

On the maximal and minimal exponent of the prime power divisors of integers

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Abstract. For some integer n and prime p let $\nu_p(n)$ be the largest non-negative integer for which $p^{\nu_p(n)}$ is a divisor of n . Let $h(n) = \min_{p|n} \nu_p(n)$, $H(n) = \max_{p|n} \nu_p(n)$. The mean value of h , H over some subsets of integers is investigated.

1. Let $\mathcal{P}$ be the set of primes, and for a prime divisor p of n let $\nu_p(n)$ be defined as $p^{\nu_p(n)} \parallel n$. Then

$$n = \prod_p p^{\nu_p(n)}.$$

As usual let $\pi(x, k, l)$ be the number of primes $p \leq x$ in the arithmetical progression $\equiv l \pmod{k}$. Let

$$H(n) := \max_{p|n} \nu_p(n), \quad h(n) = \min_{p|n} \nu_p(n).$$

NIVEN proved in [1] that

$$\sum_{n \leq x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)} \sqrt{x} + o(\sqrt{x}) \quad (x \rightarrow \infty), \quad (1.1)$$

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and W. SCHWARZ and J. SPILKER in [2], that

$$\sum_{n \leq x} H(n) = \mathcal{M}(H)x + O\left(x^{3/4} \exp\left(-\gamma\sqrt{\log x}\right)\right), \quad (1.2)$$

$$\sum_{n \leq x} \frac{1}{H(n)} = \mathcal{M}\left(\frac{1}{H}\right)x + O\left(x^{3/4} \exp\left(-\gamma\sqrt{\log x}\right)\right), \quad (1.3)$$

where $\gamma > 0$ is a suitable constant.

D. SURYANARAYANA and SITA RAMACHANDRA RAO [8] proved that

$$\sum_{n \leq x} H(n) = \mathcal{M}(H)x + O\left(\sqrt{x} \exp\left(-\gamma(\log x)^{3/5}(\log \log x)^{-1/5}\right)\right). \quad (1.4)$$

Furthermore they proved that

$$\sum_{i \leq x} \frac{1}{H(i)} = cx + O\left(\sqrt{x} \exp\left(-\gamma(\log x)^{3/5}(\log \log x)^{-1/5}\right)\right), \quad (1.5)$$

$$\sum_{i \leq x} h(i) = c_1x + c_2x^{1/2} + c_3x^{1/3} + c_4x^{1/4} + c_5x^{1/5} + O\left(x^{1/6}\right) \quad (1.6)$$

$$\sum_{i \leq x} \frac{1}{h(i)} = d_1x + d_2x^{1/2} + d_3x^{1/3} + d_4x^{1/4} + d_5x^{1/5} + O\left(x^{1/6}\right),$$

hold with suitable constants c, c_j, d_j .

Remark. Gu Tongxing and Cao Huizhong in their paper entitled “On sums of exponents of factoring integers” published in Journal of Mathematical Research and Exposition **13**(2), 1993 page 166 announced that they can improve the error form in (1.4) to $O(\sqrt{x} \exp(-c(\log x)^{3/5}(\log \log x)^{1/5}))$ by using some results of A. IVIČ and P. SHIU in Illinois J. Math., **26** (1982), 576–590.

2. Let $\zeta(s) = \sum 1/n^s$ be the Riemann zeta function, and

$$\eta(s) := \frac{1}{\zeta(s)} - 1.$$

The following assertion (which is quoted now as Lemma 1) is an unpublished result due to Michael Filaseta who communicated to JEAN-MARIE DE KONINCK and with the allowance of Dr. Filaseta his proof has been written in the paper [9].

