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On Stetkær type functional equations and Hyers–Ulam stability

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Abstract. Let G be a locally compact group, K a compact subgroup of morphisms of G, $\chi : K \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$ a continuous homomorphism and μ a K-invariant bounded measure on G. In this paper we study functional equations of the form

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)h(y), \quad x, y \in G,$$

in which $f, g, h \in C_b(G)$ are unknown functions. These equations may be viewed as a generalization of the functional equations considered by Stetkær in many of his works. We show how the solutions g and h are closely related to the solutions of Badora's functional equation solved in [4] and [13]. We treat examples and we give some applications. The case where G is a Lie group is considered. Furthermore, we investigate the Hyers–Ulam stability problem of these functional equations.

1. Introduction

Let G be a locally compact group endowed with a left Haar measure dx, and K a compact subgroup of morphisms of G i.e. of mappings k of G onto itself that are either automorphisms and homeomorphisms ($k \in K^+$), or antiautomorphisms and homeomorphisms ($k \in K^-$). The action of $k \in K$

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on $x \in G$ will be denoted by $k \cdot x$. The mapping $\chi : K \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$ is a continuous homomorphism. For μ a complex bounded measure on G, $\check{\mu}$ (resp. $\overline{\mu}$) will denote the measure defined by $\langle \check{\mu}, f \rangle = \langle \mu, \check{f} \rangle$ (resp. $\langle \overline{\mu}, f \rangle = \overline{\langle \mu, \overline{f} \rangle}$), where $\check{f}(x) = f(x^{-1})$, $\overline{f}(x) = \overline{f(x)}$ for all continuous and bounded functions f on G. $C(G)$ (resp. $C_b(G)$) designates the space of continuous (resp. continuous and bounded) complex valued functions. We assume that K has a topology making it a compact Hausdorff group with the property that the canonical map $K \times G \longrightarrow G$ sending each pair (k, x) onto $k \cdot x$ is continuous. For any $k \in K$, and for any function f on G, we put $(k \cdot f)(x) = f(k^{-1} \cdot x)$, and we say that f is K-invariant if $k \cdot f = f$ for all $k \in K$. The algebra of all regular and complex bounded measures on G will be denoted by $M(G)$. We recall that the convolution of $M(G)$ is given by

$$\langle \mu * \nu, f \rangle = \int_G \int_G f(ts) d\mu(t) d\nu(s), \quad \text{for all } f \in C_b(G).$$

For any $\mu \in M(G)$ and any $k \in K$, we put $\langle k \cdot \mu, f \rangle = \langle \mu, k \cdot f \rangle$ for all $f \in C_b(G)$, and we say that μ is K-invariant if $k \cdot \mu = \mu$ for all $k \in K$. A function $f \in C_b(G)$ is bi-μ-invariant if $f_\mu = f$, where f_μ is the continuous and bounded function defined by

$$f_\mu(x) = \int_G \int_G f(sxt) d\mu(s) d\mu(t), \quad \text{for all } x \in G.$$

We notice that if $\mu * \mu = \mu$, then f is bi-μ-invariant if and only if it is both left and right μ invariant, i.e. $\int_G f(tx) d\mu(t) = \int_G f(xt) d\mu(t) = f(x)$ for all $x \in G$.

Finally, $L_1(G, dx)$ designates the Banach algebra of all integrable functions on G.

Definition 1.1 ([2]). Let $\mu \in M(G)$; μ is said to be a Gelfand measure if $\check{\overline{\mu}} = \mu * \mu = \mu$ and the Banach algebra $L_1^\mu(G) = \mu * L_1(G) * \mu$ is commutative under the convolution.

A non-zero function $\phi \in C_b(G)$ is a μ-spherical function if it satisfies the functional equation

$$\int_G \phi(xty) d\mu(t) = \phi(x)\phi(y), \quad x, y \in G. \tag{1.1}$$

We will say that a function $f \in C_b(G)$ satisfying

$$\int_G f(xty)d\mu(t) = f(x)\phi(y), \quad x, y \in G \tag{1.2}$$

is associated to the μ-spherical function ϕ.

The μ-spherical functions and related notions have been introduced by M. Akkouchi and A. Bakali [2]. When H is a compact subgroup of G and dh is the normalized Haar measure of H, then dh is a Gelfand measure on G if and only if (G, H) is a Gelfand pair [11]. A function $f \in C_b(G)$ satisfies a Kannappan type condition $K(\mu)$ [25], [12] if

$$\int_G \int_G f(zsxty)d\mu(s)d\mu(t) = \int_G \int_G f(zsytx)d\mu(s)d\mu(t), \quad x, y, z \in G.$$

In the series of papers [28]–[32], a number of results has been obtained by Stetkær for functional equations of the form

$$\int_K f(xk \cdot y)\overline{\chi(k)}dk = \sum_{i=1}^{n} g_i(x)h_i(y), \quad x, y \in G, \tag{1.3}$$

where the functions $f, g_1, \dots, g_n, h_1, \dots, h_n$ to be determined are continuous complex-valued functions on a locally compact group G and K is a compact subgroup of automorphisms of G.

