

On the second moment of $S(T)$ in the theory of the Riemann zeta function

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Abstract. We obtain better asymptotic formulas for the second moment of $S(T)$ under the Riemann Hypothesis and a quantitative form of the Twin Prime Conjecture. It goes beyond the Random Matrix Theory prediction.

1. Introduction

Denote by $\rho = \beta + i\gamma$ the nontrivial zeros of the Riemann zeta function $\zeta(s)$. For $T \neq \gamma$,

$$S(T) := \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right), \quad (1)$$

where the argument is obtained by continuous variation along the horizontal line $\sigma + iT$ starting with the value zero at $\infty + iT$. For $T = \gamma$, we define

$$S(T) := \lim_{\epsilon \rightarrow 0} \frac{1}{2} \{S(T + \epsilon) + S(T - \epsilon)\}.$$

The Riemann Hypothesis (abbreviated as RH) asserts that the real parts of all nontrivial zeros of $\zeta(s)$ satisfy $\beta = 1/2$. In [4], GOLDSTON proved

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that, under RH,

$$\begin{aligned} \int_0^T |S(t)|^2 dt &= \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left[\int_1^\infty \frac{F(\alpha, T)}{\alpha^2} d\alpha + C_0 \right. \\ &\quad \left. + \sum_{m=2}^\infty \sum_p \left(\frac{-1}{m} + \frac{1}{m^2} \right) \frac{1}{p^m} \right] + o(T) \end{aligned} \quad (2)$$

where C_0 is Euler's constant, and

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma') \quad (3)$$

is Montgomery's pair correlation function with $w(u) = \frac{4}{4+u^2}$. Note: A minus sign is missing in GOLDSTON [4].

Here and throughout this paper, $\sum_{\gamma, \gamma'}$ denotes a sum over pairs of imaginary parts γ, γ' of nontrivial zeros of $\zeta(s)$. Also, p will denote a prime and sums over p are over all primes. We put $L := \log \frac{T}{2\pi e}$ and we shall use the following modification of (3):

$$F(\alpha) = F(\alpha, T) := \left(\frac{TL}{2\pi} \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} \left(\frac{T}{2\pi e} \right)^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma').$$

In [3], the author studied $F(\alpha)$ under RH and the following quantitative form of Twin Prime Conjecture (abbreviated as TPC): For any $\epsilon > 0$,

$$\sum_{n=1}^N \Lambda(n) \Lambda(n+d) = \mathfrak{S}(d)N + O(N^{1/2+\epsilon})$$

uniformly in $|d| \leq N$. Here $\Lambda(n)$ is the von Mangoldt lambda function; and $\mathfrak{S}(d) = 2 \prod_{p>2} (1 - \frac{1}{(p-1)^2}) \prod_{p|d, p>2} \frac{p-1}{p-2}$ when d is even, and $\mathfrak{S}(d) = 0$ when d is odd.

Combining [2] and [3], the author proved that, under RH and TPC,

$$F(\alpha) = \begin{cases} \alpha + T^{-2\alpha}L + O(\alpha T^{\alpha-1}) \\ \quad + O(T^{-(1/2-\epsilon)\alpha}L^{-1}), & \text{if } 0 \leq \alpha \leq 1 - \frac{3 \log \log T}{\log T}, \\ \alpha + O(T^{\alpha-1}L^{-1}), & \text{if } 1 - \frac{3 \log \log T}{\log T} \leq \alpha \leq 1 \end{cases} \quad (4)$$

for any $\epsilon > 0$. Using this and more careful calculations, we have

Theorem 1.1. *Assume RH and (4).*

$$\begin{aligned} \int_0^T |S(t)|^2 dt &= \frac{T}{2\pi^2} \log \log \frac{T}{2\pi e} + \frac{T}{2\pi^2} \left[\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha + C_0 \right. \\ &\quad \left. - \sum_{m=2}^\infty \sum_p \left(\frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + O\left(\frac{T}{L^2}\right). \end{aligned}$$

Theorem 1.2. *Assume RH and TPC.*

$$\begin{aligned} \int_0^T |S(t)|^2 dt &= \frac{T}{2\pi^2} \log \log \frac{T}{2\pi e} + \frac{T}{2\pi^2} \left[\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha + C_0 \right. \\ &\quad \left. - \sum_{m=2}^\infty \sum_p \left(\frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + \frac{C_1 T}{2\pi^2 L^2} + O\left(\frac{T \log L}{L^3}\right) \end{aligned} \quad (5)$$

where

$$\begin{aligned} C_1 &= -\frac{4}{3\pi e} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + \int_1^\infty \frac{\sin(2\pi e v)}{v^2} dv \right] \\ &\quad + \frac{1}{4\pi^2 e^2} \sum_{h=1}^\infty \frac{\mathfrak{S}(h)}{h^2} \left[1 - \cos(2\pi e) + 2\pi e \int_1^\infty \frac{\sin(2\pi e v)}{v^2} dv \right] \\ &\quad + \frac{1}{\pi e} \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv dy \\ &\quad + \frac{1}{\pi e} \int_1^\infty \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv dy \\ &\quad + \frac{3}{4} - \frac{3}{2} \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha. \end{aligned}$$

These improve Goldston's result (2) and we shall compare them with the Random Matrix Theory prediction in the last section.

2. Preparations

First, let us recall two lemmas from [4].

