

On p -nilpotency of finite groups with some c -supplemented subgroups of prime power order

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Abstract. A subgroup H of a group G is said to be c -supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is contained in H_G , the core in G of H . In this paper we give some sufficient conditions of p -nilpotency of a finite group under the assumption that some subgroups of prime square order of the group are c -supplemented. These are the duals of some recent results, such as WANG's [14] and GUO and SHUM's [9].

1. Introduction

Let G be a finite group. The relationship between the properties of subgroups of the Sylow subgroups of G and the structure of G has been investigated by a number of authors (for example, see [4], [8], [12]–[15]). In particular, SRINIVASAN [12] proved that a finite group is supersolvable if every maximal subgroup of every Sylow subgroup is normal. Later on, WALL [13] gave a complete classification of finite groups under the assumption of Srinivasan. In [14] and [9], the finite group G in which some maximal subgroups of the Sylow subgroups of G are c -supplemented were

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investigated. It is known, the concepts of maximal subgroup and minimal subgroup are dual in finite group theory, so in the meanwhile, the structure of a finite group G in which some minimal subgroups of the Sylow subgroups of G are c -supplemented was investigated (see [15]). Furthermore, in [14] WANG showed: Let G be a finite group and p the smallest prime dividing $|G|$. If G is A_4 -free and every second maximal subgroup of a Sylow p -subgroup of G is c -supplemented in G , then $G/O_p(G)$ is p -nilpotent ([14, Theorem 4.2]). In [9], this was generalized as follows: Let G be a finite group and p the smallest prime dividing $|G|$. If that G is A_4 -free and every second maximal subgroup of a Sylow p -subgroup of G is c -supplemented in G , then G is p -nilpotent ([9, Theorem 3.4]). In this paper we first continue the discussion in [14], [9]. Then we investigate the structure of a group G with some subgroups of prime square order of a Sylow subgroup, and the dual concept of 2-maximal subgroups of a Sylow subgroup, of G c -supplemented in G . We get some sufficient conditions for the p -nilpotency of a finite group.

Recall that a formation $\mathcal{F}$ of groups is a class of groups which is closed under homomorphic images such that $G/M \cap N \in \mathcal{F}$ whenever M, N are normal subgroups of a group G with $G/M \in \mathcal{F}$ and $G/N \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ (see [10, Ch VI]). It is easy to see that the class of groups with Sylow tower of supersolvable type is a saturated formation. For a formation $\mathcal{F}$, each group has a smallest normal subgroup N such that G/N is in $\mathcal{F}$. This uniquely determined normal subgroup of G is called the $\mathcal{F}$ -residual subgroup of G and is denoted by $G^{\mathcal{F}}$. Usually $\mathcal{N}$ will denote the class of all nilpotent groups.

Throughout this paper all groups are finite. Our notions and notation are standard, see e.g. ROBINSON [11].

2. Preliminaries

Recall that a subgroup H of a group G is said to be c -supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq \text{core}_G(H) = H_G$, or equivalently, $H \cap K = \text{core}_G(H) = H_G$, where H_G is the core in G of H ([4]). We first cite several lemmas that will be useful in the sequel.

Lemma 2.1 ([4, Lemma 2.1]). *Let H be a subgroup of a group G . Then the following statements hold:*

- (1) *Let K be a subgroup of G such that H is contained in K . If H is c -supplemented in G then H is c -supplemented in K ;*
- (2) *Let N be a normal subgroup of G such that N is contained in H . Then H is c -supplemented in G if and only if H/N is c -supplemented in G/N ;*
- (3) *Let π be a set of primes. Let N be a normal π' -subgroup of G and H a π -subgroup of G . If H is c -supplemented in G then HN/N is c -supplemented in G/N . Furthermore, if N normalizes H , then the converse statement also holds;*
- (4) *Let L be a subgroup of G and $H \leq \Phi(L)$. If H is c -supplemented in G then H is normal in G and $H \leq \Phi(G)$.*

Lemma 2.2 ([14, Lemma 4.1]). *Let G be a finite group and p a prime dividing the order of G such that $(|G|, p-1) = 1$. Assume that the order of G is not divisible by p^3 and G is A_4 -free. Then G is p -nilpotent.*

Now we give a generalization of the above lemma.