In the present paper we study a generalization of the equation (1.3)

$$\int_G \int_K f(xtk \cdot y)\overline{\chi(k)}dkd\mu(t) = g(x)h(y), \quad x, y \in G, \tag{1.4}$$

where μ is a complex bounded measure on G and K is a compact subgroup of morphisms of G, not just a compact subgroup of automorphisms of G.

Our approach is to consider $\mu = \delta_e$: The Dirac measure concentrated at the identity element e as a complex bounded measure on G, and then the functional equation (1.3) can be written in the form

$$\int_G \int_K f(xtk \cdot y)\overline{\chi(k)}dkd\delta_e(t) = \sum_{i=1}^{n} g_i(x)h_i(y), \quad x, y \in G. \tag{1.5}$$

It is the same point of view as in [12] and [16] except that the compact subgroup K of morphisms of G is new; it was $\{I, \sigma\}$ in [12], [16] (σ is a continuous involution of G), where Elqorachi and Akkouchi have introduced and studied the functional equation

$$\int_G f(xty)d\mu(t) \mp \int_G f(xt\sigma(y))d\mu(t) = 2\sum_{i=1}^{n} g_i(x)h_i(y). \tag{1.6}$$

The class of equations (1.4) contains also the functional equation of spherical functions

$$\int_K f(xk \cdot y)dk = f(x)f(y), \quad x, y \in G, \tag{1.7}$$

which has attracted the attention of many mathematicians. The first significant results were obtained in [9], [4], [31] and [32] for bounded and continuous solutions. For continuous solutions of (1.7), recently Shin'ya [27] described the non-zero solutions in the following form:

$$f(x) = \int_K \varphi(k \cdot x)dk \quad \text{for all } x \in G,$$

where $\varphi : G \longrightarrow \mathbb{C}\backslash\{0\}$ is a continuous homomorphism of the abelian group G, (cf. [27] Corollary 3.12). Badora's functional equation is considered in [4]:

$$\int_G \int_K f(x + t + k \cdot y)dkd\mu(t) = f(x)f(y), \quad x, y \in G. \tag{1.8}$$

The non-zero essentially bounded solutions of equation (1.8) are of the form

$$f(x) = \int_K (\varphi * \mu_{k \cdot x})(e)dk, \quad x \in G, \tag{1.9}$$

where φ is a character of G and e is the identity element of the abelian group G (cf. [4]). For G non necessarily abelian and μ a K-invariant generalized Gelfand measure, the non-zero continuous and bounded solutions of (1.8) are given by

$$f(x) = \int_K \varphi(k \cdot x)dk, \quad x \in G,$$

where φ is a μ-spherical function on G (cf. [13]).

We shall notice here that the additional assumption that every closed ideal of the commutative Banach algebra $\mu * L_1(G, dx) * \mu$ is contained in some maximal ideal of $\mu * L_1(G, dx) * \mu$, used in Section 3 of [13], is superfluous, because the commutative Banach algebra $\mu * L_1(G, dx) * \mu$ approximates the identity.

Equation (1.4) contains also the functional equation of μ-spherical functions

$$\int_G f(xty)d\mu(t) = f(x)f(y), \quad x, y \in G, \tag{1.10}$$

which was studied in [2] and [3]. It should be motioned here that if $\mu \in M(G)$, then the continuous solutions of (1.10) are only given when G is compact (cf. [3]). They are of the form

$$f(x) = \langle \pi(x)\xi, \eta \rangle, \tag{1.11}$$

where $(\pi, \mathcal{H})$ is an irreducible, continuous and unitary representation of G such that $\pi(\mu)$ is of rank one, $\eta \in \mathcal{H}\backslash\{0\}$ and ξ is a unit vector in $\Im(\pi(\mu))$, the range of the operator $\pi(\mu)$.

The classical examples of equation (1.4) with $K = \{I, -I\}$ and $\chi = 1$ are: D'Alembert's equation [25], [7], [10]

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G, \tag{1.12}$$

and Wilson's equation [37], [38]

$$f(x + y) + f(x - y) = 2f(x)g(y), \quad x, y \in G. \tag{1.13}$$

Other references and informations on detailed discussions of classical equations can be found in the monographs by Aczél and Dhombres (cf. [1]). An example of transformation groups K other than those two, and an example of homomorphisms other than $\chi = 1$ in connection with functional equation (1.4) is $K = \mathbb{Z}_n$ acting on $G = \mathbb{C}$

$$\frac{1}{n}\sum_{j=1}^{n-1} f(x + \omega^j y) = f(x)f(y), \quad x, y \in \mathbb{C}, \tag{1.14}$$

where $\omega = \exp(2\pi i/n)$. This functional equation occurs in Förg-Rob and Schwaiger [18] and Stetkær [30].