Lemma 2.1. Assume RH. For $t \geq 1$, $t \neq \gamma$, $x \geq 4$,

$$\begin{aligned} S(t) = & -\frac{1}{\pi} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right) \\ & + \frac{1}{\pi} \sum_{\gamma} \sin((t - \gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t - \gamma) \log x)^2} \frac{du}{\sinh u} \\ & + O\left(\frac{x^{1/2}}{t^2 \log x}\right) + O\left(\frac{1}{t \log x}\right), \end{aligned} \quad (6)$$

where

$$f(u) = \frac{\pi}{2} u \cot\left(\frac{\pi}{2} u\right), \quad (7)$$

and $\Lambda(n) = \log p$ if $n = p^m$, for p a prime and $m \geq 1$, and $\Lambda(n) = 0$ otherwise.

Lemma 2.2. Let

$$R(x) := \int_1^T \left| \frac{1}{\pi} \sum_{\gamma} \sin((t - \gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t - \gamma) \log x)^2} \frac{du}{\sinh u} \right|^2 dt.$$

Then, for $x \geq 4$ and $T \geq 2$,

$$R(x) = \frac{1}{\pi^2 \log x} \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) + O(\log^3 T), \quad (8)$$

where

$$k(u) := \begin{cases} \left(\frac{1}{2u} - \frac{\pi^2}{2} \cot(\pi^2 u) \right)^2, & \text{if } |u| \leq \frac{1}{2\pi}, \\ \frac{1}{4u^2}, & \text{if } |u| > \frac{1}{2\pi}, \end{cases}$$

and $\hat{k}$ denotes the Fourier transform of k ,

$$\hat{k}(y) = \int_{-\infty}^{\infty} k(u) e(-uy) du, \quad e(u) := e^{2\pi i u}.$$

By straightforward calculations, we have

Lemma 2.3. *Let $f(a^+) := \lim_{x \rightarrow a, x > a} f(x)$ and $f(a^-) := \lim_{x \rightarrow a, x < a} f(x)$. Then*

$$\begin{aligned} k'(0) &= 0, & k''(0) &= \frac{\pi^8}{18}, \\ k'\left(\frac{1^+}{2\pi}\right) &= -4\pi^3, & k'\left(\frac{1^-}{2\pi}\right) &= -4\pi^3 + \pi^5, \\ k''\left(\frac{1^+}{2\pi}\right) &= 24\pi^4, & k''\left(\frac{1^-}{2\pi}\right) &= \frac{\pi^8}{2} - 4\pi^6 + 24\pi^4. \end{aligned}$$

Lemma 2.4.

$$\hat{k}(y) = -\frac{1}{(2\pi y)^2} \int_{-\infty}^{\infty} k''(u) e(-uy) du + \frac{\pi^3}{2y^2} \cos y.$$

PROOF. It is easy to check that $k(u) \ll \min(1, 1/u^2)$, $k'(u) \ll \min(1, 1/u^3)$ and $k''(u) \ll \min(1, 1/u^4)$ except at $u = 1/2\pi$. Integrating by parts twice,

$$\begin{aligned} \hat{k}(y) &= \int_{-\infty}^{\infty} k(u) e(-uy) du \\ &= \frac{-1}{2\pi iy} \int_{-\infty}^{\infty} k(u) d e(-uy) = \frac{1}{2\pi iy} \int_{-\infty}^{\infty} e(-uy) k'(u) du \\ &= \frac{-1}{(2\pi iy)^2} \int_{-\infty}^{\infty} k'(u) d e(-uy) = \frac{1}{(2\pi iy)^2} \int_{-\infty}^{\infty} e(-uy) dk'(u) \\ &= \frac{1}{(2\pi iy)^2} \int_{-\infty}^{\infty} k''(u) e(-uy) du + \left(k'\left(\frac{1^+}{2\pi}\right) - k'\left(\frac{1^-}{2\pi}\right) \right) e\left(\frac{y}{2\pi}\right) \\ &\quad + \left(k'\left(-\frac{1^+}{2\pi}\right) - k'\left(-\frac{1^-}{2\pi}\right) \right) e\left(-\frac{y}{2\pi}\right) \\ &= \frac{-1}{(2\pi y)^2} \int_{-\infty}^{\infty} k''(u) e(-uy) du + \frac{\pi^3}{2y^2} \cos y \end{aligned}$$

by Lemma 2.3 and the fact that $k(u)$ is even. □

Our key improvement is the following

Lemma 2.5. *Let $x = (\frac{T}{2\pi e})^\beta$. For any $\beta > 0$,*

$$\begin{aligned} & \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) \frac{(\gamma - \gamma')^2}{4 + (\gamma - \gamma')^2} \\ &= \frac{\pi^2 T}{16L} \frac{F(\beta)}{\beta^2} - \frac{T}{64\pi^4 L \beta^3} \int_{-\infty}^{\infty} F(\alpha) k'' \left(\frac{\alpha}{2\pi\beta} \right) d\alpha, \end{aligned}$$

where $F(\alpha)$ is given by (3).

PROOF. By Lemma 2.4, the above sum is

$$\begin{aligned} &= \frac{\pi^3}{2(\log x)^2} \sum_{0 < \gamma, \gamma' \leq T} \cos((\gamma - \gamma') \log x) \frac{1}{4 + (\gamma - \gamma')^2} \\ &\quad - \frac{1}{(2\pi \log x)^2} \sum_{0 < \gamma, \gamma' \leq T} \int_{-\infty}^{\infty} k''(u) e(-u(\gamma - \gamma') \log x) du \frac{1}{4 + (\gamma - \gamma')^2} \\ &= \frac{\pi^3}{8(\log x)^2} \sum_{0 < \gamma, \gamma' \leq T} x^{i(\gamma - \gamma')} w(\gamma - \gamma') \\ &\quad - \frac{1}{(4\pi \log x)^2} \int_{-\infty}^{\infty} \left(\sum_{0 < \gamma, \gamma' \leq T} e(-u(\gamma - \gamma') \log x) w(\gamma - \gamma') \right) k''(u) du \\ &= \frac{\pi^3 \frac{TL}{2\pi}}{8(\log x)^2} F(\beta) - \frac{\frac{TL}{2\pi}}{(4\pi \log x)^2} \int_{-\infty}^{\infty} F(2\pi u\beta) k''(u) du \end{aligned}$$

which gives the lemma after substituting $\alpha = 2\pi u\beta$ in the integral. $\square$

