^i$$

where $\sigma_r = \partial_r \sigma$. Also from (3.2) and (4.1), we have

$$(4.8) \quad \bar{\sigma}_{jk}^{(p)i} = \sigma_{jk}^{(p)i},$$

which shows that $\sigma_{jk}^{(p)i}$ is also conformally invariant.

Using equations (4.5), (4.6), (4.7) and (4.8), equations (4.2), (4.3) and (4.4) become

$$(4.9) \quad \bar{F}_{jk}^{(p)i} = \Gamma_{jk}^{*i} + U_{jk}^{ir}(s_r + \frac{1}{2}a_r - \sigma_r) + \delta_j^i s_k,$$

$$(4.10) \quad \bar{N}_k^{(p)i} = G_k^i + B_k^{ir}(s_r + \frac{1}{2}a_r - \sigma_r) + y^i s_k,$$

$$(4.11) \quad \bar{C}_{jk}^{(p)i} = C_{jk}^{(p)i}.$$

If we put $s_r = \sigma_r - \frac{1}{2}a_r$ then (4.9) and (4.10) become

$$(4.12) \quad \bar{F}_{jk}^{(p)i} = \Gamma_{jk}^{*i} + \delta_j^i s_k$$

$$(4.13) \quad \bar{N}_k^{(p)i} = G_k^i + y^i s_k.$$

From these observations we have the following

Theorem 4.1. *By any conformal transformation σ , a Berwald space becomes a generalized h -recurrent Wagner space with respect to the vector $(\sigma_r - \frac{1}{2}a_r)$ and the recurrence vector $a_j(x)$.*

2. In the Finsler space with metric $\bar{L}$ a generalized h -recurrent Finsler connection $(\bar{F}_{jk}^{(p)i}, \bar{N}_k^{(p)i}, \bar{C}_{jk}^{(p)i})$ is obtained from (4.2), (4.3) and (4.4) by putting $s_j = 0$ in them. Thus

$$(4.14) \quad \bar{F}_{jk}^{(p)i} = \Gamma_{jk}^{*i} + \frac{1}{2}\bar{U}_{jk}^{ir}a_r,$$

$$(4.15) \quad \bar{N}_k^{(p)i} = \bar{G}_k^i + \frac{1}{2}\bar{B}_k^{ir}a_r,$$

$$(4.16) \quad \bar{C}_{jk}^{(p)i} = \bar{C}_{jk}^i + \bar{\sigma}_{jk}^{(p)i}.$$

Substituting (4.5), (4.6), (4.7) and (4.8) in the above we have for $a_r = 2\sigma_r$

$$(4.17) \quad \bar{F}_{jk}^{(p)i} = \Gamma_{jk}^{*i},$$

$$(4.18) \quad \bar{N}_k^{(p)i} = G_k^i,$$

$$(4.19) \quad \bar{C}_{jk}^{(p)i} = C_{jk}^{(p)i}.$$

Hence we have the following

Theorem 4.2. *By any conformal transformation σ , a Berwald space becomes a generalized h -recurrent Berwald space with respect to the recurrence gradient vector $2\sigma_r$.*

3. The proof of the following theorem can be obtained by checking the axioms (C2) and (C5).

Theorem 4.3. *Let a generalized h -recurrent Finsler connection $(F_{jk}^{(p)i}, N_k^{(p)i}, C_{jk}^{(p)i})$ with torsion be given in a Finsler space (F^n, L) . If for a conformal transformation $\bar{L} = e^\sigma L$ we put*

$$(4.20) \quad \bar{F}_{jk}^{(p)i} = F_{jk}^{(p)i} + \delta_j^i(\sigma_k + \frac{1}{2}a_k),$$

$$(4.21) \quad \bar{N}_k^{(p)i} = N_k^{(p)i} + y^i(\sigma_k + \frac{1}{2}a_k),$$

$$(4.22) \quad \bar{C}_{jk}^{(p)i} = C_{jk}^{(p)i},$$

then the coefficients $(\bar{F}_{jk}^{(p)i}, \bar{N}_k^{(p)i}, \bar{C}_{jk}^{(p)i})$ define a generalized Finsler connection with torsion in a Finsler space $(F^n, \bar{L})$.

From the above theorem and theorem (3.7) it follows that if $F_{jk}^{(p)i}$ depends on the position alone, then $\bar{F}_{jk}^{(p)i}$ also depends on the position alone. Thus we have the following

Theorem 4.4. *A generalized h -recurrent Berwald space with torsion with respect to the recurrence vector a_j transforms to a generalized Berwald space with torsion by any conformal transformation.*

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