Lemma 1. *Let $k \geq 2$ be an integer. Let $g(x)$ be a function satisfying $1 \leq g(x) \leq \log x$ for x sufficiently large, and set*

$$h = x^{\frac{1}{2k+1}} g(x)^3.$$

Then the number of k -free integers in the interval $(x, x + h]$ is

$$\frac{h}{\zeta(k)} + O\left(\frac{h \cdot \log x}{g(x)^3}\right) + O\left(\frac{h}{g(x)}\right).$$

As a direct consequence we formulate the following

Corollary. *Let $Y = x^{\frac{1}{2r+1}} \log x$. Using the abbreviation $\eta(s)$ defined at the beginning of this section, we have*

$$\#\{n \in [x, x + Y] \mid H(n) = r\} = Y(\eta(r + 1) - \eta(r)) + O\left(\frac{Y}{\log x}\right).$$

3. An asymptotic formula for the number of primes p with a fixed value $H(p + 1) = r$ is given in

Theorem 1. *Let $\varepsilon > 0$ be fixed, $Y = x^{\frac{7}{12} + \varepsilon}$. Let $r \geq 1$. Then*

$$\#\{p \in \mathcal{P}, p \in [x, x + Y] \mid H(p + 1) = r\} = e(r) \frac{Y}{\log x} + O\left(\frac{Y}{(\log x)^2}\right),$$

where

$$e(1) = \prod_p \left(1 - \frac{1}{p(p-1)}\right), \quad \text{and for } r \geq 2,$$

$$e(r) = \prod_p \left(1 - \frac{1}{(p-1)p^r}\right) - \prod_p \left(1 - \frac{1}{(p-1)p^{r-1}}\right).$$

PROOF. We shall estimate

$$E_r(x, Y) := \#\{p \in [x, x+Y], p \in \mathcal{P}, H(p+1) \leq r\}.$$

Let $z = (\log x)^3$. Let F be the number of those primes p in $[x, x+Y]$, for which $q^{r+1} \mid p+1$ holds for some prime $q > z$. We deduce that

$$F = O\left(\frac{H}{\log x}\right). \quad (3.1)$$

We have $F \leq F_1 + F_2 + F_3$, where

$$F_1 := \sum_{\substack{z < q < Y^{\frac{1}{2r+2}} \\ q \in \mathcal{P}}} (\pi(x+Y, q^{r+1}, -1) - \pi(x, q^{r+1}, -1)),$$

$$F_2 := \sum_{Y^{1/2r+2} \leq q < Y^{1/r+1}} (\pi(x+Y, q^{r+1}, -1) - \pi(x, q^{r+1}, -1))$$

and

$$F_3 = \sum_{Y^{\frac{1}{r+1}} \leq q} (\pi(x+Y, q^{r+1}, -1) - \pi(x, q^{r+1}, -1)).$$

By using the Brun–Titchmarsh inequality (see HALBERSTAM RICHERT [5], Theorem 3.7, page 107) we can obtain that

$$F_1 < \frac{cY}{\log x} \sum_{\substack{q > z \\ q \in \mathcal{P}}} 1/q^{r+1} \ll \frac{Y}{(\log x)^3}, \quad \text{say.}$$

Furthermore

$$F_2 \ll \sum_{q^{r+1} > \sqrt{Y}} \left(\left[\frac{Y}{q^{r+1}} \right] \right) \ll \frac{Y}{(\log x)^3}.$$

Let us estimate F_3 . If $q^{r+1} > Y$, $q^{r+1} \mid p+1$, then $p+1 = q^{r+1}\nu$, $\nu < \frac{2x}{Y}$. For a fixed $\nu < \frac{2x}{Y}$ the number of those primes q for which

$x \leq q^{r+1}\nu \leq x + Y$ is less than $c(\frac{x}{\nu})^{1/r+1} \cdot \frac{Y}{x}$, and so

$$F_3 \ll \sum_{\nu < \frac{2x}{Y}} \left(\frac{x}{\nu}\right)^{\frac{1}{r+1}} \cdot \frac{Y}{x}$$

whence we can obtain that $F_3 \ll Y^{\frac{1}{r+1}}$.

Let us observe that $H(p+1) \leq r$ if and only if $\sum_{d^{r+1}|p+1} \mu(d) = 1$.

From the argument used earlier we obtain that

$$E_r(x, Y) = \sum_{p \in [x, x+Y]} \sum_{\substack{d^{r+1}|p+1 \\ P(d) < z}} \mu(d) + O\left(\frac{Y}{(\log x)^2}\right).$$

Here $P(n)$ denotes the largest prime divisor of n .