Lemma 2.3. *Let G be a group and p a prime dividing the order of G , such that G is A_4 -free and $(|G|, p-1) = 1$. Assume that N is a normal subgroup of G with G/N p -nilpotent and the order of N not divisible by p^3 . Then G is p -nilpotent.*

PROOF. Applying Lemma 2.2 to the subgroup N of G we have that N is p -nilpotent. Then N has a normal p -complement $N_{p'}$, which is also normal in G . Consider the factor group $G/N_{p'}$. If $N_{p'} \neq 1$, then by induction $G/N_{p'}$ is p -nilpotent, thus G is p -nilpotent. So we can suppose that N is a p -group of order not greater than p^2 . Since G/N is p -nilpotent, G/N has a normal p -complement, H/N say. Then we can write $H = NH_{p'}$ by the Schur–Zassenhaus Theorem. By Lemma 2.2, we have that H is p -nilpotent, thus $H_{p'}$ is normal in H and then it is also normal in G . It is easy to see that $H_{p'}$ is the p -complement of G , so G is p -nilpotent, as desired. $\square$

Lemma 2.4 ([10, IV, 5.4, p. 434]). *Suppose that G is a group which is not p -nilpotent but whose proper subgroups are all p -nilpotent. Then*

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Lemma 2.5 ([10, III,5.2, p. 281]). Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then

- (i) G has a normal Sylow p -subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q -subgroup for some prime $q \neq p$;
- (ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$;
- (iii) If P is non-abelian and $p \neq 2$, then the exponent of P is p ;
- (iv) If P is non-abelian and $p = 2$, then the exponent of P is 4;
- (v) If P is abelian, then P is of exponent p ;
- (vi) $\Phi(P) \times \Phi(Q) = Z(G) = \Phi(G)$.

Lemma 2.6 ([9, Theorem 3.4]). Let G be a group and p the smallest prime dividing $|G|$. Assume that G is A_4 -free and every second maximal subgroup of a Sylow p -subgroup of G is c -supplemented in G . Then G is p -nilpotent.

Lemma 2.7 ([15, Lemma 2.3]). Let G be a group. Assume that N is a normal subgroup of G ($N \neq 1$) such that $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G which are contained in $F(N)$. In particular, if $\Phi(G) = 1$, then $F(G)$ is the direct product of the minimal normal subgroups of G which are contained in $F(G)$.

Lemma 2.8 ([3, Theorem 1 and Proposition 1]). Let $\mathcal{F}$ be a saturated formation. Assume that G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G . Then:

- (1) $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G ;
- (2) $G^{\mathcal{F}}$ is a p -subgroup for some prime p ;
- (3) $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$;
- (4) $G^{\mathcal{F}}$ is either elementary abelian or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$.

Lemma 2.9 ([8, Lemma 3.16]). Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and P a normal p -subgroup of a group G such that $G/P \in \mathcal{F}$ for some prime p . If G is A_4 -free and $|P| \leq p^2$, then G belongs to $\mathcal{F}$.

3. Main results

We first continue the discussion of WANG's in [14] and GUO and SHUM's in [9], that is, we investigate the structure of a finite group with some c -supplemented 2-maximal subgroups of a Sylow p -subgroup, and generalize some results of GUO and SHUM, such as [9, Corollary 3.5 and 3.6].

Theorem 3.1. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G such that $G/N \in \mathcal{F}$. Suppose G is A_4 -free. If for every prime p dividing the order of N and $P \in \text{Syl}_p(N)$, every 2-maximal subgroup of P is c -supplemented in G , then G belongs to $\mathcal{F}$.*

PROOF. It is easy to see that N is a Sylow tower group of supersolvable type by Lemmas 2.1 and 2.6. Let r be the largest prime in $\pi(N)$ and $R \in \text{Syl}_r(N)$. Then R is normal in G and $(G/R)/(N/R) \simeq G/N$ is a Sylow tower group of supersolvable type. Let $\overline{P} = PR/R$ be a Sylow p -subgroup of N/R with $r \neq p$. We may assume that P is a Sylow p -subgroup of N . If $\overline{P}_1$ is a 2-maximal subgroup of $\overline{P}$, then, without loss of generality, we may assume that $\overline{P}_1 = P_1R/R$ with P_1 a 2-maximal subgroup of P . Since P_1 is c -supplemented in G , we know that $\overline{P}_1$ is c -supplemented in G/R by Lemma 2.1 (3). Therefore, G/R satisfies the hypotheses of our theorem for the normal subgroup N/R . Thus, by induction, G/R is a Sylow tower group of supersolvable type, and of course, every 2-maximal subgroup of R is c -supplemented in G .

