

The set of pseudo solutions of the differential equation $x^{(m)} = f(t, x)$ in Banach spaces

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Abstract. In this paper we prove the existence theorem for the equation $x^{(m)} = f(t, x(t))$ in Banach spaces where f is weakly-weakly sequentially continuous. Moreover, we prove that the set of pseudo-solutions of our equation is compact and connected.

1. Introduction

In this paper we will deal with the Cauchy problem

$$\begin{cases} x^{(m)} = f(t, x(t)) \\ x(0) = 0, \\ x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}, \end{cases} \quad t \in I = \langle 0, a \rangle, \quad a \in \mathbb{R}_+ \quad (1.1)$$

where $\eta_1, \dots, \eta_{m-1} \in E$, $m \in \mathbb{N}$.

Throughout this paper $(E, \|\cdot\|)$ will be denote a real Banach space, E^* the dual space, $(R) \int_0^t f(s)ds$ the weak Riemann integral, $(P) \int_0^t f(s)ds$ the Pettis integral ([8], [9], [12], [16]).

By $(C(I, E), \omega)$ we will denote the space of all continuous functions from I to E endowed with the topology $\sigma(C(I, E), C(I, E)^*)$.

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This paper is divided into two main sections. In Section 1 we prove an existence theorem for the problem (1.1). In Section 2 we prove that, the set of pseudo-solutions of the equation (1.1) is compact and connected.

The result presented in this paper extends the results for CICHÓN [5], CICHÓN, KUBIACYK [6], CRAMER, LAKSMIKANTHAM and MITCHELL [7], O'REGAN [15], SZUFLA [17], SZUFLA and SZUKAŁA [18].

Assume that $B = \{x \in E : \|x\| < b, b > 0\}$ and $f : I \times B \rightarrow E$. Moreover, let $M = \sup\{\|f(t, x)\| : t \in I, x \in B\}$. Choose a positive number d such that $d \leq a$, $\sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M \frac{d^m}{m!} < b$, $d^m < 1$, ($m > 1$).

Let $J = \langle 0, d \rangle$. We set $\tilde{B} = \{x \in C(J, E) : x(t) \in B, t \in J\}$.

We will consider the problem

$$x(t) = p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1, \quad (1.2)$$

where $p(t) = \begin{cases} 0, & m = 1 \\ \sum_{j=1}^{m-1} \eta_j \cdot \frac{t^j}{j!}, & m > 1 \end{cases}$ is a continuous function.

Now we recall the notion of the pseudo-solution. For such solutions, the problem (1.1) is equivalent to the integral problem (1.2).

Fix $x^* \in E^*$. Let us introduce the following definition:

Definition 1.1. A function $x : I \rightarrow E$ is said to be a *pseudo-solution* of the equation (1.1) if it satisfies the following conditions:

- (i) x is a strongly absolutely continuous, $(m-1)$ -times weakly differentiable,
- (ii) $\forall_{x^* \in E^*} \exists_{\substack{\text{mes } A(x^*)=0 \\ A(x^*) \subset I}} A(x^*) \ x^* x : I \rightarrow E$ is m -times differentiable,
- (iii) $(x^* x^{(m-1)})'(t) = x^* f(t, x(t))$ for each $t \notin A(x^*)$ and $x(0) = 0$,
 $x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1}$.

In this paper we will use the measure of weak noncompactness developed by DEBLASI [3]. The proofs of properties of the measure of weak noncompactness see in [2].

Let A be a bounded nonvoid subset of E .

The *de Blasi measure of weak noncompactness* $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{there exist } C \in K^\omega \text{ such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball.

The properties of measure of weak noncompactness $\beta(A)$ are:

- (i) if $A \subset B$ then $\beta(A) \leq \beta(B)$;
- (ii) $\beta(A) = \beta(\overline{A})$, where $\overline{A}$ denotes the closure of A ;
- (iii) $\beta(A) = 0$ if and only if A is a weakly relatively compact;
- (iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
- (v) $\beta(\lambda A) = |\lambda|\beta(A)$, ($\lambda \in \mathbb{R}$);
- (vi) $\beta(A + B) \leq \beta(A) + \beta(B)$;
- (vii) $\beta(\text{conv } A) = \beta(A)$.

We can construct many other measures of noncompactness with the above properties, by using a scheme from [1], [4].