Our discussion in the present paper is organized as follows. In Section 2 we establish some general properties of the solutions of (1.4). We show how they are closely related to the solutions of Badora's equation. This is an extension of Stetkær's results ([30], III, Theorem 1). The conclusion is the same if we replace the functional equation of spherical functions in [30] by Badora's functional equation, but the assumptions are weaker. K is not assumed to act by homomorphisms only, but by homomorphisms and also antihomomorphisms, and f satisfies $K(\mu)$. In the case when K is a compact subgroup of the group $\mathrm{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms and μ is a Gelfand K-invariant measure, we prove that the solutions g and h of (1.4) are associated to $\mu \otimes dk$-spherical functions on the semi-direct product group $K \propto G$. In Section 3 we treat examples. In Section 4 we study the functional equation

$$\int_G \int_K f(xtk \cdot y)\overline{\chi(k)}dkd\mu(t) = g(x)f(y), \quad x, y \in G, \tag{1.15}$$

as a particular case of (1.4). In Section 5, G is a connected Lie group and μ is a K-invariant idempotent measure with compact support. We show that the solutions of (1.4) are the eigenfunctions of a system of operators associated to left invariant differential operators on G. This extends the previous results obtained by Stetkær for equation (1.4) ([30], II, Theorem 2) and those of the authors in [13] to Badora's equation. In the last section we deal with the stability of Badora's functional equation and of the equation (1.15).

The results obtained in this paper may be viewed as a continuation and a generalization of Badora's work [4], [5], Förg-Rob's and Schwaiger's work [18], [19] and Stetkær's work [30].

2. On the second generalization of functional equations of Stetkær type

In this section we study the properties of the functional equation

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)h(y), \quad x, y \in G. \tag{2.1}$$

The ideas are inspired by the Stetkær's work [30] just mentioned.
By easy computation, we get the following

Proposition 2.1. *Let μ be a K-invariant measure. Let $f, g, h \in C_b(G)$ be a solution of* (2.1) *such that f satisfies the Kannappan type condition $K(\mu)$, $g \neq 0$ and $h \neq 0$. Then, for all $x, y \in G$, we have*

$$\int_G h(xty)d\mu(t) = \int_G h(ytx)d\mu(t)$$

and g satisfies $K(\mu)$.

Theorem 2.2. *Let μ be a K-invariant measure. Let $f, g, h \in C_b(G)$ be a solution of* (2.1) *such that f satisfies the Kannappan type condition $K(\mu)$, $g \neq 0$ and $h \neq 0$. Then*

i) *$h(k \cdot x) = \chi(k)h(x)$ for all $k \in K$, $x \in G$,*

ii) *there exists a function ϕ, solution of Badora's functional equation, such that*

$$\int_G \int_K g(xtk \cdot y)dkd\mu(t) = g(x)\phi(y), \quad x, y \in G \tag{2.2}$$

and

$$\int_G \int_K \check{h}(xtk \cdot y)dkd\check{\mu}(t) = \check{h}(x)\check{\phi}(y), \quad x, y \in G. \tag{2.3}$$

iii) *If G is a unimodular group, K a compact subgroup of automorphisms of G and μ a Gelfand K-invariant measure, then ϕ is a $\mu \otimes \omega_K$-spherical function and g (resp. $\check{h}$) is associated to ϕ (resp. $\check{\phi}$).*

Proof. Let $x, y \in G$ and let $k_0 \in K$, then we have

$$g(x)h(k_0 \cdot y) = \int_G \int_K f(xtkk_0 \cdot y)\overline{\chi}(k)dkd\mu(t)$$

$$= \int_K f(xtk\cdot y)\overline{\chi}(kk_0^{-1})dkd\mu(t) = \chi(k_0)g(x)h(y),$$

from which we deduce (i).

Let $x_0, y_0 \in G$ such that $g(x_0) \neq 0$ and $h(y_0) \neq 0$, then by using K(μ), the K-invariance of μ and equation (2.1), we get

$$\begin{aligned}
&h(y_0)\int_G\int_K g(x_0tk\cdot x)dkd\mu(t)\\
&= \int_G\int_K\int_G\int_K f(x_0tk\cdot xsk_1\cdot y_0)\overline{\chi}(k_1)dk_1d\mu(s)dkd\mu(t)\\
&= \int_G\int_K\int_G\int_{K^+} f(x_0tk\cdot xsk_1\cdot y_0)\overline{\chi(k_1)}dk_1dkd\mu(s)d\mu(t)\\
&\quad + \int_G\int_K\int_G\int_{K^-} f(x_0tk_1\cdot y_0sk\cdot x)\overline{\chi(k_1)}dk_1dkd\mu(s)d\mu(t)\\
&= \int_G\int_K\int_G\int_{K^+} f(x_0tk_1\cdot [k_1^{-1}k\cdot xsy_0])\overline{\chi}(k_1)dk_1d\mu(s)dkd\mu(t)\\
&\quad + \int_G\int_K\int_G\int_{K^-} f(x_0tk_1\cdot [k_1^{-1}k\cdot xsy_0])\overline{\chi}(k_1)dk_1dkd\mu(s)d\mu(t)\\
&= \int_G\int_K\int_G\int_K f(x_0tk_1\cdot (k\cdot xsy_0))\overline{\chi}(k_1)dk_1d\mu(t)dkd\mu(s)\\
&= g(x_0)\int_G\int_K h(k\cdot xsy_0)dkd\mu(s).
\end{aligned}$$