Using Lemmas 2.2 and 2.5, we have the following improvement of Lemma 3 in [4]:

Lemma 2.6. *Let $\beta > 0$ and $x = (\frac{T}{2\pi e})^\beta$. Then*

$$\begin{aligned} R(x) &= \frac{T}{(2\pi^2\beta)^2} \int_{-\infty}^{\infty} F(\alpha) k \left(\frac{\alpha}{2\pi\beta} \right) d\alpha + \frac{T}{16L^2} \frac{F(\beta)}{\beta^3} \\ &\quad - \frac{T}{64\pi^6 \beta^4 L^2} \int_{-\infty}^{\infty} F(\alpha) k'' \left(\frac{\alpha}{2\pi\beta} \right) d\alpha + O(L^3). \end{aligned} \tag{9}$$

PROOF. One simply notes that

$$\begin{aligned} \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) &= \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) w(\gamma - \gamma') \\ &+ \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) \frac{(\gamma - \gamma')^2}{4 + (\gamma - \gamma')^2} = \Sigma_1 + \Sigma_2. \end{aligned}$$

By Lemma 2.5, we have

$$\Sigma_2 = \frac{\pi^2 T F(\beta)}{16L \beta^2} - \frac{T}{64\pi^4 L \beta^3} \int_{-\infty}^{\infty} F(\alpha) k'' \left(\frac{\alpha}{2\pi\beta} \right) d\alpha. \quad (10)$$

Using the definition of $\hat{k}$,

$$\begin{aligned} \Sigma_1 &= \int_{-\infty}^{\infty} k(u) \sum_{0 < \gamma, \gamma' \leq T} e(-u(\gamma - \gamma') \log x) w(\gamma - \gamma') du \\ &= \frac{TL}{2\pi} \int_{-\infty}^{\infty} F(2\pi u \beta) k(u) du = \frac{TL}{(2\pi)^2 \beta} \int_{-\infty}^{\infty} F(\alpha) k \left(\frac{\alpha}{2\pi\beta} \right) d\alpha. \end{aligned} \quad (11)$$

Putting (10) and (11) into (8), we have the lemma. $\square$

Lemma 2.7. *For any $\beta > 0$,*

$$\int_0^\beta \left(\frac{T}{2\pi e} \right)^{-2\alpha} k'' \left(\frac{\alpha}{2\pi\beta} \right) d\alpha = 16\pi^2 \beta^2 L^2 \int_0^\beta \left(\frac{T}{2\pi e} \right)^{-2\alpha} k \left(\frac{\alpha}{2\pi\beta} \right) d\alpha.$$

PROOF. Integrating by parts twice. $\square$

Lemma 2.8. *Assume RH and (4). For any $\epsilon > 0$ and $0 < \beta < 1$,*

$$\begin{aligned} \int_{-\infty}^{\infty} F(\alpha) k \left(\frac{\alpha}{2\pi\beta} \right) d\alpha &= 2\pi^2 \beta^2 \left[1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} + \int_1^{\infty} \frac{F(\alpha)}{\alpha^2} d\alpha - \log \beta \right] \\ &+ 2L \int_0^\beta \left(\frac{T}{2\pi e} \right)^{-2\alpha} k \left(\frac{\alpha}{2\pi\beta} \right) d\alpha + O \left(\frac{1}{\beta^2 L^4} \right) \\ &+ O \left(\frac{L}{T^{(1/2-\epsilon)\beta}} \right) + O \left(\frac{\beta^2}{L^2} \right). \end{aligned}$$

PROOF. Let $\epsilon_T = 3 \frac{\log \log T}{\log T}$. Since F and k are even,

$$\begin{aligned} & \int_{-\infty}^{\infty} F(\alpha) k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &= 2 \left(\int_0^{\beta} + \int_{\beta}^{1-\epsilon_T} + \int_{1-\epsilon_T}^1 + \int_1^{\infty} \right) F(\alpha) k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &= 2(I_1 + I_2 + I_3 + I_4). \end{aligned}$$

By (4),

$$\begin{aligned} I_1 &= \int_0^{\beta} \left[\alpha + \frac{L}{T^{2\alpha}} \right] \left[\frac{\pi\beta}{\alpha} - \frac{\pi^2}{2} \cot\left(\frac{\pi\alpha}{2\beta}\right) \right]^2 d\alpha \\ &\quad + O\left(\int_0^{\beta} \alpha T^{\alpha-1} \frac{\alpha^2}{\beta^2} d\alpha\right) + O\left(\int_0^{\beta} \frac{T^{-(1/2-\epsilon)\alpha}}{L} \frac{\alpha^2}{\beta^2} d\alpha\right) \\ &= \int_0^{\beta} \alpha \left[\frac{\pi\beta}{\alpha} - \frac{\pi^2}{2} \cot\left(\frac{\pi\alpha}{2\beta}\right) \right]^2 d\alpha + L \int_0^{\beta} \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &\quad + O\left(\frac{\beta T^{\beta-1}}{L}\right) + O\left(\frac{1}{\beta^2 L^4}\right) \end{aligned}$$

because $\cot(x) = 1/x + O(x)$ when $0 \leq x \leq \pi/2$. The first integral can be evaluated by elementary means. This gives

$$I_1 = \pi^2 \beta^2 \left[1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} \right] + L \int_0^{\beta} \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + O\left(\frac{1}{\beta^2 L^4}\right).$$