We recall that a function $f : I \times \tilde{B} \rightarrow E$ is called a *Carathéodory function* if for each $x \in \tilde{B}$, $f(t, x)$ is measurable in t and for almost all $t \in I$, $f(t, x)$ is continuous. A function $f : I \rightarrow E$ is said to be *weakly continuous* if it is continuous from I to E endowed with its weak topology.

A function $g : E \rightarrow E_1$, where E and E_1 are Banach spaces, is said to be *weakly – weakly sequentially continuous* if for each weakly convergent sequence $(x_n) \subset E$, a sequence $(g(x_n)) \subset E_1$.

2. Existence of solution

We will use the following lemmas:

Lemma 2.1 ([14]). *Let $H \subset C(I, E)$ be a family of strongly equicontinuous functions. Then $\beta_C(H) = \sup_{t \in I} \beta(H(t)) = \beta(H(I))$, where $\beta_C(H)$ denotes the measure of weak noncompactness in $C(I, E)$ and the function $t \rightarrow \beta(H(t))$ is continuous.*

Lemma 2.2 ([6]). *Let (X, d) be a metric space and let $g : X \rightarrow (E, \omega)$ be sequentially continuous. If $A \subset X$ is a connected subset in X , then $g(A)$ is a connected subset in (E, ω) .*

Similar as in [10] we can prove the following lemma.

Lemma 2.3. *For each bounded, equicontinuous set $X \subset C(I, E)$ and for each $c, d \in I$ we have*

$$\beta\left(\int_c^d X(s)ds\right) \leq \int_c^d \beta(X(s))ds,$$

where $\int_c^d X(s)ds = \left\{ \int_c^d x(s)ds : x \in X \right\}$.

In the proof of the main theorem we will apply the following fixed point theorem.

Theorem 2.1 ([13]). *Let D be a closed convex subset of E , and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

$$\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively weakly compact}, \quad (2.1)$$

Now we prove an existence theorem for the problem (1.1).

Theorem 2.2. *Assume, that for each strongly absolutely continuous function $x : J \rightarrow E$, $f(\cdot, x(\cdot))$ is Pettis integrable, $f(t, \cdot)$ is weakly-weakly sequentially continuous and*

$$\beta(f(J \times X)) \leq h(\beta(X)) \quad \text{for each } X \subset B, \quad (2.2)$$

where h is a function such that $h(u) < u$ for $u \in \mathbb{R}_+$. Then there exists a pseudo-solution of the problem (1.1) on J .

PROOF. By F_x we define a mapping

$$F_x(t) = p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1,$$

$$\text{where } p(t) = \begin{cases} 0, & m = 1, \\ \sum_{j=1}^{m-1} \eta_j \cdot \frac{t^j}{j!}, & m > 1. \end{cases}$$

We require that $F_x : \tilde{B} \rightarrow \tilde{B}$ is weakly sequentially continuous.

(i) For any $x^* \in E^*$ such that $\|x^*\| \leq 1$ and for any $x \in B$ as $|x^* f(t, x(t))| \leq M$ we have

$$\begin{aligned} & |x^* F_x(t)| \\ &= \left| x^* \left[p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right] \right| \end{aligned}$$

$$\begin{aligned}
&\leq \|x^*\| \cdot \sum_{j=1}^{m-1} \|\eta_j\| \frac{\|t^j\|}{j!} \\
&\quad + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} |x^*(f(t_m, x(t_m)))| dt_m \dots dt_2 dt_1 \\
&\leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} M dt_m \dots dt_2 dt_1 \\
&\leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + \frac{M \cdot d^m}{m!} < b.
\end{aligned}$$

Hence

$$\sup\{|x^* F_x(t)| : x^* \in E^*, \|x^*\| \leq 1\} \text{ and } \|F_x(t)\| \leq b \text{ so } F_x \in \tilde{B}.$$

(ii) Now we will prove that the set $F_x(\tilde{B})$ is equicontinuous.

Because

$$\begin{aligned}
&\|F_x(t) - F_x(s)\| \leq \|p(t) - p(s)\| \\
&\quad + \left\| (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\| \\
&\leq \|p(t) - p(s)\| + \frac{M d^{m-1}}{(m-1)!} |t - s|, \quad \text{for each } x \in C(J, E),
\end{aligned}$$

so $F_x(\tilde{B})$ is strongly equicontinuous.