Now, in view of Proposition 2.1 and the K-invariance of μ, we obtain

$$\begin{aligned}
&= g(x_0)\int_G\int_K h(k\cdot xsy_0)dkd\mu(s)\\
&= g(x_0)\int_G\int_{K^+} h(k\cdot (xtk^{-1}\cdot y_0))dkd\mu(t)\\
&\quad + g(x_0)\int_G\int_{K^-} h(k\cdot (k^{-1}\cdot y_0tx))dkd\mu(t)\\
&= g(x_0)\int_G\int_{K^+} \chi(k)h(xtk^{-1}\cdot y_0)dkd\mu(t)\\
&\quad + g(x_0)\int_G\int_{K^-} \chi(k)h(k^{-1}\cdot y_0tx)dkd\mu(t)
\end{aligned}$$

$$= g(x_0)\int_G\int_K \chi(k)h(xtk^{-1}\cdot y_0)dkd\mu(t)$$
$$= g(x_0)\int_G\int_K \overline{\chi(k)}h(xtk\cdot y_0)dkd\mu(t),$$

from which we get

$$\int_G\int_K g(xtk\cdot y)dkd\mu(t) = g(x)\phi(y), \quad x,y\in G,$$

where ϕ is given by

$$\phi(x) = \frac{1}{g(x_0)}\int_G\int_K g(x_0tk\cdot x)dkd\mu(t)$$
$$= \frac{1}{h(y_0)}\int_G\int_K h(xtk\cdot y_0)\overline{\chi}(k)dkd\mu(t).$$

Now, using Proposition 2.1 and the definition of ϕ, we show that ϕ is a solution of Badora's functional equation.

$$\int_G\int_K \phi(xtk\cdot y)dkd\mu(t) = \phi(x)\phi(y), \quad x,y\in G,$$

and h,ϕ satisfy the equation

$$\int_K\int_G \check{h}(xtk\cdot y)dkd\check{\mu}(t) = \check{h}(x)\check{\phi}(y).$$

This proves (ii). For iii), let K be a compact subgroup of $\mathrm{Aut}(G)$, and let $K\propto G$ be the semi-direct product group with the group law

$$(k_1,x)(k_2,y) = (k_1k_2, xk_1\cdot y), \quad k_1,k_2\in K,\ x,y\in G.$$

A function $F: K\propto G\longrightarrow\mathbb{C}$ that is bi-$\mu\otimes dk$-invariant can be regarded as a function $F(k,x)=f(x)$ on G such that f is both bi-μ-invariant and K-invariant. Accordingly, we obtain the bijection

$$L_1^{\mu\otimes dk}(K\propto G)\longrightarrow L_1^{\mu}(G)\cap L_1^{K}(G)$$
$$F\longrightarrow f,$$

where $L_1^K(G) = \{f \in L_1(G) : k \cdot f = f,\ k \in K\}$, so that $L_1^{\mu\otimes dk}(K \propto G) \cong L_1^{\mu}(G) \cap L_1^K(G) = \mu * L_1^K(G) * \mu = M_K(\mu * L_1(G) * \mu)$, where $M_K(f)(x) = \int_K f(k{\cdot}x)dk$, $x \in G$, and $f \in L_1(G)$. Then $\mu\otimes dk$ is a Gelfand measure on $K \propto G$. Furthermore, by using ([13], Theorem 2.2), we get that the $\mu \otimes dk$-spherical functions are solutions of Badora's functional equation. $\square$

Remark 2.3. In Theorem 2.2 it is not necessary to assume that f satisfies the condition $K(\mu)$ if K is a compact subgroup of homomorphisms of G.

Corollary 2.4. *Let G be a locally compact group and let H be a compact subgroup of G such that H is K-invariant (i.e. $K \cdot H \subset H$). Let $(f, g, l) \in C_b(G)$ be a solution of*

$$\int_H \int_K f(xhk \cdot y)\overline{\chi}(k)dkdh = g(x)l(y), \quad x, y \in G, \tag{2.4}$$

such that $g \neq 0$, $l \neq 0$ and f satisfies a Kannappan type condition

$$\int_H \int_H f(zh_1xh_2y)dh_1dh_2 = \int_K \int_K f(zh_1yh_2x)dh_1dh_2, \quad x, y, z \in G.$$

Then

i) $l(k \cdot x) = \chi(k)l(x)$ *for all* $k \in K$, $x \in G$,

ii) *there exists a function ϕ solution of the functional equation*

$$\int_H \int_K \phi(xhk \cdot y)dkdh = \phi(x)\phi(y), \quad x, y \in G, \tag{2.5}$$

such that

$$\int_H \int_K g(xhk \cdot y)dkdh = g(x)\phi(y), \quad x, y \in G, \tag{2.6}$$

and

$$\int_H \int_K \check{l}(xhk \cdot y)dkdh = \check{l}(x)\check{\phi}(y), \quad x, y \in G. \tag{2.7}$$

iii) *If G is a unimodular group and K a compact subgroup of* $\mathrm{Aut}(G)$, *then ϕ is a $K \propto H$-spherical function.*

