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Lie ideals and commuting mappings in prime rings

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Abstract. J. Vukman [6, Theorem 1] proved that if R is a prime ring of characteristic different from two and if d is a derivation of R such that $[[d(x), x], x] = 0$ for all x in R then either $d = 0$ or R is commutative. This result extends one due to E. Posner [5, Theorem 2]. In this paper our object is to generalize the above mentioned result of J. Vukman to Lie ideals and we prove the following:

Theorem. *Let R be a prime ring of characteristic different from two, and let d be a derivation of R. Let U be a Lie ideal of R such that $[[d(u), u], u] = 0$ for all $u \in U$. Then either $d = 0$ or $U \subset Z$, where Z is the center of R.*

1. Introduction

A theorem of J. Vukman [6] states that if R is a prime ring of characteristic different from 2, and if d is a derivation of R such that $(d(x)x - xd(x))x - x(d(x)x - xd(x)) = 0$ for all x in R then either $d = 0$ or R is commutative. In this paper we extend this result to Lie ideals.

Throughout the paper we assume that R is an associative ring of characteristic not 2. The center of R is denoted by Z. We also assume that d is a derivation of R, i.e, an additive mapping of R into itself such that $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. For $x, y \in R$, let

$$[x, y] = xy - yx \quad \text{and} \quad f(x) = [x, d(x)]$$

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$.

2. The main theorem

In this section we prove the main theorem of this paper which generalizes a result of J. Vukman [6, theorem 1] to Lie ideals. We begin with the main theorem.

Theorem 1. *Let R be a prime ring,* char $\neq 2$, *and let U be a Lie ideal of R. If d is a derivation of R such that $[[d(u),u],u]=0$ for all $u\in U$, then either $d=0$ or $U\subset Z$.*

Proof. By hypothesis,

$$[[d(u),u],u]=0 \quad \text{for all } u\in U. \tag{1}$$

In (1), replace u by $u+v$ where $v\in U$ and use (1) to get

$$[[d(u),u],v]+[[d(u),v],u]+[[d(v),u],u]+[[d(v),v],u]+ \\ +[[d(v),u],v]+[[d(u),v],v]=0.$$

Replace u by $-u$ then

$$[[d(u),u],v]+[[d(u),v],u]+[[d(v),u],u]-[[d(v),v],u]- \\ -[d(v),u],v]-[[d(u),v],v]=0.$$

Adding the last two equations and dividing by 2, we have

$$[[d(u),u],v]+[[d(u),v],u]+[[d(v),u],u]=0 \\ \text{for all } \ u,v\in U. \tag{2}$$

Suppose now that $v,w\in U$ are such that vw is also in U. By replacing v by vw in (2), where $w\in U$, and after expanding we get

$$v[[d(u),u],w]+[[d(u),u],v]w+v[[d(u),w],u]+[v,u][d(u),w]+ \\ +[d(u),v][w,u]+[[d(u),v],u]w+d(v)[[w,u],u]+[d(v),u][w,u]+ \\ +[d(v),u][w,u]+[[d(v),u],u]w+v[[d(w),u],u]+[v,u][d(w),u]+ \\ +[v,u][d(w),u]+[[v,u],u]d(w)=0.$$

In view of (2), the last equation reduces to

$$[v,u][d(u),w]+[d(u),v][w,u]+d(v)[[w,u],u]+2[d(v),u][w,u]+ \\ +2[v,u][d(w),u]+[[v,u],u]d(w)=0.$$

For any $r\in R$, the element $w=vr-rv$ satisfies the criterion $vw\in U$, hence by the above equation, we get

$$[v,u][d(u),[v,r]]+[d(u),v][[v,r],u]+d(v)[[[v,r],u],u]+ \\ +2[d(v),u][[v,r],u]+2[v,u]\ [d[v,r],u]+[[v,u],u]\ d[v,r]=0 \\ \text{for all } u,v\in U,\ r\in R. \tag{3}$$

Let $v=u$ in (3). Then

$$3[u,d(u)]\ [u,[u,r]]+d(u)\ [u,[u,[u,r]]]=0.$$

For $u \in U$, let $I(r) = [u, r]$ for all $r \in R$. Then I is an inner derivation of R.