If $p+1$ is such a number for which there is a $d > z^2$, such that $d^{r+1} | p+1$, and $P(d) < z$, then there is a divisor δ of d such that $\delta \in [z, z^2]$, and so $\delta^{r+1} | p+1$.

Since

$$\begin{aligned} \sum_{\delta \in [z, z^2]} (\pi(x+Y, \delta^{r+1}, -1) - \pi(x, \delta^{r+1}, -1)) \\ \ll \frac{Y}{\log x} \sum_{z \leq \delta \leq z^2} \frac{1}{\varphi(\delta^{r+1})} \ll \frac{Y}{(\log x)^2}, \end{aligned}$$

therefore

$$\begin{aligned} E_r(x, Y) &= \sum_{\substack{d < z^2 \\ P(d) < z}} \mu(d) (\pi(x+Y, d^{r+1}, -1) - \pi(x, d^{r+1}, -1)) \\ &\quad + O\left(\frac{Y}{(\log x)^2}\right), \end{aligned}$$

and so by the prime number theorems for short intervals due to HUXLEY [7] we obtain that

$$E_r(x, Y) = \frac{Y}{\log x} \sum_{\substack{d < z^2 \\ P(d) \leq z}} \frac{\mu(d)}{\varphi(d^{r+1})} + O\left(\frac{Y}{(\log x)^2}\right).$$

One can observe furthermore that

$$\sum_{\substack{d \leq z^2 \\ P(d) \leq z}} \frac{\mu(d)}{\varphi(d^{r+1})} = \prod_p \left(1 - \frac{1}{(p-1)p^r} \right) + O\left(\frac{1}{\log x}\right)$$

say.

If $r \geq 2$, then $H(n) = r$ holds if and only if $H(n) \leq r$, and $H(n) \leq r-1$ does not hold, therefore

$$\#\{p \in [x, x+Y], H(p+1) = r\} = E_r(x, Y) - E_{r-1}(x, Y).$$

Furthermore

$$\#\{p \in [x, x+Y], H(p+1) = 1\} = E_1(x, Y),$$

thus our theorem immediately follows. $\square$

4. Let $1 \leq Y \leq \sqrt{x}$. The number of those $n \leq x$ which have a divisor p^2 such that $p > Y$, is less than $O(\frac{x}{Y})$. Hence, we can deduce easily that

$$\begin{aligned} x^{-1} \#\{n \leq x \mid H(n+j) = r_j, j = 0, \dots, s\} \\ = c(r_0, r_1, \dots, r_s) + O((\log x)^{-2}), \end{aligned} \quad (4.1)$$

say, with a constant $c(r_0, \dots, r_s)$.

Similarly, one can get that

$$\begin{aligned} \frac{1}{\log x} \#\{p \leq x \mid H(p+l) = r_l, l = 1, \dots, s\} \\ = d(r_1, \dots, r_s) + O((\log x)^{-2}). \end{aligned} \quad (4.2)$$

The relations (4.1), (4.2) hold for every choice of $r_0, r_1, \dots, r_s \in \mathbb{N}$.

(4.1) readily follows from the relation

$$\begin{aligned} x^{-1} \#\{n \leq x \mid H(n+j) \leq r_j, j = 0, \dots, s\} \\ = d(r_0, r_1, \dots, r_s) + O((\log x)^{-2}) \end{aligned} \quad (4.3)$$

which is almost a direct consequence of the relation

$$\sum_{d^t \mid m} \mu(d) = \begin{cases} 1 & \text{if } H(m) < t, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4)$$

Hence

$$\begin{aligned} & \#\{n \leq x \mid H(n+j) \leq r_j, j = 0, \dots, s\} \\ = & \sum_{d_0, d_1, \dots, d_s} \mu(d_0) \dots \mu(d_s) \#\{n \leq x \mid n+j \equiv 0 \pmod{d_j^{r+j}}, j = 0, \dots, s\} \end{aligned}$$

The right hand side can be evaluated simply, since we can drop all those $d_0, \dots, d_s$ for which $\max d_j \geq (\log x)^2$, their contribution is $O(x/(\log x)^2)$.