(iii) Now we will show weakly sequentially continuity of F_x .

Let $x_n \rightarrow x$ in $(C(I, E), \omega)$.

$$\begin{aligned}
&|x^*[F_{x_n}(t) - F_x(t)]| \\
&= \left| x^* \left[p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x_n(t_m)) dt_m \dots dt_2 dt_1 \right. \right. \\
&\quad \left. \left. - p(t) - (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right] \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| x^* \left[(R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} [f(t_m, x_n(t_m)) - f(t_m, x(t_m))] dt_m \dots dt_2 dt_1 \right] \right| \\
&\leq (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} |x^*[f(t_m, x_n(t_m)) - f(t_m, x(t_m))]| dt_m \dots dt_2 dt_1.
\end{aligned}$$

Because $x_n \rightarrow x$ in $(C(I, E), \omega)$ and f is weakly sequentially continuous so F_x is weakly sequentially continuous.

Suppose that $\overline{V} = \overline{\text{conv}}(F_x(V) \cup \{0\})$ for some $V \subset \widetilde{B}$.

We will prove that V is relatively weakly compact, thus (2.1) is satisfied. As $F_x(V)$ is equicontinuous, the function $v(t) \rightarrow \beta(V(t))$ is continuous (by Lemma 2.1).

By the definition of V , the mean valued theorem for the Pettis integral, Lemma 2.3, the strongly equicontinuity of the family of Riemann integrals, by the properties of β and (2.2) we obtain:

$$\begin{aligned}
\beta(V(t)) &= \beta(\overline{\text{conv}}(F_x(V) \cup \{0\})) \leq \beta(F_x(V)) \\
&= \beta \left(p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right) \\
&\leq \beta \left((R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right) \\
&\leq (R) \int_0^t (R) \int_0^{t_1} \dots (R) \int_0^{t_{m-2}} \beta \left[(P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \right] dt_{m-1} \dots dt_2 dt_1 \\
&\leq (R) \int_0^t (R) \int_0^{t_1} \dots (R) \int_0^{t_{m-2}} \beta[t_{m-1} \cdot \overline{\text{conv}} f(J \times V(J))] dt_{m-1} \dots dt_1 \\
&\leq (R) \int_0^t (R) \int_0^{t_1} \dots (R) \int_0^{t_{m-2}} t_{m-1} \cdot h(\beta(V(J))) dt_{m-1} \dots dt_1 \\
&\leq \frac{d^m}{m!} \cdot h(\beta(V(J))).
\end{aligned}$$

By our assumptions about the function h we have

$$\beta(V(t)) \leq \frac{d^m}{m!} \beta(V(J)).$$

So

$$\beta(V(J)) \leq \frac{d^m}{m!} \beta(V(J)).$$

Because $d^m < 1$, we get $v(t) = \beta(V(t)) = 0$ for $t \in J$.

By Arzelá–Ascoli’s theorem, V is relatively weakly compact. So, by Theorem 2.1 F_x has a fixed point in $\tilde{B}$ which is actually a pseudo-solution of the problem (1.1). $\square$

3. Compactness and connectedness

In this part we show that the set of pseudo-solutions of our equation (1.1) is compact and connected.

Theorem 3.1. *Under the assumptions of Theorem 2.2 the set S of all pseudo-solutions of the Cauchy problem (1.1) on J is compact and connected in $(C(J, E), \omega)$.*

PROOF. As $S = F_x(S)$, by repeating the above argument, with $V = S$ we can show that S is relatively compact in $(C(J, E), \omega)$. Since F is weakly continuous on $\overline{S(J)^\omega}$, S is weakly closed and consequently weakly compact.

For any $\eta > 0$ denotes by S_η the set of all functions $u : J \rightarrow E$ satisfying the following conditions:

$$(i) \quad u(0) = 0, \quad u'(0) = \eta_1, \dots, u^{(m-1)}(0) = \eta_{m-1},$$

$$\|u(t) - u(s)\| \leq K|t - s|, \text{ for } t, s \in J, \text{ where } K = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{j!} + \frac{Md^{m-1}}{(m-1)!},$$

$$(ii) \quad \sup_{t \in J} \left\| u(t) - p(t) - (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\| < \eta.$$

The set S_η is nonempty as $S \subset S_\eta$.