If r is the largest prime dividing the order of G , then it is clear that G is a Sylow tower group of supersolvable type. In this case, we may assume that q is the largest prime dividing the order of G with $q > r$. Let Q be a Sylow q -subgroup of G . Since G/R is a Sylow tower group, we see that RQ is normal in G . By the Frattini argument we have $G = RN_G(Q)$.

If $RQ < G$, then RQ is a Sylow tower group of supersolvable type by induction on $|G|$, thus $Q \triangleleft RQ$, then $G = N_G(Q)$, i.e., Q is normal in G . Now we consider the quotient group G/Q and its normal subgroup NQ/Q . For any prime p dividing the order of NQ/Q , then $p < q$. For any 2-maximal subgroup $\overline{P}_2$ of a Sylow p -subgroup $\overline{P}$ of NQ/N , we can write $\overline{P}_2 = P_2Q/Q$, where P_2 is a 2-maximal subgroup of some Sylow subgroup P of N . By the hypotheses, P_2 is c -supplemented in G , then

$\overline{P_2}$ is c -supplemented in G/Q by Lemma 2.1(3). So G/Q with its normal subgroup NQ/N satisfies the hypotheses of our theorem. By induction, G/Q is a Sylow tower group and therefore G must be a Sylow tower group of supersolvable type.

So suppose $G = RQ$. Now let L be a minimal normal subgroup of G with $L \leq R$. Then it is easy to see that the quotient group G/L satisfies the hypotheses of our theorem for the normal subgroup of R/L . By induction we see that G/L is a Sylow tower group of supersolvable type. By a trivial argument, we may assume that L is the unique minimal normal subgroup of G which is contained in R . If $L \leq \Phi(G)$, then it follows that G is a Sylow tower group of supersolvable type. Thus, we may further assume that $R \cap \Phi(G) = 1$ and therefore $L = F(R) = R$ is an abelian minimal normal subgroup of G by Lemma 2.7.

If R is a cyclic group of order r , then because $\text{Aut}(R)$ is a cyclic group of order $r-1$ and $G/C_G(P) \leq \text{Aut}(R)$, we see that $|Q| \mid |C_G(R)|$, therefore we may assume that $Q \leq C_G(R)$ and then $G = R \times Q$. Thus G is a Sylow tower group of supersolvable type. If $|R| > r^2$, then let R_1 be a 2-maximal subgroup of R . Now, by our hypotheses there exists a subgroup K of G such that $G = R_1K$ and $R_1 \cap K = 1$ since L is the unique minimal normal subgroup of G contained in R . Thus $R = R_1(R \cap K)$. Since $R \cap K$ is normal in K and R is abelian, $R \cap K$ is a normal subgroup of G . The minimality of $R = L$ implies that $R \cap K = R$, and therefore $R_1 = 1$, a contradiction. Hence R is an elementary abelian group of order r^2 . Since R is normal in G , any element g of Q induces an automorphism σ of R . When $r = 2$, if $\sigma \neq 1$, noticing that $|\text{Aut}(R)| = (r+1)r(r-1)^2$, the order of σ must be 3 ($q = r+1 = 3$) as $q > r$. Then $R < \sigma > \cong A_4$, contrary to the hypothesis. So suppose that $r > 2$, noticing that $r+1$ is not a prime, hence we see that $\sigma = 1$ and therefore $G = R \times Q$, so G is a Sylow tower group of supersolvable type. The proof is now complete. $\square$

Corollary 3.2. *Let G be a group which is A_4 -free, and N a normal subgroup of G such that G/N is supersolvable. If, for every prime p dividing the order of N and $P \in \text{Syl}_p(N)$, every 2-maximal subgroup of P is c -supplemented in G , then G is supersolvable.*

Corollary 3.3 ([9, Corollary 3.6]). *Let G be a group of odd order, and N a normal subgroup of G such that G/N is a Sylow tower group*

of supersolvable type. If, for every prime p dividing the order of N and $P \in \text{Syl}_p(N)$, every 2-maximal subgroup of P is c -supplemented in G , then G is a Sylow tower group of supersolvable type.

In the sequel, we discuss the influence of the properties of subgroups of prime square order of a Sylow subgroup, and the dual concept of a 2-maximal subgroup of a Sylow subgroup, on the structure of G .