We can argue similarly, by the proof (4.2). Here we should use the Siegel–Walfisz theorem also, which asserts that

$$\pi(x, k, l) = \frac{\text{li } x}{\varphi(k)} \left(1 + O\left(e^{-c\sqrt{\log x}}\right)\right)$$

uniformly as $(l, k) = 1$, $k \leq (\log x)^B$, where $c > 0$ is a suitable, and $B > 0$ is an arbitrary large constant (see [10], Theorem 8.3 Chapter IV).

Theorem 2. *Let $g \in \mathbb{Z}[x]$, irreducible over $\mathbb{Q}$, $r = \deg g$, $r \geq 3$. Assume that $g(n) \in \mathbb{N}$ for $n > 0$.*

Then

$$\frac{1}{x} \#\{n \leq x \mid H(g(n)) \leq s\} = c(g, s) + O((\log \log x)^{-1}), \quad (4.5)$$

if $s \geq r - 2$, $c(g, s)$ is a suitable constant, and

$$\frac{1}{\text{li } x} \#\{p \leq x \mid H(g(p)) \leq s\} = d(g, s) + O((\log \log x)^{-1}), \quad (4.6)$$

if $s \geq r - 1$, $d(g, s)$ is a suitable constant

Remark. The proof is based upon an important theorem of C. HOOLEY [4] which is quoted now as

Lemma 2. *Let g be a polynomial satisfying the conditions of Theorem 2. Let $N''(x)$ be the number of those $n \leq x$ for which q^{r-1} divides $g(n)$ for at least one $q \geq \frac{1}{6} \log x$. Then*

$$N''(x) \ll x \cdot (\log x)^{\frac{2}{r+1}-1}.$$

Let $S(x)$ be the number of those primes $p \leq x$ for which there exists a prime $q > \log x$ such that $q^r | g(p)$, then

$$S(x) = O\left(\frac{\text{li } x}{\log \log x}\right).$$

PROOF OF THEOREM 2. By using Lemma 2

$$\sum_{\substack{n \leq x \\ H(g(n)) \leq s}} 1 = \sum_{n \leq x} \sum_{d^{s+1} | g(n)}^* \mu(d) + O(N''(x)),$$

where d runs over those square free integers the largest prime factor of which is smaller than $\frac{1}{6} \log x$. Hence (4.5) can be deduced immediately. (See [4], Chapter 4, Theorem 3) (4.6) can be proved similarly, by using Lemma 2, the Siegel–Walfisz and the Bombieri–Vinogradov theorem. $\square$

Theorem 3. *Let $\varepsilon > 0$ be a constant, $Y = x^{\frac{2}{3}+\varepsilon}$. Then, for every $s \in \mathbb{N}$, and every fixed $A > 0$,*

$$\frac{1}{Y} \sum_{\substack{n \in [x, x+Y] \\ H(n^2+1)=s}} 1 = a(s) + O((\log x)^{-A}), \quad (4.7)$$

$$\frac{1}{Y} \sum_{\substack{p \in [x, x+Y] \\ H(p^2+1)=s}} 1 = b(s) + O((\log x)^{-A}). \quad (4.8)$$

Remark. In [6] we proved that the number of those $n \in [x, x+Y]$ for which $q^2 | n^2 + 1$ holds for at least one $q \geq \sqrt{Y}$ is $O(x^{2/3} \log x)$. By using this, and standard techniques we obtain (4.7). (4.8) follows similarly, by using the Hoheisel–Tatuzawa theorem, the short interval version of the Siegel–Walfisz theorem ([10], Theorem 3.2, Chapter IX).

5. Theorem 4. *We have*

$$\begin{aligned} \left(\sum : \right) &= \#\{n \leq x \mid H(n) = 1, h(n+1) \geq 2\} \\ &= C\sqrt{x} + O\left(\frac{\sqrt{x}}{\log x}\right). \end{aligned} \quad (5.1)$$

PROOF OF THEOREM 4. If $H(n) = 1$, $h(n+1) \geq 2$, then $n+1$ can be uniquely written as am^2 , where a is cube-full, m is square-free and $am^2 - 1$ is square free.