Corollary 2.5. *Let G be a locally compact group and H a compact subgroup of G such that $K.H \subset H$. Let τ be a continuous, unitary and irreducible representation of H and let χ_τ be a normalized character of τ such that $\chi_\tau * \chi_\tau = \chi_\tau$ and $\mu_\tau = \chi_\tau dh$. Moreover, let $(f, g, l) \in C_b(G)$ be a solution of*

$$\int_H \int_K f(xhk \cdot y)\overline{\chi}(k)\chi_\tau(h)dkdh = g(x)l(y) \tag{2.8}$$

such that $g \neq 0$, $h \neq 0$ and f satisfies a Kannappan type condition

$$\int_H \int_H f(zh_1xh_2y)\chi_\tau(h_1)\chi_\tau(h_2)dh_1dh_2 = \int_K \int_K f(zh_1yh_2x)\chi_\tau(h_1)\chi_\tau(h_2)dh_1dh_2.$$

Then

i) *$l(k \cdot x) = \chi(k)l(x)$ for all $k \in K$, $x \in G$,*

ii) *there exists a function ϕ, solution of the functional equation*

$$\int_H \int_K \phi(xhk \cdot y)\chi_\tau(h)dkdh = \phi(x)\phi(y), \quad x, y \in G, \tag{2.9}$$

such that

$$\int_H \int_K g(xhk \cdot y)\chi_\tau(h)dkdh = g(x)\phi(y), \quad x, y \in G, \tag{2.10}$$

and

$$\int_H \int_K \check{l}(xhk \cdot y)\overline{\chi_\tau}(h)dkdh = \check{l}(x)\check{\phi}(y), \quad x, y \in G. \tag{2.11}$$

iii) *If G is unimodular and K a compact subgroup of $\mathrm{Aut}(G)$, then ϕ is a $K \propto H$-spherical function of type τ.*

Corollary 2.6. *Let G be an unimodular group and μ a K-invariant Gelfand measure on G. Then the corresponding $\mu \otimes dk$-spherical functions have the form*

$$\phi(x) = \int_K \omega(k \cdot x)dk, \quad x \in G,$$

for some μ-spherical function ω. Furthermore, if ϕ is integrable or G is a compact group, then ϕ has the form

$$\phi(x) = \int_K \langle \pi(\mu)\pi(k\cdot x)\xi, \eta\rangle dk, \quad x \in G,$$

where $(\pi, \mathcal{H}_\pi)$ is an irreducible, continuous and unitary representation of G, such that $\pi(\mu)$ is a rank one operator and $\xi, \eta \in \mathcal{H}_\pi$.

PROOF. By using [3] and [13], we derive the proof. □

3. Examples

The next examples extend those obtained by STETKÆR in [30].

3.1. Let K be a compact subgroup of morphisms of G. Let μ be a K-invariant measure, $\omega \in C_b(G)$ a solution of (1.1), and $a \in C(G)$. Put

$$f(x) := \int_K a(k)\omega(k\cdot x)\overline{\chi}(k)dk, \quad x \in G,$$

$$g(x) := \int_K a(k)\omega(k\cdot x)dk, \quad x \in G,$$

$$h(x) := \int_K \omega(k\cdot x)\overline{\chi}(k)dk, \quad x \in G.$$

Then (f, g, h) is a solution of (2.1) and the corresponding function ϕ given by Theorem 2.2 has the form $\phi(x) = \int_K \omega(k\cdot x)dk$, $x \in G$.

3.2. Let $\chi = 1$ and let $f \neq 0$ be a right μ-invariant function which satisfies the condition $K(\mu)$, and $(f, g, h) \in C_b(G)$ a solution of (2.1). By putting $y = e$ in (2.1) we get $f(x) = g(x)h(e)$. So $h(e) \neq 0$, and (2.1) becomes

$$\int_G \int_K f(xtk\cdot y)dkd\mu(t) = 2\frac{f(x)}{h(e)}h(y) = 2f(x)\phi(y), \quad x, y \in G. \tag{3.1}$$

By Theorem 2.2, $\phi = \frac{h}{h(e)}$ is a solution of Badora's functional equation. An example of (2.1) with $K = \{I, \sigma\}$, where σ is a continuous involution of G, is

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2g(x)h(y), \quad x, y \in G, \tag{3.2}$$

which reduces to the generalized form of Wilson's functional equation

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)\phi(y), \quad x, y \in G, \tag{3.3}$$

where $\phi(y) = \frac{h(y)}{h(e)}$, for all $y \in G$. The solutions of (3.3) and (3.2) are described in [12] and [16].

3.3. By taking G an abelian locally compact group and $\mu = \delta_e$ we may derive other examples (see [1]).

4. On the first generalization of a functional equation of Stetkær type

In this section we will study a functional equation of the form

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)f(y), \quad x, y \in G. \tag{4.1}$$

This equation is a special case of the equation (2.1) in which $f = h$. Using Theorem 2.2, we deduce the following

Theorem 4.1. *Let μ be a K-invariant measure. Let $(f, g) \in C_b(G)$ such that $g \neq 0$ and f satisfies $K(\mu)$. Then*

(1) *If (f, g) is a solution of* (4.1) *and $f \neq 0$ then g is a solution of* (1.3).