By (4) again, we have,

$$\begin{aligned} I_2 &= \int_{\beta}^{1-\epsilon_T} \left[\alpha + O(\alpha T^{\alpha-1}) + O\left(\frac{L}{T^{(1/2-\epsilon)\alpha}}\right) \right] \left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha \\ &= \pi^2 \beta^2 [\log(1-\epsilon_T) - \log \beta] + O\left(\frac{1}{L^4}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right), \\ I_3 &= \int_{1-\epsilon_T}^1 \left[\alpha + O\left(\frac{T^{\alpha-1}}{L}\right) \right] \left(\frac{\pi\beta}{\alpha}\right)^2 d\alpha \\ &= -\pi^2 \beta^2 \log(1-\epsilon_T) + O\left(\frac{\beta^2}{L} \int_{1-\epsilon_T}^1 T^{\alpha-1} d\alpha\right) \end{aligned}$$

$$= -\pi^2 \beta^2 \log(1 - \epsilon_T) + O\left(\frac{\beta^2}{L^2}\right).$$

Finally,

$$I_4 = \pi^2 \beta^2 \int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha.$$

Combining the above results for I_1 , I_2 , I_3 and I_4 , we have the lemma. $\square$

Lemma 2.9. Assume RH and (4). For any $\epsilon > 0$ and $0 < \beta < 1$,

$$\begin{aligned} \int_{-\infty}^\infty F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha &= 4\pi^6 \beta^2 - 24\pi^4 \beta^4 + 48\pi^4 \beta^4 \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha \\ &\quad + 32\pi^2 \beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &\quad + O\left(\frac{1}{L^2}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right). \end{aligned}$$

PROOF. Let $\epsilon_T = 3 \frac{\log \log T}{\log T}$. Again, since F and k are even,

$$\begin{aligned} \int_{-\infty}^\infty F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha &= 2 \left(\int_0^\beta + \int_\beta^{1-\epsilon_T} + \int_{1-\epsilon_T}^1 + \int_1^\infty \right) F(\alpha) k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &= 2(J_1 + J_2 + J_3 + J_4). \end{aligned}$$

By (4) and Lemma 2.7,

$$\begin{aligned} J_1 &= \int_0^\beta \alpha k''\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + 16\pi^2 \beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &\quad + O\left(\int_0^\beta \alpha T^{\alpha-1} d\alpha\right) + O\left(\int_0^\beta \frac{T^{-(1/2-\epsilon)\alpha}}{L} d\alpha\right) \\ &= 4\pi^2 \beta^2 \int_0^{1/2\pi} u k''(u) du + 16\pi^2 \beta^2 L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha \\ &\quad + O\left(\frac{T^{\beta-1}}{L^2}\right) + O\left(\frac{1}{L^2}\right) \end{aligned}$$

since $k''(x) \ll 1$ when $0 \leq x \leq 1/2\pi$. Using integration by parts twice and Lemma 2.3 to compute the first integral, we have

$$J_1 = 2\pi^4(\pi^2 - 6)\beta^2 + 16\pi^2\beta^2L^3 \int_0^\beta \left(\frac{T}{2\pi e}\right)^{-2\alpha} k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha + O\left(\frac{1}{L^2}\right).$$

By (4) again, we have

$$\begin{aligned} J_2 &= \int_\beta^{1-\epsilon_T} \left[\alpha + O(\alpha T^{\alpha-1}) + O\left(\frac{L}{T^{(1/2-\epsilon)\alpha}}\right) \right] \left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha \\ &= 12\pi^4\beta^4 \left[\frac{1}{\beta^2} - \frac{1}{(1-\epsilon_T)^2} \right] + O\left(\frac{1}{L^4}\right) + O\left(\frac{L}{T^{(1/2-\epsilon)\beta}}\right), \\ J_3 &= \int_{1-\epsilon_T}^1 \left[\alpha + O\left(\frac{T^{\alpha-1}}{L}\right) \right] \left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha \\ &= 12\pi^4\beta^4 \left[\frac{1}{(1-\epsilon_T)^2} - 1 \right] + O\left(\frac{\beta^4}{L} \int_{1-\epsilon_T}^1 T^{\alpha-1} d\alpha\right) \\ &= 12\pi^4\beta^4 \left[\frac{1}{(1-\epsilon_T)^2} - 1 \right] + O\left(\frac{\beta^4}{L^2}\right). \end{aligned}$$

Finally,

$$J_4 = 24\pi^4\beta^4 \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha.$$

Combining the above results for J_1 , J_2 , J_3 and J_4 , we have the lemma. $\square$

Combining Lemmas 2.6, 2.8 and 2.9, we have

Lemma 2.10. Assume RH and (4). For $0 < \beta < 1$ and $x = (\frac{T}{2\pi e})^\beta$,

$$\begin{aligned} R(x) &= \frac{T}{2\pi^2} \left[1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} + \int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha - \log \beta \right] + \frac{3T}{8\pi^2 L^2} \\ &\quad - \frac{3T}{4\pi^2 L^2} \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha + O\left(\frac{T}{L^2}\right) + O\left(\frac{T}{\beta^4 L^4}\right). \end{aligned}$$

Note: This is more precise than Lemma 4 of [4]. Also, we keep some of the T/L^2 terms explicit because one can actually make the $O(T/L^2)$ error

term $= C_1 T/L^2 + O(T \log L/L^3)$ for some constant C_1 using Theorem 1.1 of [3]. We defer this discussion to Section 4.