Theorem 3.4. *Let G be a group and p a prime dividing the order of G . Suppose that $(|G|, p-1) = 1$ and G is A_4 -free. If there exists a normal subgroup N of G such that G/N is p -nilpotent and every subgroup of order p^2 of every Sylow p -subgroup of N is c -supplemented in G , then G is p -nilpotent.*

PROOF. Assume that the theorem is false and let G be a counterexample of minimal order. Then we may make the following claims:

(i) The hypotheses are inherited by all proper subgroups of G , thus G is a group which is not p -nilpotent but whose proper subgroups are all p -nilpotent.

In fact, for all $H < G$, $H/H \cap N \cong HN/N$, thus $H/H \cap N$ is p -nilpotent. If $|H \cap N|_p \leq p^2$, then H is p -nilpotent by Lemma 2.3. So suppose that $|H \cap N|_p > p^2$. Then we can take an arbitrary subgroup P_2 of order p^2 of any Sylow p -subgroup of $H \cap N$. Obviously P_2 is also a subgroup of order p^2 of some Sylow p -subgroup of N . Thus it is c -supplemented in G by the hypotheses and then it is c -supplemented in H by Lemma 2.1. Hence H satisfies the hypotheses of the theorem. The minimal choice of G implies that H is p -nilpotent, thus G is a group which is not p -nilpotent but whose proper subgroups are all p -nilpotent.

(ii) $G = PQ$, where $P \triangleleft G$ and Q is not a normal subgroup of G . Furthermore, p^3 divides the order of P .

These can be induced by Lemmas 2.4, 2.5 and 2.2.

(iii) Every subgroup of order p^2 of P is normal in G .

Suppose there exists a subgroup P_2 of order p^2 which is not normal in G . By the hypotheses, P_2 is c -supplemented in G , so there exists a subgroup K of G such that $G = P_2K$ and $P_2 \cap K = (P_2)_G < P_2$. Then K is a proper subgroup of G , thus K is nilpotent. Denote $K = K_p \times K_{p'}$, then $P = P_2K_p$. Now consider $N_G(K_p)$. Since $K \leq N_G(K_p)$, we have

that $[G : N_G(K_p)] \leq p$. If $[G : N_G(K_p)] = 1$, then K_p is normal in G . By Lemma 2.5 (ii), $K_p \leq \Phi(P)$ or $K_p = P$. If $K_p \leq \Phi(P)$, then $P = P_2$, contrary to (ii). If $K_p = P$, then $K = G$, a contradiction. Now suppose that $[G : N_G(K_p)] = p$, then we can write $N_G(K_p) = P_1 \times K_{p'}$, where P_1 is a maximal subgroup of P containing K_p . Now $N_G(P_1)$ contains P and $K_{p'}$, so P_1 is normal in G , then, again by Lemma 2.5(ii), $P_1 \leq \Phi(P)$ or $P_1 = P$, which implies that $P = P_2$ or P_1 , a contradiction.

(iv) Every subgroup of order p^2 of P is contained in $\Phi(P)$, thus in $Z(G)$.

Suppose P_2 is an arbitrary subgroup of order p^2 of P , then P_2 is normal in G by (iii), therefore $P_2\Phi(G) = P$ or $P_2\Phi(P) = \Phi(P)$ by Lemma 2.5(ii). If $P_2\Phi(G) = P$, then $P = P_2$, contrarily to (ii), so P_2 is contained in $\Phi(P)$, hence it is contained in $Z(G)$ by Lemma 2.5(vi).

(v) $\Phi(P) = 1$.

If $\Phi(P) \neq 1$, we can pick an element a of order p in $\Phi(P)$. If $\exp(P) = p$, then for any element b of P not in $\langle a \rangle$, $\langle a \rangle \langle b \rangle$ is a group of order p^2 , so $\langle a \rangle \langle b \rangle$ is contained in $Z(G)$, thus $P \leq Z(G)$. Therefore $G = P \times Q$, a contradiction. So we may suppose that $p = 2$ and $\exp(P) = 4$. For any element b of P not in $\langle a \rangle$, if b is of order 2, then $\langle a \rangle \langle b \rangle$ is a group of order 4, hence $\langle a \rangle \langle b \rangle \leq \Phi(P)$ by (iv), so $b \in \Phi(P)$; if b is of order 4, then $\langle b \rangle$ is contained in $\Phi(P)$ by (iv), which again implies $b \in \Phi(P)$. Hence $P = \Phi(P) \leq Z(G)$ and from here $G = P \times Q$, a contradiction.