(2) *(f, g) is a solution of* (4.1) *if and only if*

  i) *$f(k \cdot x) = \chi(k)f(x)$ for all $k \in K$, $x \in G$, and*

  ii) $$\int_G \int_K \check{f}(xtk \cdot y)dkd\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G. \tag{4.2}$$

Corollary 4.2 ([12])**.** *Let σ be a continuous involution of G. Let μ be a σ-invariant measure. Let $(f, g) \in C_b(G) \setminus \{0\}$ such that f satisfies $K(\mu)$. The solutions of the functional equation*

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2g(x)f(y), \quad x, y \in G \tag{4.3}$$

are given as follows:

i) *there exists a $\check{\mu}$-spherical function φ such that $g = \frac{\varphi+\varphi\circ\sigma}{2}$,*

ii) *if $\varphi\circ\sigma \neq \varphi$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $f = \alpha\frac{\varphi+\varphi\circ\sigma}{2} + \beta\frac{\varphi-\varphi\circ\sigma}{2}$,*

iii) *if $\varphi \circ \sigma = \varphi$, then there exists $\gamma \in \mathbb{C}$ such that $f = \gamma\varphi + l$, where $l \circ \sigma = -l$ and l is a solution of the functional equation*

$$\int_G l(xty)d\check{\mu}(t) = l(x)\varphi(y) + l(y)\varphi(x), \quad x, y \in G. \tag{4.4}$$

Proof. By taking $\chi = 1$ in (4.1) and by using Theorem 4.1 and ([12], Theorem 3.1), we get the proof. □

5. On generalized functional equations of Stetkær type on Lie groups

In this section we characterize the solutions $f, g \in C^\infty(G)$ of the functional equation

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)h(y), \quad x, y \in G \tag{5.1}$$

on a connected Lie group G as joint eigenfunctions of certain operators associated to the left invariant differential operators, where in this case K is a compact subgroup of the group $\mathrm{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms. This extends the previous results obtained by Stetkær in [30] to equation (1.7) and those of the authors in [13] to Badora's functional equation.

To formulate our results, we need the following notations:
Let G be a connected Lie group and K a compact subgroup of the group $\mathrm{Aut}(G)$ of all mappings of G onto G that are simultaneously automorphisms and homeomorphisms. $\mathbb{D}(G)$ denotes the algebra of the left invariant differential operators on G, i.e. for all $D \in \mathbb{D}(G)$, $a \in G$, and for all $f \in C^\infty(G)$ we have $(L_aD)f = D(L_af)$, where $(L_af)(x) = f(a^{-1}x)$ for all $x \in G$. We recall (see [30], Proposition II.3) that K has a Lie group structure, the canonical map $K \times G \longrightarrow G$ sending (k, x) onto $k.x$ is C^∞, and if $f \in C^\infty$ then so does $k \cdot f$ for any $k \in K$, because continuous homomorphisms between Lie groups automatically are C^∞. Throughout the rest of the present section, we assume that μ satisfies the following conditions:

i) μ is a K-invariant measure with compact support on G and

ii) $\mu * \mu = \mu$.

The symbol $C_\mu^\infty(G) = \check{\mu} * C^\infty(G) * \Delta\check{\mu}$ will stand for all functions $f \in C^\infty(G)$ which are μ-invariant on G. The subspace of $C_\mu^\infty(G)$ consisting of the functions which are K-invariant will be denoted $C_{\mu,K}^\infty(G)$. For any $D \in C^\infty(G)$, we define the new operator $D_\mu^K f$ by

$$(D_\mu^K f)(x) = D\{M_K(L_{x^{-1}} f)_\mu\}(e)$$

for all $f \in C^\infty(G)$ and $x \in G$ [13]. In view of ([13] Proposition 4.1 and Proposition 4.2), D_μ^K has the following properties:

Theorem 5.1. i) *D_μ^K is left invariant.*

ii) *$k \cdot D_\mu^K f = D_\mu^K k \cdot f$, for all $k \in K$ and $f \in C^\infty(G)$,*

iii) *$(D_\mu^K f)(e) = D(M_K f_\mu)(e)$. In particular if f is a bi-μ-invariant and K-invariant function on G, then we have $(D_\mu^K f)(e) = (Df)(e)$.*

iv) *g and $h \in C^\infty(G)$.*

v) *If (f, g, h) is a solution of (5.1), such that $g \neq 0$, $h \neq 0$ and satisfying $\int_G \check{h}(xt)d\check{\mu}(t) = \check{h}(x)$ and $\int_G g(xt)d\mu(t) = g(x)$, then $D_\mu^K g = (D\phi)(e)g$ and $D_{\check{\mu}}^K \check{g} = (D\check{\phi})(e)\check{g}$, where ϕ is a solution of the functional equation (3.1). Consequently g and h are analytic.*

vi) *If $D \in \mathbb{D}(G)$, then for all $f \in C_{\mu,K}^\infty(G)$ we have*

$$D_\mu^K f = M_K(Df * \Delta\check{\mu}).$$

In particular, the restriction of D_μ^K to $C_{\mu,K}^\infty(G)$ is an endomorphism.