Thus from the above equation, we get

$$(4) \qquad 3f(u)I^2(r) + d(u)I^3(r) = 0 \quad \text{for all} \quad r \in R,\ u \in U.$$

In (4), let $r = ur$. Then, as $I(u) = 0$,

$$3f(u)uI^2(r) + d(u)uI^3(r) = 0.$$

From (1) $uf(u) = f(u)u$. So by (4) and the last equation, we get

$$(5) \qquad f(u)I^3(r) = 0 \quad \text{for all} \ \ r \in R,\ u \in U.$$

Similarly, after replacing v by $wv = [v, r]v$ in (2), and then putting $v = u$, we obtain

$$(6) \qquad 3I^2(r)\ f(u) + I^3(r)d(u) = 0 \quad \text{for all} \ \ r \in R,\ u \in U.$$

Replace r by $rd(u)$ in (4) and use (4) to get

$$3f(u)\ \{2I(r)I(d(u)) + rI^2(d(u))\} + d(u)\ \{3I^2(r)\ I(d(u)) + \\ +3I(r)I^2(d(u)) + rI^3(d(u))\} = 0.$$

In view of (1), $I^2(d(u)) = 0$, and $I(d(u)) = [u, d(u)] = f(u)$. Thus, we get

$$(7) \qquad 6f(u)I(r)f(u) + 3d(u)I^2(r)f(u) = 0 \quad \text{for all} \ \ r \in R,\ u \in U.$$

Similarly, after replacing r by $d(u)r$ in (4), we have

$$(8) \qquad 6f(u)f(u)I(r) + 3f(u)d(u)I^2(r) = 0 \quad \text{for all } r \in R,\ u \in U.$$

In (6) replace r by $rd(u)$ and use (6). Then

$$(9) \qquad 6I(r)f(u)f(u) + 3I^2(r)f(u)d(u) = 0 \quad \text{for all} \ \ r \in R,\ u \in U.$$

Similarly, after replacing r by $d(u)r$ in (6), we get

$$(10) \qquad 6f(u)I(r)f(u) + 3f(u)I^2(r)d(u) = 0 \quad \text{for all} \ \ r \in R,\ u \in U.$$

In (4), let $r = rs$ where $s \in R$ and use (4) to get

$$(11) \qquad 3f(u)\{2I(r)I(s) + rI^2(s)\} + d(u)\{3I^2(r)I(s) + 3I(r)I^2(s) + \\ +rI^3(s)\} = 0 \quad \text{for all} \ \ r, s \in R, u \in U.$$

Replace r by $f(u)$ in (11). Then, as $I(f(u)) = 0$ from (1) and $f(u)I^3(s) = 0$ from (5), we get

$$(12) \qquad 3f(u)f(u)I^2(s) = 0 \quad \text{for all} \ \ s \in R, u \in U.$$

Write $s = rs$, $r \in R$ in (12) and use (12) to obtain

$$3f(u)f(u)\{2I(r)I(s) + rI^2(s)\} = 0 \quad \text{for all} \;\; r, s \in R, \;\; u \in U. \tag{13}$$

From (11) and (13), we get

$$f(u)d(u)\{3I^2(r)I(s) + 3I(r)I^2(s) + rI^3(s)\} = 0 \quad \text{for all} \;\; r, s \in R, \;\; u \in U. \tag{14}$$