Following [4], we need to compute the mean value of the Dirichlet series in Lemma 2.1, and the cross term obtained from multiplying $S(t)$ with this series. Let

$$G(T) := \int_1^T \left| \frac{1}{\pi} \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right) \right|^2 dt$$

and

$$H(T) := \frac{2}{\pi} \int_1^T S(t) \sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2}} \frac{\sin(t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right) dt,$$

where f is defined as in (7). We need a lemma.

Lemma 2.11. *For $C \geq 2$ and $k \geq 1$,*

$$\sum_{n=1}^{\infty} \frac{n^k}{C^n} \ll_k \frac{1}{C}.$$

PROOF. First, we note that $u^k C^{-u}$ is decreasing when $u > \frac{k}{\log C}$. So,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^k}{C^n} &= \sum_{n \leq k/\log C} \frac{n^k}{C^n} + \sum_{n > k/\log C} \frac{n^k}{C^n} \\ &\leq \left(\frac{k}{\log C}\right)^k \frac{1}{C-1} + \int_1^{\infty} u^k C^{-u} du \\ &\ll_k \frac{1}{C} + \frac{1}{\log C} C^{-1} \end{aligned}$$

by integration by parts. This gives the lemma. $\square$

From p. 165–166 of [4], we have, assuming RH,

$$G(T) = \frac{T}{2\pi^2} \sum_{n \leq x} \frac{\Lambda^2(n)}{n \log^2 n} f^2\left(\frac{\log n}{\log x}\right) + O(x^2), \quad (12)$$

$$H(T) = -\frac{T}{\pi^2} \sum_{n \leq x} \frac{\Lambda^2(n)}{n \log^2 n} f\left(\frac{\log n}{\log x}\right) + O(x^{2+\epsilon}) \quad (13)$$

for any $\epsilon > 0$. Adding (12) and (13), we get

$$\begin{aligned}
G(T) + H(T) &= \frac{T}{2\pi^2} \left[\sum_{p \leq x} \frac{1}{p} f^2 \left(\frac{\log p}{\log x} \right) - 2 \sum_{p \leq x} \frac{1}{p} f \left(\frac{\log p}{\log x} \right) \right. \\
&\quad \left. - \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1}{m^2 p^m} + \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1}{m^2 p^m} \left(f \left(\frac{m \log p}{\log x} \right) - 1 \right)^2 \right] + O(x^{2+\epsilon}) \\
&= \frac{T}{2\pi^2} [S_1 - 2S_2 - S_3 + S_4] + O(x^{2+\epsilon}), \\
S_3 &= \sum_{m=2}^{\infty} \sum_p \frac{1}{m^2 p^m} + O \left(\sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{n \geq x^{1/m}} \frac{1}{n^m} \right) \\
&= \sum_{m=2}^{\infty} \sum_p \frac{1}{m^2 p^m} + O \left(\sum_{m=2}^{\infty} \frac{x^{-(m-1)/m}}{m^2(m-1)} \right) = \sum_{m=2}^{\infty} \sum_p \frac{1}{m^2 p^m} + O \left(\frac{1}{x^{1/2}} \right).
\end{aligned}$$

By Taylor's expansion of $\tan x$, we have $f(u) = 1 + O(u^2)$ when $0 \leq u \leq 1$. Thus,

$$\begin{aligned}
S_4 &\ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{p^m \leq x} \frac{\log^4 p}{p^m} \ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{i=1}^{\infty} \sum_{2^i \leq p \leq 2^{i+1}} \frac{i^4}{2^{mi}} \\
&\ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{i=1}^{\infty} \frac{i^4}{(2^{m-1})^i} \ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} \frac{m^2}{2^{m-1}} \ll \frac{1}{\log^4 x}
\end{aligned}$$

by applying Lemma 2.11 twice.

We now define

$$T(u) := \sum_{2 \leq p \leq u} \frac{1}{p}.$$

Then

$$\begin{aligned}
T(u) &= \log \log u + C_0 + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) + r(u) \\
&= \log \log u + C_0 - \sum_{m=2}^{\infty} \sum_p \frac{1}{m p^m} + r(u)
\end{aligned}$$

where $r(u) \ll \log u / \sqrt{u}$ under RH. By the Riemann–Stieltjes integral,

$$\begin{aligned} S_1 &= \int_2^x f^2 \left(\frac{\log u}{\log x} \right) dT(u) \\ &= \int_2^x f^2 \left(\frac{\log u}{\log x} \right) \frac{du}{u \log u} + \int_2^x f^2 \left(\frac{\log u}{\log x} \right) dr(u) = I_1 + I_2. \end{aligned}$$

Similarly,

$$S_2 = \int_2^x f \left(\frac{\log u}{\log x} \right) \frac{du}{u \log u} + \int_2^x f \left(\frac{\log u}{\log x} \right) dr(u) = J_1 + J_2.$$

Thus,

$$S_1 - 2S_2 = (I_1 - 2J_1) + (I_2 - 2J_2).$$

By integration by parts,

$$\begin{aligned} I_2 - 2J_2 &= -r(2^-) \left[f^2 \left(\frac{\log 2}{\log x} \right) - 2f \left(\frac{\log 2}{\log x} \right) \right] \\ &\quad - \int_2^x r(u) \left[2f \left(\frac{\log u}{\log x} \right) f' \left(\frac{\log u}{\log x} \right) - f' \left(\frac{\log u}{\log x} \right) \right] \frac{du}{u \log x} \\ &= r(2^-) - r(2^-) \left[f \left(\frac{\log 2}{\log x} \right) - 1 \right]^2 \\ &\quad - \frac{2}{\log x} \int_2^x r(u) f' \left(\frac{\log u}{\log x} \right) \left[f \left(\frac{\log u}{\log x} \right) - 1 \right] \frac{du}{u} \\ &= -\log \log 2 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m} + O \left(\frac{1}{\log^4 x} \right) \\ &\quad + O \left(\frac{1}{\log x} \int_2^x \frac{\log u}{\sqrt{u}} \left(\frac{\log u}{\log x} \right) \left(\frac{\log u}{\log x} \right)^2 \frac{du}{u} \right) \\ &= -\log \log 2 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m} + O \left(\frac{1}{\log^4 x} \right) \end{aligned}$$

because $f(u) = 1 + O(u^2)$ and $f'(u) \ll u$ when $0 \leq u \leq 1$. The integrals in I_1 and J_1 can be evaluated by elementary means. Using $u \cot u =$