The next theorem extends the result obtained by the authors to Badora's functional equation ([13]).

Theorem 5.2. *Let $\mu \in M(G)$ be a K-invariant, idempotent measure on G with compact support. If $(f, g) \in C(G) \setminus \{0\}$, then the following statements are equivalent:*

(1) *(f, g) is a solution of*

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k)dkd\mu(t) = g(x)f(y), \quad x, y \in G, \tag{5.2}$$

(2) a) $f(k \cdot y) = \chi(k) f(y)$,
b) f *and* $g \in C^\infty(G)$,
c) f *and* g *are analytic,*
d) $\int_G \check{f}(xt) d\check{\mu}(t) = \check{f}(x)$ *for all* $x \in G$,
e) $D_{\check{\mu}}^K \check{f} = (D\check{g})(e)\check{f}$ *for all* $D \in \mathbb{D}(G)$.

Proof. (1) $\Longrightarrow$ (2) follows directly from Theorem 5.1. Conversely, suppose that (a), (b), (c), (d) and (e) hold. For a fixed $x \in G$, we define the function

$$F(y) = \int_G \int_K \check{f}(xtk \cdot y) dk d\check{\mu}(t), \quad y \in G.$$

It is easy to verify that F is K-invariant. Furthermore, since $\mu * \mu = \mu$, μ is K-invariant and $\check{f}$ is right μ-invariant hence F is bi-μ-invariant. Now $F(y)$ can be written

$$F(y) = \int_G \int_K (L_{(k^{-1}.xt)})^{-1} \check{f})(y) \overline{\chi}(k) dk d\check{\mu}(t).$$

Consequently, for all $D \in \mathbb{D}(G)$ we have

$$(D_{\check{\mu}}^K F)(y) = D(\check{g})(e) F(y).$$

In particular, for $y = e$ we have

$$(D_{\check{\mu}}^K F)(e) = D(\check{g})(e) F(e).$$

Hence, by Theorem 5.1, it follows that

$$(DF)(e) = D(\check{g})(e) F(e)$$

i.e

$$D(F - F(e)\check{g})(e) = 0$$

for all $D \in \mathbb{D}(G)$. Since $F - F(e)\check{g}$ is an analytic function on the connected Lie group G, by [21] we obtain

$$F - F(e)\check{g} \equiv 0$$

on G. We conclude that

$$\int_G \int_K \check{f}(xtk \cdot y) dk d\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G.$$

Finally, by using (a) we obtain

$$\int_G \int_K f(xtk \cdot y)\overline{\chi}(k) dk d\mu(t) = g(x)f(y), \quad x, y \in G.$$

This ends the proof of the theorem. □

6. Hyers–Ulam stability of generalized equations of Stetkær type

In 1940 S. M. Ulam posed the following problem on the stability of homomorphisms:

> Given a group G_1, a metric group (G_2, d), and a positive number ε, does there exist a $\lambda > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies the inequality
>
> $$d(f(xy), f(x)f(y)) < \varepsilon$$
>
> for all $x, y \in G$, then a homomorphism $a : G_1 \longrightarrow G_2$ exists with
>
> $$d(f(x), a(x)) < \lambda \quad \text{for all } x \in G?$$

See S. M. Ulam [35] or [36] for a discussion of such problems, as well as D. H. Hyers [22], D. H. Hyers and S. M. Ulam [24], Th. M. Rassias [26], D. H. Hyers, G. I. Isac and T. M. Rassias [23]. Later, the above question became a source of stability theory in the Hyers–Ulam sense. The first affirmative answer to Ulam's question was given by D. H. Hyers in [22], under the assumption that G_1 and G_2 are Banach spaces. The Hyers–Ulam–Rassias stability was taken up by a number of mathematicians and the study of this area has grown to be one of the central subjects in mathematical analysis. There is a strong stability phenomenon which is known as superstability. An equation is called superstable if for any approximate homomorphism, (i.e. $d(f(xy), f(x)f(y)) \leq \delta$), either f is bounded or f is a true homomorphism. This property was first observed when the following theorem was proved by J. Baker, J. Lawrence, and F. Zorzitto [8]:

Theorem. *Let V be a vector space. If a function $f : V \longrightarrow \mathbb{R}$ satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in V$, then either f is a bounded function or $f(x+y) = f(x)f(y)$, for all $x, y \in V$.

Later this result was generalized by J. Baker [7] and L. Székelyhidi [33], [34].

The aim of the present section is to investigate the stability of the following family of functional equations:

$$\int_K \int_G f(xtk \cdot y)dkd\mu(t) = f(x)g(y), \quad x, y \in G, \tag{6.1}$$

$$\int_K \int_G f(xtk \cdot y)\overline{\chi(k)}dkd\mu(t) = f(y)g(x), \quad x, y \in G, \tag{6.2}$$

where $\mu \in M(G)$ is a K-invariant measure with compact support on G.