Replacing r by ur in (14) and using (14), as $uf(u) = f(u)u$ from (1), we get

$$f(u)f(u)\left\{3I^2(r)I(s) + 3I(r)I^2(s) + rI^3(s)\right\} = 0.$$

In view of (12), the last equation reduces to

$$f(u)f(u)\{3I(r)I^2(s) + rI^3(s)\} = 0$$

i.e,

$$2f(u)f(u)\{3I(r)I^2(s) + rI^3(s)\} = 0.$$

In (13), putting $s = I(s)$ and then subtracting, from the last equation, we have $f(u)f(u)RI^3(s) = 0$. Since R is prime, if $f(u)f(u) \neq 0$ for some u in U, then $I^3(s) = 0$ for all $s \in R$. Thus, from (6), $3I^2(r)f(u) = 0$ and so, from (9), $6I(r)f(u)f(u) = 0$, i.e, $3I(r)f(u)f(u) = 0$ for all $r \in R$. Suppose R is of characteristic different from 3. Then $I(r)f(u)f(u) = 0$ and so $0 = \{I(r)s + rI(s)\}f(u)f(u) = I(r)Rf(u)f(u) = [u, R]Rf(u)f(u)$. Since $f(u)f(u) \neq 0$ and R is prime, then $u \in Z$ and so $f(u)f(u) = 0$. Thus $f(u)f(u) = 0$ for all $u \in U$.

Suppose now that R is of characteristic 3. Then from (4), $d(u)I^3(r) = 0$. Thus

$$0 = d(u)[u, [u, [u, r]]] = d(u)[u^3, r] \quad \text{for all} \;\; r \in R, \;\; u \in U. \tag{15}$$

In (15), replace u by $u + v$ where $v \in U$ and use (15). Then

$$\{d(u) + d(v)\}[u^2v + uvu + vu^2 + uv^2 + vuv + v^2u, r] + \\ + d(u)[v^3, r] + d(v)[u^3, r] = 0.$$

Replace u by $-u$, then

$$\{-d(u) + d(v)\}[u^2v + uvu + vu^2 - uv^2 - vuv - v^2u, r] - \\ - d(u)[v^3, r] - d(v)[u^3, r] = 0.$$

Adding the last two equations and dividing by 2, we have

$$d(u)[uv^2+vuv+v^2u,r]+d(v)[u^2v+uvu+vu^2,r]=0 \quad \text{for all } r\in R \text{ and } u,v\in U. \tag{16}$$

Replacing r by ru in (16) and using (16) we get

$$d(u)r[uv^2+vuv+v^2u,u]+d(v)r[u^2v+uvu+vu^2,u]=0$$

or

$$d(u)r[uv^2+vuv+v^2u,u]+d(v)r\{(u^2v+uvu+vu^2)u-u(u^2v+uvu+vu^2)\}=0$$

or

$$d(u)r[uv^2+vuv+v^2u,u]+d(v)r[v,u^3]=0 \quad \text{for all } r\in R, \text{ and } u,v\in U. \tag{17}$$

In (15) let $r=rs$, then $0=d(u)\{[u^3,r]s+r[u^3,s]\}=d(u)R\{u^3,s]$. So, if for some $u\in U$, $d(u)\neq 0$ then $[u^3,s]=0$ for all $s\in R$, since R is prime. Thus from (17) $d(u)R[uv^2+vuv+v^2u,u]=0$, i.e, $[uv^2+vuv+v^2u,u]=0$ for all $v\in U$, as $d(u)\neq 0$ and R is prime. Thus

$$\begin{aligned}0&=I(uv^2+vuv+v^2u)\\&=I\{(uv-vu)v-v(uv-vu)\}, \text{as } \mathrm{char}R=3,\\&=I[I(v),v]=[I^2(v),v]\end{aligned}$$