Let

$$\sum_a := \sum_{\substack{m^2 \leq \frac{x}{a} \\ (m,a)=1}} |\mu(m)| |\mu(am^2 - 1)|. \quad (5.2)$$

Then

$$\sum = \sum_{a < x} \sum_a = \sum_{a < Y} \sum_a + \sum_{a \geq Y} \sum_a = \sum^{(1)} + \sum^{(2)}.$$

Since $\sum_a \leq \sqrt{\frac{x}{a}}$, and summing over cube-full a $\sum_{a \geq Y} \frac{1}{\sqrt{a}} \ll \frac{1}{Y^{1/6}}$, we obtain that $\sum^{(2)} \ll \frac{\sqrt{x}}{\log x}$, if $Y = (\log x)^6$.

To evaluate (5.2) for a fixed a , first we overestimate the number of those m for which $p > Z$ and either $p^2 | m$, or $p^2 | am^2 - 1$. $Z = \frac{1}{6} \log x$. We have

$$\sum_{m < \sqrt{\frac{x}{a}}} \sum_{\substack{p^2 | m \\ p > Z}} 1 \leq 2\sqrt{\frac{x}{a}} \sum_{p > Z} \frac{1}{p^2} \ll 2 \left(\sqrt{\frac{x}{a}} \right) \frac{1}{Z},$$

furthermore

$$\begin{aligned} \sum_{m < \sqrt{\frac{x}{a}}} \sum_{\substack{p^2 | am^2 - 1 \\ Y < p < \sqrt{\frac{x}{a}}}} 1 &\leq 2\sqrt{\frac{x}{a}} \sum_{p > Y} \frac{1}{p^2} \leq 3\sqrt{\frac{x}{a}} \cdot \frac{1}{Y}, \\ \sum_{m < \sqrt{\frac{x}{a}}} \sum_{\substack{p | am^2 - 1 \\ \sqrt{\frac{x}{a}} \leq p < (\sqrt{x})(\log x)^c}} 1 &\leq 2\sqrt{\frac{x}{a}} \sum_{\sqrt{\frac{x}{a}} \leq p < \sqrt{x} \cdot (\log x)^c} \frac{1}{p} \\ &\leq 2\sqrt{\frac{x}{a}} \frac{\log \log x}{\log x}. \end{aligned}$$

For fixed integers $a, b > 0$ the number of solutions m, n , $am^2 \leq x$ of the equation $am^2 - bn^2 = 1$ is no more than $O(\log x)$. This follows from the identity

$$\left(\sqrt{\frac{a}{b}} - \frac{n}{m} \right) \left(\sqrt{\frac{a}{b}} + \frac{n}{m} \right) = \frac{1}{bm^2},$$

whence one get that if n, m is a solution, then n/m is an approximant from the continued fraction of $\sqrt{\frac{a}{b}}$. As it is known, no more than $O(\log \sqrt{\frac{x}{a}}) = O(\log x)$ such m, n pairs exist.

Collecting our inequalities, we get that

$$\begin{aligned} \sum_a &= \sum_{m < \sqrt{\frac{x}{a}}} \left(\sum_{\delta_1^2 | m}^* \mu(\delta_1) \right) \left(\sum_{\delta_2^2 | am^2 - 1}^* \mu(\delta_2) \right) \\ &\quad + O \left(x_1^{-1} \sqrt{\frac{x}{a}} \right) + O \left(\sqrt{\frac{x}{a}} \frac{\log \log x}{\log x} \right) \\ &\quad + O \left(\sqrt{\frac{x}{a}} (\log x)^{-c} \right), \end{aligned}$$

where c is an arbitrary fixed positive constant. The asterisk means that we sum over those δ_1, δ_2 the largest prime factor of which is no larger than $\frac{1}{6} \log x$. In this case $\delta_1 \leq x^{\frac{1}{6} + \varepsilon}$, $\delta_2 \leq x^{\frac{1}{6} + \varepsilon}$. For fixed δ_1, δ_2 , $(\delta_1, \delta_2) = 1$ we have to sum over those $\nu \leq \frac{1}{\delta_1^2} \sqrt{\frac{x}{a}}$ for which $a\delta_1^4 \nu^2 - 1 \equiv 0 \pmod{\delta_2^2}$ which is equivalent to $\nu^2 \equiv a \pmod{\delta_2^2}$ if $(a, \delta_2) = 1$.