(vi) The final contradiction.

Take $a \in P$, then a is of order p . Now we can find an element b of order p which is not in $\langle a \rangle$ by (ii), then the order of the subgroup $\langle a \rangle \langle b \rangle$ is p^2 , thus it is contained in $\Phi(P)$ by (iv), which is contrary to (v), the final contradiction.

If we choose the subgroup N in Theorem 3.4 as $G^{\mathcal{N}}$, the nilpotent residual of G , then we can see that the following is an equivalent form of Theorem 3.4. □

Corollary 3.5. *Let p be a prime number dividing the order of a group G such that $(|G|, p-1) = 1$ and let G be A_4 -free. Suppose P is a Sylow p -subgroup of G . If every subgroup of order p^2 of $P \cap G^{\mathcal{N}}$ is c -supplemented in G , then G is p -nilpotent.*

Remark 3.6. We observe that the assumption $(|G|, p-1) = 1$ cannot be removed in Corollary 3.5. In fact, assume G is a non-cyclic group of order 21 and $p = 7$. Then G is A_4 -free and there does not exist a subgroup of order 7^2 in G , but G is not 7-nilpotent. It is easy to see that the assumption that G is A_4 -free cannot be removed either in our result, because A_4 is a counterexample.

Corollary 3.7. *Let G be a group. If, for every prime p dividing the order of G and $P \in \text{Syl}_p(G)$, every subgroup of order p^2 of P is c -supplemented in G and G is A_4 -free, then G is a Sylow tower group of supersolvable type.*

PROOF. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and therefore Corollary 3.7 follows immediately from Theorem 3.4. $\square$

Now we generalize Corollary 3.7 as follows.

Theorem 3.8. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and H a normal subgroup of a group G such that $G/H \in \mathcal{F}$. If G is A_4 -free and all subgroups of prime square order of every Sylow subgroup of H are c -supplemented in G , then G is in $\mathcal{F}$.*

PROOF. Suppose the result is false and let G be a counterexample of minimal order. Then by Corollary 3.7, we can see that H has a Sylow tower of supersolvable type. Let p be the largest prime in $\pi(H)$ and $P \in \text{Syl}_p(H)$. Then P is a normal subgroup of G . Now consider the factor group G/P . It is easy to see that all subgroups of prime square order of every Sylow subgroup of H/P are c -supplemented in G/P and G/P is A_4 -free. Thus, by the minimal choice of G , we have $G/P \in \mathcal{F}$ and every subgroup of order p^2 of P is c -supplemented in G .

So $G^{\mathcal{F}}$ is a p -subgroup. By [1, Theorem 3.5], there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \text{Soc}(G \text{ mod } \Phi(G))$ and $G/M_G \notin \mathcal{F}$. Hence $G = MG^{\mathcal{F}} = MF(G)$ (since $G^{\mathcal{F}}$ is a p -group). It is obvious that M satisfies the hypotheses of the theorem on its normal subgroup $M \cap P$. By the minimal choice of G , we have that M lies in $\mathcal{F}$.

Now, by Lemma 2.8, $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G , $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^{\mathcal{F}}$ or $(G^{\mathcal{F}})' = \Phi(G^{\mathcal{F}}) = Z(G^{\mathcal{F}})$ is elementary abelian.

Now we distinguish two cases:

Case 1 $\Phi(G^{\mathcal{F}}) = 1$.

In this case, $G^{\mathcal{F}}$ is a minimal normal subgroup of G . If $|G^{\mathcal{F}}| \leq p^2$, then $G \in \mathcal{F}$ by Lemma 2.9. So suppose $|G^{\mathcal{F}}| \geq p^3$, then we can take a subgroup P_2 of order p^2 of $G^{\mathcal{F}}$ and P_2 is c-supplemented in G by the hypotheses. So there exists a subgroup K of G such that $G = P_2K$ and $P_2 \cap K = (P_2)_G = 1$. Therefore $G^{\mathcal{F}} = P_2(K \cap G^{\mathcal{F}})$. Since $G^{\mathcal{F}}$ is elementary abelian, it is easy to see that $G^{\mathcal{F}} \cap K$ is normal in G , thus $G^{\mathcal{F}} \cap K = 1$ or $G^{\mathcal{F}}$ by the minimality of $G^{\mathcal{F}}$. If $G^{\mathcal{F}} \cap K = 1$, then $G^{\mathcal{F}} = P_2$ and it is of order p^2 , while $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $G^{\mathcal{F}} \cap K = G^{\mathcal{F}}$, then $P_2 = P_2 \cap K = (P_2)_G = 1$, a contradiction too.