Linearize the last equation on v to get

$$[I^2(v),w]+[I^2(w),v]=0 \text{ for all } v,w\in U.$$

Replacing w by $I(w)$, as $I^3(w)=[u,[u,[u,w]]]=[u^3,w]=0$, we get $[I^2(v), I(w)]=0$ for all $v,w\in U$. Suppose that $u\notin Z$, then by Theorem 4 of [2], as $I\neq 0$ and $U\not\subset Z$, $I^2(v)\in Z$ for all $v\in U$. Now by Theorem 5 of [2], as $I\neq 0$, $U\subset Z$. Thus $u\in Z$ and so $f(u)=[u,d(u)]=0$. Hence for all $u\in U$, $f(u)=0$. Thus, in all cases

$$f(u)f(u)=0 \quad \text{for all } u\in U. \tag{18}$$

Linearizing (3) on v, we get

$$\begin{aligned}&[v,u][d(u),[w,r]]+[w,u][d(u),[v,r]]+[d(u),v][[w,r],u]+\\&+[d(u),w][[v,r],u]+d(v)[[w,r],u],u]+d(w)[[[v,r],u],u]+\\&+2[d(v),u][[w,r],u]+2[d(w),u][[v,r],u]+2[v,u][d[w,r],u]+\\&+2[w,u][d[v,r],u]+[[v,u],u]d[w,r]+[[w,u],u]d[v,r]=0\\&\text{for all } u,v,w\in U.\end{aligned} \tag{19}$$

Let v=u=$f(u)$ in (19). Then, since $f(u)f(u) = 0$ and $f(u)d(f(u))\times f(u)$=0, we get

$$\begin{aligned}(20)\qquad &[w, f(u)][d(f(u)), [[f(u), r]] + 3[d(f(u)), f(u)][w, r]f(u)+\\ &+2[d(f(u)), w]\ f(u)r\ f(u) - 2d(f(u))\ f(u)[w, r]\ f(u)+\\ &+4[d(w), f(u)]\ f(u)rf(u) + 2[w, f(u)]\ d[f(u), r]f(u) = 0\\ &\text{for all}\quad r \in R\ \text{ and }\ u, w \in U.\end{aligned}$$

Multiplying by $f(u)$ from the right, the last equation yields

$$[w, f(u)]\ [d(f(u)),\ [f(u), r]]\ f(u) = 0$$

i.e,

$$f(u)U\{f(u)r\ d(f(u))\ f(u) - d(f(u))\ f(u)r\ f(u)\} = 0 \quad \text{for all}\ \ r \in R,\ u \in U.$$

If for some $u \in U$, $f(u) \neq 0$ then $u \notin Z$ and so by Lemma 4 of [3] the last equation yields

$$f(u)r\ d(f(u))f(u) = d(f(u))\ (f(u))r\ f(u)\ \text{for all}\ r \in R.$$

By using a result of Martindale (Corollary of Lemma 1.3.2 in [4]) we conclude that, as $f(u) \neq 0$, $d(f(u))\ f(u) = \delta(u)\ f(u)$ where $\delta(u) \in C$, the extended centroid of R (See p.22 of [4] for the notion of extended centroid). Moreover, $f(u)d(f(u)) = -d(f(u))f(u) = -\delta(u)\ f(u)$.

Let $r = f(u)r$ in (20). Since $0 = f(u)f(u) = f(u)d(f(u))\ f(u)$, $d(f(u))f(u) = \delta(u)f(u)$ and $f(u)d(f(u)) = -\delta(u)f(u)$, we get

$$\begin{aligned}&-[w, f(u)][d(f(u)),\ f(u)r\ f(u)] + 6\delta(u)\ f(u)w\ f(u)r\ f(u)-\\ &-2\delta(u)f(u)wf(u)rf(u) + 2[w, f(u)]\ d\{f(u)[f(u), r]\}f(u) = 0\end{aligned}$$

i.e.

$$\begin{aligned}&f(u)w\,d(f(u))\ f(u)r\ f(u) - f(u)w\ f(u)r\ f(u)\ d(f(u))+\\ &+4\delta(u)f(u)wf(u)r\ f(u) + 2[w, f(u)]\{d(f(u))[f(u), r]+\\ &+f(u)[d(f(u)), r] + f(u)[f(u), d(r)]\}f(u) = 0\end{aligned}$$

i.e.

$$\begin{aligned}&6\delta(u)f(u)w\ f(u)r\ f(u) + 2[w, f(u)]\ d(f(u))\ f(u)r\ f(u)-\\ &-2\ f(u)w\ f(u)[d(f(u)), r]\ f(u) = 0\end{aligned}$$

i.e.