Let $\rho = \min(a, \eta/K)$. For any $\varepsilon \in (0, \rho)$ let $v(\cdot, \varepsilon) : J \rightarrow E$ be defined by the formula:

$$v(t, \varepsilon) = \begin{cases} p(t), & \text{for } 0 \leq t \leq \varepsilon \\ p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots \\ (P) \int_0^{t_{m-1}-\varepsilon} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1, & \text{for } \varepsilon < t \leq d \end{cases}$$

Clearly $v(\cdot, \varepsilon)$ satisfies (i).

Furthermore we have:

$$\begin{aligned} & \left\| v(t, \varepsilon) - p(t) - (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\| \\ &= \begin{cases} \left\| (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\|, & \text{for } 0 \leq t \leq \varepsilon \\ \left\| (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_{t_{m-1}-\varepsilon}^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\|, & \text{for } \varepsilon < t \leq d \end{cases} \\ &\leq \frac{M \cdot \varepsilon \cdot d^{m-1}}{(m-1)!} < \eta \end{aligned}$$

thus $v(\cdot, \varepsilon)$ satisfies (ii).

Now, we will prove that S_η is connected. Define

$$v_\varepsilon(t) = \begin{cases} p(t), & \text{for } 0 \leq t \leq \varepsilon \\ F_x(v_\varepsilon)(t - \varepsilon), & \text{for } \varepsilon < t \leq d \end{cases}$$

where $v_\varepsilon = v(\cdot, \varepsilon)$. We will show that the mapping $\varepsilon \rightarrow v_\varepsilon(\cdot)$ is sequentially continuous from $(0, \rho)$ into $(C(J, E), \omega)$.

Let $0 < \varepsilon < \delta \leq d$ (when $\delta \leq \varepsilon$ the argument is similar).

For $t \in \langle 0, \varepsilon \rangle$

$$|x^*(v_\varepsilon(t) - v_\delta(t))| = 0. \quad (3.1)$$

For $t \in (\varepsilon, \delta)$

$$\begin{aligned} & |x^*(v_\varepsilon(t) - v_\delta(t))| \\ &= \left| x^* \left[(R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}-\varepsilon} f(t_m, v_\varepsilon(t_m)) dt_m \dots dt_2 dt_1 \right. \right. \\ &\quad \left. \left. - (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}-\delta} f(t_m, v_\varepsilon(t_m)) dt_m \dots dt_2 dt_1 \right] \right| \\ &\leq \|x^*\| \left\| (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_{t_{m-1}-\delta}^{t_{m-1}-\varepsilon} f(t_m, v_\varepsilon(t_m)) dt_m \dots dt_2 dt_1 \right\| \\ &\leq \|x^*\| \cdot |\delta - \varepsilon| \cdot M \frac{d^{m-1}}{(m-1)!}. \end{aligned} \quad (3.2)$$

For $t \in (\delta, 2\delta)$

$$\begin{aligned} |x^*(v_\varepsilon(t) - v_\delta(t))| &= |x^*(F_x(v_\varepsilon)(t - \varepsilon) - F_x(v_\delta)(t - \delta))| \\ &\leq |x^*[F_x(v_\varepsilon)(t - \varepsilon) - F_x(v_\varepsilon)(t - \delta)]| \\ &\quad + |x^*[F_x(v_\varepsilon)(t - \delta) - F_x(v_\delta)(t - \delta)]| \\ &\leq |x^*[F_x(v_\varepsilon)(t - \delta) - F_x(v_\delta)(t - \delta)]| \\ &\quad + \|x^*\| \cdot M \cdot \frac{d^{m-1}}{(m-1)!} |t - \varepsilon - t\delta| \\ &= |x^*(F_x(v_\varepsilon)(t - \delta) - F_x(v_\delta)(t - \delta))| \\ &\quad + \|x^*\| \cdot M \cdot \frac{d^{m-1}}{(m-1)!} |\delta - \varepsilon|. \end{aligned} \quad (3.3)$$

Let (δ_n) be a sequence such that $\delta_n \rightarrow \varepsilon$ ($\delta_n \geq \varepsilon$).