$1 - u^2/3 + O(u^4)$ and $\sin u = u - u^3/6 + O(u^5)$, one has

$$I_1 = \log \log x - \log \log 2 - \frac{\pi^2}{8} + 1 - \log \frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{12 \log^2 x} + O\left(\frac{1}{\log^4 x}\right),$$

$$J_1 = \log \log x - \log \log 2 - \log \frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{24 \log^2 x} + O\left(\frac{1}{\log^4 x}\right).$$

Hence,

$$S_1 - 2S_2 = -\log \log x + \log \frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m} + O\left(\frac{1}{\log^4 x}\right).$$

Therefore,

$$\begin{aligned} G(T) + H(T) &= \frac{T}{2\pi^2} \left[-\log \log x + \log \frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 \right. \\ &\quad \left. + \sum_{m=2}^{\infty} \sum_p \left(\frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + O\left(\frac{T}{\log^4 x}\right). \end{aligned} \quad (14)$$

3. Proof of Theorem 1.1

Suppose $x = (\frac{T}{2\pi e})^\beta$ and β is a fixed positive number less than $1/2$. By Lemma 2.1, (6) holds except on a countable set of points. Hence, on squaring both sides of (6) and integrating from 1 to T ,

$$\int_1^T S(t)^2 dt + H(T) + G(T) = R(x) + O(T^{1/2}x^{1/2}),$$

where the error term is obtained by the Cauchy–Schwarz inequality since $R \ll T$. The lower limit of integration may be replaced by zero since $\int_0^1 S(t)^2 dt \ll 1$. Then, Lemma 2.10 and (14) give the theorem.

4. Proof of Theorem 1.2

First, let us recall some definitions in [3]. Let

$$\epsilon(u) := \sum_{h \leq u} \mathfrak{S}(h) - u + \frac{1}{2} \log u, \quad (15)$$

and

$$f(y) := \int_0^y \epsilon(u) - \frac{B}{2} du \quad \text{with} \quad B := -C_0 - \log 2\pi. \quad (16)$$

Putting (15) into (16), we have

$$f(y) = \sum_{h \leq y} \mathfrak{S}(h)(y-h) - \frac{1}{2}y^2 + \frac{1}{2}y \log y - \left(\frac{1+B}{2}\right)y. \quad (17)$$

If one traces the proof of Theorem 1.1, the only ambiguous T/L^2 term comes from the error term of I_3 in Lemma 2.8. Thus, we need a more precise formula for $F(\alpha)$ when $1 - 3\frac{\log \log T}{\log T} \leq \alpha \leq 1$. Let $\tau = T/(2\pi e)$ and $\epsilon_T = 3\frac{\log \log T}{\log T}$. From Theorem 1.1 of [3], one has

$$\begin{aligned} F(\alpha) = & \alpha - \frac{4\tau^{\alpha-1}}{3\pi eL} \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv \\ & + \frac{\tau^{2\alpha-2}}{2\pi^2 e^2 L} \left[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \right] (1 - \cos(2\pi e\tau^{1-\alpha})) \\ & + \frac{2}{L} \int_1^{\infty} \left[-\frac{1}{2y} - \frac{4f(y)}{y^2} + \frac{2}{y^3} \int_0^y f(u) du \right. \\ & \left. + 6y \int_y^{\infty} \frac{f(u)}{u^4} du \right] \frac{\sin(2\pi e\tau^{1-\alpha}y)}{2\pi e\tau^{1-\alpha}y} dy + O\left(\frac{1}{L^M}\right) \end{aligned} \quad (18)$$

for some large $M > 0$. Note: It is here that we require the full strength of TPC. By (17), one can simplify (18) to

$$\begin{aligned} F(\alpha) = & \alpha - \frac{4\tau^{\alpha-1}}{3\pi eL} \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv \\ & + \frac{\tau^{2\alpha-2}}{2\pi^2 e^2 L} \left[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \right] (1 - \cos(2\pi e\tau^{1-\alpha})) \\ & + \frac{2}{L} \int_1^{\infty} \left(\frac{\sum_{h \leq y} \mathfrak{S}(h)h^2}{y^3} - \frac{1}{3} \right) \frac{\sin(2\pi e\tau^{1-\alpha}y)}{2\pi e\tau^{1-\alpha}y} dy \\ & + \frac{2}{L} \int_1^{\infty} \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \frac{\sin(2\pi e\tau^{1-\alpha}y)}{2\pi e\tau^{1-\alpha}y} dy + O\left(\frac{1}{L^M}\right). \end{aligned} \quad (19)$$

Hence, the error term of I_3 in Lemma 2.8 can be replaced by

$$\begin{aligned}
& -\frac{4(\pi\beta)^2}{3\pi eL} \int_{1-\epsilon_T}^1 \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{\tau^{\alpha-1}}{\alpha^2} d\alpha \\
& + \frac{(\pi\beta)^2}{2\pi^2 e^2 L} \left[\sum_{h=1}^{\infty} \frac{\mathfrak{S}(h)}{h^2} \right] \int_{1-\epsilon_T}^1 (1 - \cos(2\pi e\tau^{1-\alpha})) \frac{\tau^{2\alpha-2}}{\alpha^2} d\alpha \\
& + \frac{2(\pi\beta)^2}{L} \int_{1-\epsilon_T}^1 \int_1^{\infty} \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{\sin(2\pi e\tau^{1-\alpha} y)}{2\pi e\tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\
& + \frac{2(\pi\beta)^2}{L} \int_{1-\epsilon_T}^1 \int_1^{\infty} \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \frac{\sin(2\pi e\tau^{1-\alpha} y)}{2\pi e\tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\
& + O\left(\frac{\beta^2}{L^M}\right). \tag{20}
\end{aligned}$$