$$6\delta(u)\ f(u)w\ f(u)r\ f(u) - 2f(u)w\, d(f(u))\ f(u)r\ f(u) - \\ -2f(u)w\, f(u)\, d(f(u))r\ f(u) + 2f(u)wf(u)r\ d(f(u))\ f(u) = 0$$

i.e.

$$8\delta(u)\ f(u)wf(u)rf(u) = 0 \quad \text{for all}\ \ w \in U,\ r \in R.$$

Now, if $\delta(u) \neq 0$ then $f(u)\ U\ f(u)\ R\ f(u) = 0$ and so $f(u)\ U\ f(u) = 0$, as $f(u) \neq 0$ and R is prime. Az $f(u) \neq 0$ then $u \notin Z$, so again by Lemma 4 of [3] $f(u) = 0$, a contradiction. Thus $\delta(u) = 0$. Therefore, $d(f(u))\ f(u) = f(u)d(f(u)) = 0$. Hence from (20), we get

$$\begin{aligned} &f(u)w\{d(f(u))rf(u) + f(u)rd(f(u))\} + 2d(f(u)wf(u)r\ f(u) - \\ &-4f(u)d(w)f(u)rf(u) - 2f(u)w\{d(f(u))r\ f(u) + \\ &+f(u)d(r)f(u)\} = 0 \qquad \text{for}\ r \in R,\ w \in U. \end{aligned} \tag{21}$$

In (2), let $u = f(u)$. Then we get

$$d(f(u))vf(u) + f(u)vd(f(u)) - 2f(u)d(v)f(u) = 0 \quad \text{for all}\ v \in U. \tag{22}$$

Thus from (21), we get

$$\{2d(f(u))wf(u) - 4f(u)d(w)f(u) - 2f(u)wd(f(u)\}vf(u) = 0 \\ \text{for all}\ \ v, w \in U.$$

Again, $f(u) \neq 0$ implies $u \notin Z$. Hence, by Lemma 4 of [3],

$$d(f(u))wf(u) - f(u)wd(f(u)) - 2f(u)d(w)f(u) = 0 \quad \text{for all}\ w \in U. \tag{23}$$

In view of (22) & (23), we get $2f(u)wd(f(u)) = 0$ for all $w \in U$. Then $f(u)Ud(f(u)) = 0$. Thus $d(f(u)) = 0$, since $f(u) \neq 0$. So from (22), $f(u)d(v)f(u) = 0$ for all $v \in U$. Hence from (21), $f(u)Uf(u)d(r)f(u) = 0$ for all $r \in R$. As $f(u) \neq 0$, we get $f(u)d(r)f(u) = 0$ for all $r \in R$. Replace r by ru, then $0 = f(u)\{d(r)u + rd(u)\}f(u) = f(u)r\ d(u)\ f(u)$, as $uf(u) = f(u)u$. Since R is prime and $f(u) \neq 0$, $d(u)f(u) = 0$. Also $0 = d[u, f(u)] = [d(u), f(u)] = f(u)d(u)$. Thus we conclude that if for some $u \in U$, $f(u) \neq 0$ then $f(u)d(u) = d(u)f(u) = 0$. Hence

$$f(u)d(u) = d(u)f(u) = 0 \quad \text{for all}\ u \in U$$

i.e,

$$[u, d(u)]d(u) = d(u)[u, d(u)] = 0 \quad \text{for all}\ u \in U. \tag{24}$$

Linearizing (24) on u and using a similar approach as in the proof of (2) we get

$$[u,d(u)]d(v)+[u,d(v)]d(u)+[v,d(u)]d(u)=0 \quad \text{for all } u,v\in U. \tag{25}$$