By (3.1) and (3.2), it follows that $v_{\delta_n}(t)$ converges weakly to $v_\varepsilon(t)$, uniformly for $t \in \langle 0, \delta \rangle$. So $F_x(v_{\delta_n})(t) \rightarrow F_x(v_\varepsilon)(t)$ weakly on $\langle 0, \delta \rangle$. Now, by (3.3) $v_{\delta_n}(t)$ tends to $v_\varepsilon(t)$ weakly for each $t \in \langle 0, 2\delta \rangle$.

By repeating the above argument and using induction, we obtain that the map $\varepsilon \rightarrow v_\varepsilon(t)$ from $(0, d)$ into $(C(J, E), \omega)$ is sequentially continuous. Therefore, by Lemma 2.2, the set $\{v_\varepsilon(\cdot) : 0 < \varepsilon < d\}$ is connected in $(C(J, E), \omega)$.

Let $x \in S_\eta$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < d$ and

$$\sup_{t \in J} \left\| x(t) - p(t) - (R) \int_0^t (R) \int_0^{t_1} \dots (P) \int_0^{t_{m-1}} f(t_m, x(t_m)) dt_m \dots dt_2 dt_1 \right\| + M\varepsilon \cdot \frac{d^{m-1}}{(m-1)!} < \eta.$$

For any q , $0 \leq q \leq d$ let $y(\cdot, q) : J \rightarrow E$ be defined by the formula:

$$y(t, q) = \begin{cases} x(t), & \text{for } 0 \leq t \leq q \\ x(q) + \frac{p(t) - x(q)}{\varepsilon}(t - q), & \text{for } q < t \leq \min(d, q + \varepsilon) \\ p(t) + (R) \int_0^t (R) \int_0^{t_1} \dots \\ \quad (P) \int_q^{t_{m-1}-\varepsilon} f(t_m, y(t_m, q)) dt_m \dots dt_2 dt_1, & \text{for } \min(d, q + \varepsilon) < t < d \end{cases}$$

By repeating the above consideration, with $y(\cdot, q)$ in the place of $v(\cdot, \varepsilon)$, one can show that $y(\cdot, q) \in S_\eta$ for each $q \in \langle 0, d \rangle$ and the mapping $q \rightarrow y(\cdot, q)$ from J into $(C(J, E), \omega)$ is sequentially continuous. Consequently, by Lemma 2.2, the set $T_x = \{y(\cdot, q) : 0 \leq q \leq d\}$ is connected in $(C(J, E), \omega)$.

As $y(\cdot, 0) = v(\cdot, \varepsilon) \in V \cap T_x$, the set $V \cup T_x$ is connected, and therefore the set $W = \bigcup_{x \in S_\eta} T_x \cup V$ is connected in $(C(J, E), \omega)$.

Moreover $S_\eta \subset W$, because $x = y(\cdot, d) \in T_x$ for each $x \in S_\eta$. On the other hand $W \subset S_\eta$, since $T_x \subset S_\eta$ and $V \subset S_\eta$. Finally $S_\eta \subset W$ is a connected subset of $(C(J, E), \omega)$.

Suppose that the set S is not connected. As S weakly compact, there exist nonempty weakly compact sets W_1 and W_2 such that $S = W_1 \cup W_2$

and $W_1 \cap W_2 = \emptyset$. Consequently there exists two disjoint weakly open sets U_1, U_2 such that $W_1 \subset U_1, W_2 \subset U_2$. Suppose that for every $n \in N$, there exists a $u_n \in V_n \setminus U$, where $V_n = \overline{S_{1/n}^\omega}$ and $U = U_1 \cup U_2$.

Put $H = \overline{\{u_n : n \in N\}^\omega}$. Since $u_n - F_x(u_n) \rightarrow 0$ in $C(J, E)$ as $n \rightarrow \infty$ and $H(t) \subset \{u_n(t) - F_x(u_n)(t) : u_n \in H\} + F_x(H)(t)$ repeating the argument from Theorem 2.2, one can show that there exists $u_0 \in H$ such that $u_0 = F_x(u_0)$, i.e. $u_0 \in S \setminus U$. Furthermore, $S \subset (C(J, E), \omega) \setminus U$, since U is weakly open and hence $u_0 \in S$, a contradiction.

Therefore, there is $m \in N$ such that $V_m \subset U$. Since $U_1 \cap V_m \neq \emptyset \neq U_2 \cap V_m$, V_m is not connected, a contradiction with the connectedness of each V_n . Consequently, S is connected in $(C(J, E), \omega)$. $\square$

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