Lemma 4.1.

$$\begin{aligned}
& \int_{1-\epsilon_T}^1 \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{\tau^{\alpha-1}}{\alpha^2} d\alpha \\
& = \frac{1}{L} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + \int_1^{\infty} \frac{\sin(2\pi e v)}{v^2} dv + O(\epsilon_T) \right].
\end{aligned}$$

PROOF. By integration by parts, the left hand side is

$$\begin{aligned}
& = \frac{1}{L} \int_{1-\epsilon_T}^1 \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv \frac{1}{\alpha^2} d\tau^{\alpha-1} \\
& = \frac{1}{L} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O\left(\frac{1}{L^3}\right) \right. \\
& \quad \left. + \int_{1-\epsilon_T}^1 \tau^{\alpha-1} \left[\frac{2}{\alpha^3} \int_0^{2\pi e\tau^{1-\alpha}} \frac{\sin v}{v} dv + \frac{L}{\alpha^2} \sin(2\pi e\tau^{1-\alpha}) \right] d\alpha \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{L} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{1}{(1 - \log v/L)^2} \frac{\sin(2\pi ev)}{v^2} dv \right] \\
&= \frac{1}{L} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{\sin(2\pi ev)}{v^2} (1 + O(\epsilon_T)) dv \right] \\
&= \frac{1}{L} \left[\int_0^{2\pi e} \frac{\sin v}{v} dv + O(\epsilon_T) + \int_1^{\tau^{\epsilon_T}} \frac{\sin(2\pi ev)}{v^2} dv \right]
\end{aligned}$$

which gives the lemma as $\int_{\tau^{\epsilon_T}}^{\infty} \frac{\sin(2\pi ev)}{v^2} dv \ll \frac{1}{\tau^{\epsilon_T}} \ll \epsilon_T$. $\square$

Lemma 4.2.

$$\begin{aligned}
&\int_{1-\epsilon_T}^1 (1 - \cos(2\pi e\tau^{1-\alpha})) \frac{\tau^{2\alpha-2}}{\alpha^2} d\alpha \\
&= \frac{1}{2L} \left[1 - \cos(2\pi e) + 2\pi e \int_1^{\infty} \frac{\sin(2\pi ev)}{v^2} dv + O(\epsilon_T) \right].
\end{aligned}$$

PROOF. By integration by parts, the left hand side is

$$\begin{aligned}
&= \frac{1}{2L} \int_{1-\epsilon_T}^1 \frac{1 - \cos(2\pi e\tau^{1-\alpha})}{\alpha^2} d\tau^{2\alpha-2} \\
&= \frac{1}{2L} \left[1 - \cos(2\pi e) + O\left(\frac{1}{L^6}\right) \right. \\
&\quad \left. + \int_{1-\epsilon_T}^1 \tau^{2\alpha-2} \left[\frac{\sin(2\pi e\tau^{1-\alpha})}{\alpha^2} 2\pi e\tau^{1-\alpha} L + \frac{2(1 - \cos(2\pi e\tau^{1-\alpha}))}{\alpha^3} \right] d\alpha \right] \\
&= \frac{1}{2L} \left[1 - \cos(2\pi e) + 2\pi e L \int_{1-\epsilon_T}^1 \frac{\sin(2\pi e\tau^{1-\alpha})}{\alpha^2 \tau^{1-\alpha}} d\alpha + O(\epsilon_T) \right] \\
&= \frac{1}{2L} \left[1 - \cos(2\pi e) + 2\pi e \int_1^{\tau^{\epsilon_T}} \frac{1}{(1 - \log v/L)^2} \frac{\sin(2\pi ev)}{v^2} dv + O(\epsilon_T) \right] \\
&= \frac{1}{2L} \left[1 - \cos(2\pi e) + 2\pi e \int_1^{\infty} \frac{\sin(2\pi ev)}{v^2} dv + O(\epsilon_T) \right]
\end{aligned}$$

by the same argument as in the proof of Lemma 4.1. $\square$

Recall a theorem in MONTGOMERY and SOUNDARARAJAN [7].

Lemma 4.3. For any $\epsilon > 0$,

$$\sum_{h \leq y} (y - h) \mathfrak{S}(h) = \frac{1}{2} y^2 - \frac{1}{2} y \log y - \left(\frac{1+B}{2} \right) y + O(y^{1/2+\epsilon}) \quad (21)$$

where $B = -C_0 - \log 2\pi$.