Suppose now that $v,w\in U$ are such that $vw\in U$. By replacing v by vw in (25), where $w\in U$, and using (25), after expansion we conclude

$$\begin{gathered}[u,d(u)]d(v)w+[u,d(v)]wd(u)+d(v)[u,w]d(u)+\\ +[u,v]d(w)d(u)+[v,d(u)]wd(u)+[[u,d(u)],v]d(w)=0\\ \text{for } u,v,w\in U.\end{gathered} \tag{26}$$

For any $r\in R$, the elements $w=vr-rv$ satisfies the criterion that $vw\in U$, hence by the above equation, we get

$$\begin{gathered}[u,d(u)]d(v)[v,r]+[u,d(v)][v,r]d(u)+d(v)[u,[v,r]]d(u)+\\ +[u,v]d[v,r]d(u)+[v,d(u)][v,r]d(u)+[[u,d(u),v]d[v,r]=0\\ \text{for all } r\in R \text{ and } u,v\in U.\end{gathered} \tag{27}$$

Let $v=u$ in (27). Then, in view of (1) and (24), we get

$$2f(u)I(r)d(u)+d(u)I^2(r)d(u)=0 \quad \text{for all } r\in R,\ u\in U. \tag{28}$$

Write $r=rs$ where $s\in R$ in (28). Then

$$\begin{aligned}2f(u)\{I(r)s+rI(s)\}d(u)+d(u)\{I^2(r)s+2I(r)\ I(s)+\\ +rI^2(s)\}d(u)=0 \quad \text{for all } r,s\in R \text{ and } u\in U.\end{aligned} \tag{29}$$

Replace r by u in (29). Then, as $I(u)=[u,u]=0$

$$2f(u)uI(s)d(u)+d(u)\ uI^2(s)d(u)=0 \quad \text{for all } s\in R,\ u\in U. \tag{30}$$

As $uf(u)=f(u)u$, so from (28) and (30), we get

$$f(u)I^2(r)\ d(u)=0 \quad \text{for all } r\in R,\ u\in U. \tag{31}$$

Similarly as above, from $d(u)[u,d(u)]=0$ for all $u\in U$, we can conclude

$$d(u)I^2(r)\ f(u)=0 \quad \text{for all } r\in R,\ u\in U. \tag{32}$$

Let $s=u$ in (29). Then $2f(u)I(r)ud(u)+d(u)I^2(r)ud(u)=0$. But from (28), $2f(u)I(r)d(u)u+d(u)I^2(r)d(u)u=0$. Thus, $2f(u)I(r)f(u)+d(u)I^2(r)f(u)=0$. Hence, in view of (32)

$$f(u)I(r)f(u)=0 \quad \text{for all } r\in R, u\in U. \tag{33}$$

Linearize (27) on v to get

$$
\begin{aligned}
&[u,d(u)]d(v)[w,r]+[u,d(v)][w,r]\ d(u)+d(v)\ [u,[w,r]]\ d(u)+\\
&+[u,v]\ d[w,r]\ d(u)+[v,d(u)][w,r]\ d(u)+[[u,d(u)],v]d[w,r]+\\
&+[u,d(u)]d(w)[v,r]+[u,d(w)][v,r]d(u)+d(w)[u,[v,r]]d(u)+\\
&+[u,w]d[v,r]d(u)+[w,d(u)][v,r]d(u)+[[u,d(u)],w]d[v,r]=0\\
&\text{for all } \ r\in R \ \text{ and } u,v,w\in U.
\end{aligned}
\tag{34}
$$

Write $v=u$ in (34). As $f(u)d(u)=0$ and $[f(u),u]=0$, we get

$$
\begin{aligned}
&2f(u)[w,r]d(u)+d(u)[u,[w,r]]d(u)+f(u)d(w)[u,r]+\\
&+[u,d(w)][u,r]d(u)+d(w)[u,[u,r]]d(u)+[u,w]\ d[u,r]d(u)+\\
&+[w,d(u)][u,r]d(u)+[f(u),w]\ d[u,r]=0.
\end{aligned}
$$