Case 2 $\Phi(G^{\mathcal{F}}) \neq 1$.

We consider two subcases.

Subcase 2.1. $p = 2$ and $\exp(G^{\mathcal{F}}) = 4$.

Now we can take an element x of order 4 in $G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$. Then there exists a subgroup K of G such that $G = \langle x \rangle K$ and $\langle x \rangle \cap K = \langle x \rangle_G$. Hence $G^{\mathcal{F}} = \langle x \rangle (K \cap G^{\mathcal{F}})$. Note that $x^2 \in \Phi(G^{\mathcal{F}})$, thus $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$ is a maximal subgroup of $G^{\mathcal{F}}$, so x normalizes $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$. Since $[x^2, K] \leq \Phi(G^{\mathcal{F}}) \leq \langle x^2 \rangle (K \cap G^{\mathcal{F}})$, we get that $\langle x^2 \rangle (K \cap G^{\mathcal{F}})$ is a normal subgroup of G . Hence $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}}$ by the minimality of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. If $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ then $G^{\mathcal{F}} = \langle x \rangle$, so $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $\langle x^2 \rangle (K \cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ then $G^{\mathcal{F}} = K \cap G^{\mathcal{F}}$, so $\langle x \rangle = \langle x \rangle \cap K = \langle x \rangle_G$ is normal in G , and then $\langle x \rangle = G^{\mathcal{F}}$. By Lemma 2.9 we have $G \in \mathcal{F}$, another contradiction.

Subcase 2.2 $\exp(P) = p$.

Since $G^{\mathcal{F}} \neq \Phi(G^{\mathcal{F}}) \neq 1$, we can take two elements a and b of order p such that $a \in \Phi(G^{\mathcal{F}})$ and $b \in G^{\mathcal{F}} - \Phi(G^{\mathcal{F}})$. Then $\langle a \rangle \langle b \rangle$ is a subgroup of $G^{\mathcal{F}}$ of order p^2 , so there exists a subgroup K of G such that $G = \langle a \rangle \langle b \rangle K$ and $\langle a \rangle \langle b \rangle \cap K = (\langle a \rangle \langle b \rangle)_G$. Hence $G^{\mathcal{F}} = \langle a \rangle \langle b \rangle (K \cap G^{\mathcal{F}})$. Note that $\langle a \rangle (K \cap G^{\mathcal{F}})$ is a maximal subgroup of $G^{\mathcal{F}}$, so b normalizes $\langle a \rangle (K \cap G^{\mathcal{F}})$. Since $[a, K] \leq \Phi(G^{\mathcal{F}}) \leq \langle a \rangle (K \cap G^{\mathcal{F}})$, we get that $\langle a \rangle (K \cap G^{\mathcal{F}})$ is a normal subgroup of G . Hence $\langle a \rangle (K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ or $\langle a \rangle (K \cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ by

the minimality of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. If $\langle a \rangle(K \cap G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$, then $G^{\mathcal{F}} = \langle b \rangle$, so $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. If $\langle a \rangle(K \cap G^{\mathcal{F}}) = G^{\mathcal{F}}$ then $G^{\mathcal{F}} = K \cap G^{\mathcal{F}}$, so $\langle a \rangle \langle b \rangle = \langle a \rangle \langle b \rangle \cap K = (\langle a \rangle \langle b \rangle)_G$ is normal in G which implies $G^{\mathcal{F}} = \langle a \rangle \langle b \rangle = \langle b \rangle$. By Lemma 2.9 we have $G \in \mathcal{F}$, a contradiction. $\square$

The following are immediate corollaries of Theorem 3.8.

Corollary 3.9. *Let G be a group which is A_4 -free and N a normal subgroup of G such that G/N is supersolvable. If for every prime p dividing the order of H and $P \in \text{Syl}_p(H)$, every subgroup of order p^2 of P is c -supplemented in G , then G is supersolvable.*

Corollary 3.10. *Let G be a group of odd order and N a normal subgroup of G such that G/N is a Sylow tower group of supersolvable type. If all subgroups of prime square order of every Sylow subgroup of N are c -supplemented in G , then G is a Sylow tower group of supersolvable type.*

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