Lemma 4.4. For any $\epsilon > 0$,

$$\sum_{h \leq y} \mathfrak{S}(h) = y + O(y^{1/4+\epsilon}).$$

PROOF. Apply Lemma 4.3 with $y + \Delta y$ instead of y where $\Delta y \ll y$,

$$\begin{aligned} \sum_{h \leq y + \Delta y} (y + \Delta y - h) \mathfrak{S}(h) &= \frac{1}{2} (y + \Delta y)^2 - \frac{1}{2} (y + \Delta y) \log (y + \Delta y) \\ &\quad - \left(\frac{1+B}{2} \right) (y + \Delta y) + O(y^{1/2+\epsilon}). \end{aligned} \quad (22)$$

Note that $\mathfrak{S}(h) \ll h^\epsilon$. Equations (21) and (22) give

$$\Delta y \sum_{h \leq y} \mathfrak{S}(h) + O(\Delta y^2 y^\epsilon) = y \Delta y + \frac{1}{2} \Delta y \log y + O(\Delta y) + O(y^{1/2+\epsilon})$$

which gives the lemma after setting $\Delta y = y^{1/4}$ and dividing by Δy . $\square$

Lemma 4.5.

$$\begin{aligned} &\int_{1-\epsilon_T}^1 \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{\sin(2\pi e \tau^{1-\alpha} y)}{2\pi e \tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\ &= \frac{1}{2\pi e L} \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv dy + O\left(\frac{\epsilon_T}{L}\right). \end{aligned}$$

PROOF. By Lemma 4.4, the integrand on the left hand side is absolutely convergent. Thus, it is justified to change the order of integration.

The left hand side is

$$\begin{aligned}
&= \frac{1}{2\pi e} \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{1}{y} \int_{1-\epsilon_T}^1 \frac{\sin(2\pi e \tau^{1-\alpha} y)}{\alpha^2 \tau^{1-\alpha}} d\alpha dy \\
&= \frac{1}{2\pi e L} \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{1}{y} \int_1^{\tau^{\epsilon_T}} \frac{1}{(1 - \log v/L)^2} \frac{\sin(2\pi e y v)}{v^2} dv dy \\
&= \frac{1}{2\pi e L} \int_1^\infty \left(\frac{\sum_{h \leq y} \mathfrak{S}(h) h^2}{y^3} - \frac{1}{3} \right) \frac{1}{y} \int_1^\infty \frac{\sin(2\pi e y v)}{v^2} dv dy + O\left(\frac{\epsilon_T}{L}\right)
\end{aligned}$$

by similar argument as in the proof of Lemma 4.1. The lemma follows after substituting $u = yv$. $\square$

Lemma 4.6.

$$\begin{aligned}
&\int_{1-\epsilon_T}^1 \int_1^\infty \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \frac{\sin(2\pi e \tau^{1-\alpha} y)}{2\pi e \tau^{1-\alpha} y} dy \frac{1}{\alpha^2} d\alpha \\
&= \frac{1}{2\pi e L} \int_1^\infty \left(y \sum_{h > y} \frac{\mathfrak{S}(h)}{h^2} - 1 \right) \int_y^\infty \frac{\sin(2\pi e v)}{v^2} dv dy + O\left(\frac{\epsilon_T}{L}\right).
\end{aligned}$$

PROOF. It is similar to Lemma 4.5. $\square$

PROOF OF THEOREM 1.2: Applying Lemmas 4.1, 4.2, 4.5 and 4.6 to (20) and putting the result to Lemma 2.10, we have the T/L^2 term explicitly. Hence, we have Theorem 1.2.

5. Comparison with Random Matrix Theory

It is widely believed that the non-trivial zeros of the Riemann zeta function (and other L -functions) behave like the eigenvalues of an infinite complex Hermitian matrix drawn randomly from the Gaussian unitary ensemble (GUE). Using the GUE model, KEATING and SNAITH conjectured in [5, equation (98)] that, for even integer $k \geq 2$,

$$\frac{1}{T} \int_0^T \left(\operatorname{Im} \log \zeta \left(\frac{1}{2} + it \right) \right)^k dt \sim (-i)^k \frac{d^k}{ds^k} \left[L_N(s) b \left(\frac{s}{2} \right) \right]_{s=0},$$

where

$$L_N(s) = \prod_{j=1}^N \frac{\Gamma(j)^2}{\Gamma(j+s/2)\Gamma(j-s/2)},$$

and

$$b(\lambda) = \prod_p \left[\left(1 - \frac{1}{p}\right)^{-\lambda^2} \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(1+\lambda-n)\Gamma(1-\lambda-n)(n!)^2} p^{-n} \right].$$

Here $N = \log \frac{T}{2\pi e}$ is the mean density of the zeros of the Riemann zeta function up to height T . When $k = 2$, one has (see [5, equation (63)])

$$\frac{1}{T} \int_0^T \left(\operatorname{Im} \log \zeta \left(\frac{1}{2} + it \right) \right)^2 dt = \frac{1}{2} \log N + \frac{1}{2}(C_0 + 1) + \frac{1}{24N^2} + O\left(\frac{1}{N^4}\right)$$

which gives, via (1),

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log \log \frac{T}{2\pi e} + \frac{T}{2\pi^2} (1 + C_0) + \frac{T}{24\pi^2 L^2} + O\left(\frac{T}{L^4}\right). \quad (23)$$

Both (5) and (23) have the same leading order term $\frac{T}{2\pi^2} \log \log \frac{T}{2\pi e}$. However, one begins to see some differences in the next term T . MONTGOMERY [6] conjectured that

$$F(\alpha) = 1 + o(1) \quad \text{uniformly for } 1 \leq \alpha \leq M, \quad (24)$$

for any fixed M . This implies

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha = 1 + o(1).$$

Thus, the coefficient of the T term in (23) differs from that of (5) by a sum over primes. This is not surprising because the GUE model only gives the universal statistics while the sum over primes comes from non-universal part (see also the discussion in [1]).

Next, both (5) and (23) seems to have no $\frac{T}{L}$ term and their $\frac{T}{L^2}$ terms have different coefficients ($C_1 \approx 0.006953$ by Mathematica while $1/12 = 0.08333\dots$). However, the problem is that we still do not know anything precise about

$$\int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha.$$

Even (24) above only gives $o(T)$ as error term in (5). In order to get better error term, one may need to understand $F(\alpha, T)$ for longer range of α , say $1 \leq \alpha \leq \log T$. This would be a great challenge.

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