Multiplying by $f(u)$ from the right, the last equation becomes

$$f(u)d(w)\ [u,r]f(u)+[f(u),w]\ d[u,r]\ f(u)=0$$

i.e,

$$f(u)d(w)[u,r]f(u)+[f(u),w][d(u),r]f(u)+[f(u),w][u,d(r)]f(u)=0$$

i.e,

$$f(u)d(w)[u,r](u)+f(u)wd(u)rf(u)+[f(u),w][u,d(r)]f(u)=0.$$

Replace r by $d(u)$ in the last equation. As $f(u)f(u)=0$ and $d(u)f(u)=0$, we have $0=[f(u),w][u,d^2(u)]\ f(u)=[f(u),w]d[u,d(u)]f(u)=$ $=[f(u),w]d(f(u))\ f(u)$. Now $f(u)f(u)=0$ and so $f(u)d(f(u))f(u)=0$. Hence $f(u)wd(f(u))f(u)=0$ for all $u,w\in U$. Now if for some $u\in U$, $f(u)\neq 0$ then $u\notin Z$. Thus, by Lemma 4 of [3], $d(f(u))f(u)=0$. By multiplying $f(u)$ from the right in (34), we have

$$
\begin{aligned}
&f(u)d(v)\ [w,r]f(u)+[f(u),v]\{[d(w),r]+[w,d(r)]\}\ f(u)+\\
&+f(u)d(w)[v,r]f(u)+[f(u),w]\{[d(v),r]+[v,d(r)]\}\ f(u)=0\\
&\text{for all } \ r\in R \text{ and } v,w\in U.
\end{aligned}
$$

Replace w by $[u,d(u)]=f(u)$. As $d(f(u))\ f(u)=0$, $f(u)d(f(u))=0$ and $f(u)f(u)=0$, we get

$$f(u)d(v)f(u)r\ f(u)+[f(u),v]\{[d(f(u)),r]+[f(u),d(r)]\}f(u)=0.$$

From (25) we have $f(u)d(v)f(u)=0$, since $d(u)f(u)=0$ by (24). Hence from the last equation, we get

$$[f(u),v]\{d(f(u))r\ f(u)+f(u)d(r)f(u)\}=0$$

i.e.

$$f(u)U\{d(f(u))rf(u)+f(u)\ d(r)\ f(u)\}=0\quad \text{for all } r\in R.$$

As $f(u)\neq 0$ and so $u\notin Z$, by Lemma 4 of [3], $d(f(u))\ rf(u)+f(u)d(r)\times f(u)=0$ for all $r\in R$. In particular $d(f(u))v\ f(u)+f(u)d(v)f(u)=0$ for all $v\in U$. As we have seen above, $f(u)d(v)f(u)=0$, so we conclude that $d(f(u))\ v\ f(u)=0$ for all $v\in U$, i.e, $d(f(u))Uf(u)=0$. As $f(u)\neq 0$, so by Lemma 4 of [3] $d(f(u))=0$.

Replace v by $f(u)$ in (34). As $d(f(u))=0$ and keeping in view (1) and (24), we get

$$\begin{gathered}f(u)d(w)[f(u),r]+[u,d(w)]f(u)rd(u)+\\ +d(w)[u,[f(u),r]]d(u)+[u,w][f(u),d(r)]d(u)+\\ +[w,d(u)]f(u)rd(u)+[f(u),w][f(u),d(r)]=0\\ \text{for all } r\in R,\ w\in U.\end{gathered}\tag{35}$$

Multiplying by $f(u)$ from the right in (35), as we have seen above $f(u)d(w)$ $f(u)=0$ for all $w\in U$, and we conclude that

$$f(u)wf(u)d(r)f(u)=0\quad \text{for all } w\in U,\ r\in R.$$

As $f(u)\neq 0$, again by Lemma 4 of [3], we have

$$f(u)d(r)f(u)=0 \text{ for all } \ r\in R.\tag{36}$$

In view of (36), we conclude from (35) that

$$\begin{gathered}-f(u)d(w)rf(u)+[u,d(w)]f(u)rd(u)+d(w)f(u)[u,r]d(u)+\\ +[u,w]f(u)d(r)d(u)+[w,d(u)]\ f(u)r\ d(u)+f(u)w\ f(u)\ d(r)-\\ -f(u)wd(r)f(u)=0\qquad \text{for all } r\in R,\ w\in U.\end{gathered}\tag{37}$$