Let $\rho_a(D)$ be the number of solutions of $\nu \pmod{D}$, for which $\nu^2 \equiv a \pmod{D}$, if D is square free. It is clear that ρ_a is multiplicative in D , $\rho_a(p) = 1 + \left(\frac{a}{p}\right)$ for p prime, $p \nmid D$, and so

$$\# \left\{ \nu \leq \frac{1}{\delta_1^2} \sqrt{\frac{x}{a}}, \nu^2 \equiv a \pmod{\delta_2^2} \right\} = \rho(\delta_2^2) \left(\frac{1}{\delta_1^2 \delta_2^2} \sqrt{\frac{x}{a}} + O(1) \right).$$

Thus

$$\sum_a = \sqrt{\frac{x}{a}} \sum_{\substack{(\delta_1, \delta_2)=1 \\ (a, \delta_2)=1}}^* \frac{\mu(\delta_1) \mu(\delta_2)}{\delta_1^2 \delta_2^2} \rho(\delta_2^2) + O \left(\sum_{\delta_1}^* \sum_{\delta_2}^* 1 \right).$$

The error term is $O(x^{1/3+2\varepsilon})$. For the fixed δ_2 ,

$$\begin{aligned} \sum_{(\delta_1, \delta_2)=1}^* \frac{\mu(\delta_1)}{\delta_1^2} &= \prod_{\substack{(p, \delta_2)=1 \\ p < (\log x)^{1/6}}} \left(1 - \frac{1}{p^2} \right) + O \left(\frac{1}{x^{1/2}} \right) \\ &= \frac{1}{\zeta(2)} \prod_{p | \delta_2} \frac{1}{1 - 1/p^2} + O \left(\frac{1}{(\log x)^6} \right). \end{aligned}$$

Thus

$$\sum_a = \sqrt{\frac{x}{a}} \cdot \frac{1}{\zeta(2)} \sum_{(\delta_2, a)=1} \frac{\mu(\delta_2) \rho(\delta_2^2)}{\delta_2^2} \prod_{p|\delta_2} \frac{1}{1-1/p^2} + O\left(\sqrt{\frac{x}{a}} \cdot \frac{1}{(\log x)^3}\right),$$

the sum on the right hand side

$$\begin{aligned} \sum_{(\delta_2, a)=1} \frac{\mu(\delta_2) \rho(\delta_2^2)}{\delta_2^2} \prod_{p|\delta} \frac{1}{1-1/p^2} \\ = \prod_{p \nmid a} \left\{ 1 - \frac{\rho(p^2)}{p^2} \cdot \frac{1}{1-1/p^2} \right\} = \prod_{p \nmid a} \left(1 - \frac{\rho(p^2)}{p^2-1} \right). \end{aligned}$$

Summing $\sum_a$ over the cube-full a , we obtain (5.1) easily. $\square$

One can prove similarly

Theorem 5. *Let r, s be arbitrary positive integers. Then*

$$\begin{aligned} \#\{n \leq x \mid H(n) \leq r, h(n+1) \geq s\} \\ = c(r, s) x^{1/s} + O\left(x^{1/s} / \log x\right). \end{aligned} \tag{5.3}$$

The following two conjectures seem to be quite plausible.

Conjecture 1. We have

$$\#\{n \leq x \mid h(n) \geq 2, h(n+1) \geq 2\} \geq \frac{cx^{1/4}}{\log x}. \tag{5.4}$$

Conjecture 2. We have

$$\#\{p \leq x \mid h(p+1) \geq 2\} \rightarrow \infty \quad (x \rightarrow \infty). \tag{5.5}$$

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