Replace r by ru in (37) and use (37). As $uf(u)=f(u)u$ and $d(u)f(u)=0$, we get

$$\begin{gathered}[u,d(w)]f(u)rf(u)+d(w)f(u)[u,r]f(u)+[u,w]f(u)d(r)f(u)+\\ +[u,w]\ f(u)r\ d(u)d(u)+[w,d(u)]f(u)r\ f(u)+f(u)w\ f(u)r\ d(u)=0\\ \text{for all } r\in R,\ w\in U.\end{gathered}$$

In view of (33) and (36) the last equation yields

$$(38)\ [u,d(w)]\ f(u)r\,f(u)+[u,w]\,f(u)r\,d(u)d(u)+[w,d(u)]\,f(u)r\ f(u)+ \\ +f(u)wf(u)rd(u)=0 \quad \text{for all } r\in R,\ w\in U.$$

In (38), let $r=ru$, and use (38). As $uf(u)=f(u)u$, we get

$$[u,w]f(u)rud(u)d(u)+f(u)wf(u)rf(u)-[u,w]f(u)rd(u)d(u)u=0.$$

But from (24), $ud(u)d(u)=d(u)ud(u)=d(u)d(u)u$. Hence $f(u)wf(u)rf(u)=0$ for all $w\in U$, $r\in R$. Since $f(u)\neq 0$, by Lemma 4 of [3], $f(u)rf(u)=0$ for all $r\in R$. Since R is prime, we conclude that $f(u)=0$. Thus $f(u)=0$ for all $u\in U$. So, by Theorem 7 of [2] either $d=0$ or $U\subset Z$.

3. In this section our object is to provide the affirmative answer for a question raised by J. Vukman in [6]. J. Vukman [6, Theorem 2] proved that if R is a prime ring, $\mathrm{char}R\neq 2,3$, and d is a derivation of R such that $[[d(x),x],x]\in Z$ for all $x\in R$, then either $d=0$ or R is commutative. We generalize this result in case $\mathrm{char}R=3$ and prove the following:

Theorem 2. *Let R be a prime ring of characteristic different from 2, and let d be a derivation of R such that $[[d(x),x],x]\in Z$ for all $x\in R$. Then either $d=0$ or R is commutative.*

Proof. *Case I.* If $\mathrm{char}R\neq 2,3$, the result follows from Theorem 2 of [6].

Case II. Suppose now that $\mathrm{char}R=3$. Replace x by $x+y$ where $y\in R$ in the hypothesis, then by using a similar approach as in the proof of (2) we obtain

$$(39)\qquad [[d(x),x],y]+[[d(x),y],x]+[[d(y),x],x]\in Z \quad \text{for all } x,y\in R.$$

Replace y by yx in (39), as $\mathrm{char}R=3$, and expand then

$$(40)\qquad \{[[d(x),x],y]+[[d(x),y],x]+[[d(y),x],x]\}x+ \\ +[[y,x],x]d(x)\in Z \quad \text{for all } x,y\in R$$

Commuting (40) with x, in view of (39), we get

$$[[y,x],x]d(x)x=x[[y,x],x]\ d(x) \quad \text{for all } x,y\in R.$$

Replace y by $d(x)$, then $[[d(x),x],x][d(x),x]=0$, since $[[d(x),x],x]\in Z$. Hence $[f(x),x]Rf(x)=0$ for all $x\in R$. If for some $x\in R$, $[f(x),x]\neq 0$ then $f(x)=0$ and so $[f(x),x]=0$, since R is prime. Thus $[f(x),x]=0$ for all $x\in R$. So by Theorem 1 of [6] either $d=0$ or R is commutative.

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