Particular cases of (6.1) and (6.2) are

$$\int_K f(x + k \cdot y)dk = f(x)g(y), \quad x, y \in G \tag{6.3}$$

and

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = f(y)g(x), \quad x, y \in G, \tag{6.4}$$

where G is a commutative group and $\mu = \delta_e$, the Dirac measure concentrated at the identity element of G. The stability properties of the equations (6.3), (6.4) have been obtained by Badora [5]. For K-spherical functions (i.e. (6.3) with $f = g$) with K finite this problem was solved by W. Förg-Rob and J. Schwaiger in [19] and by R. Badora in [6], and for $K = \{Id, -Id\}$, i.e. d'Alembert's functional equation, by J. Baker [7].

For the noncommutative case, some results for some particular equations of type (6.1) where obtained by Elqorachi and Akkouchi [14], [15], [17]. The stability of the classical examples

$$f(x+y) = f(x)f(y), \tag{6.5}$$

$$f(x+y)+f(x-y)=2f(x)f(y) \tag{6.6}$$

of equations (6.1) and (6.2) has attracted the attention of many mathematicians. The interested reader should refer to [23] for a thorough account on the subject of stability of functional equations.

Throughout this section μ is assumed to be a compactly supported measure on G which is K-invariant, and f satisfies the Kannappan condition $K(\mu)$.

Theorem 6.1. *Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions. Assume that there exists $\delta \geq 0$ such that*

$$\Big|\int_K\int_G f(xtk\cdot y)dkd\mu(t) - f(x)g(y)\Big| \leq \delta, \quad x, y \in G, \tag{6.7}$$

and f fulfills $K(\mu)$. Then either

i) *f, g are bounded or*

ii) *f is unbounded and g satisfies Badora's equation*

$$\int_K\int_G g(xtk\cdot y)dkd\mu(t) = g(x)g(y), \quad x, y \in G, \tag{6.8}$$

or

iii) *g is unbounded and f satisfies the equation* (6.1) *(if $f \neq 0$, then g satisfies* (6.8)*).*

Proof. The proof of the theorem is related to the one in [15], (see Theorem 3.1), where $K = \{Id, \sigma\}$ and σ is a continuous involution of G. If f is unbounded, then by using the inequality (6.7), we get

$$\begin{aligned}
&|f(z)|\Big|\int_K\int_G g(xtk\cdot y)dkd\mu(t) - g(x)g(y)\Big| \\
&\quad\leq \Big|f(z)\int_K\int_G g(xtk\cdot y)dkd\mu(t) - g(y)\int_K\int_G f(ztk\cdot x)dkd\mu(t)\Big| \\
&\qquad + |g(y)|\Big|\int_K\int_G f(ztk\cdot x)dkd\mu(t) - f(z)g(x)\Big| \\
&\quad\leq \Big|f(z)\int_K\int_G g(xtk\cdot y)dkd\mu(t) - g(y)\int_K\int_G f(ztk\cdot x)dkd\mu(t)\Big| + |g(y)|\delta,
\end{aligned}$$

for all $x, y, z \in G$. Since

$$\begin{aligned}
&\Big|\int_K\int_K\int_G\int_G f(ztk\cdot xsk'\cdot y)dkdk'd\mu(t)d\mu(s)\\
&\qquad -\int_K\int_G f(ztk\cdot x)dkd\mu(t)g(y)\Big|\\
&\quad\leq \int_K\int_G\Big|\int_K\int_G f(ztk\cdot xsk'\cdot y)dk'd\mu(s)-f(ztk\cdot x)g(y)\Big|dkd|\mu|(t)\\
&\quad\leq \delta\|\mu\|dk(K)=\delta\|\mu\|,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_K\int_K\int_G\int_G f(ztk\cdot(xsk'\cdot y))dkdk'd\mu(t)d\mu(s)\\
&\qquad - f(z)\int_K\int_G g(xsk'\cdot y)dk'd\mu(t)d\mu(s)\Big|\\
&\quad\leq \int_K\int_G\Big|\int_K\int_G f(ztk\cdot(xsk'\cdot y))dkd\mu(t)-f(z)g(xsk'\cdot y)\Big|dk'd|\mu|(s)\\
&\quad\leq \delta\|\mu\|,
\end{aligned}$$

and from the relation

$$\begin{aligned}
&\int_K\int_K\int_G\int_G f(ztk\cdot(xsk'\cdot y))dkdk'd\mu(t)d\mu(s)\\
&\quad=\int_K\int_{K^+}\int_G\int_G f(ztk\cdot xk\cdot s(kk')\cdot y)dkdk'd\mu(t)d\mu(s)\\
&\qquad+\int_K\int_{K^-}\int_G\int_G f(zt(kk')\cdot yk\cdot sk\cdot x)dkdk'd\mu(t)d\mu(s)\\
&\quad=\int_K\int_{K^+}\int_G\int_G f(ztk\cdot xs(kk')\cdot y)dkdk'd\mu(t)d\mu(s)\\
&\qquad+\int_K\int_{K^-}\int_G\int_G f(ztk\cdot xs(kk')\cdot y)dkdk'd\mu(t)d\mu(s)\\
&\quad=\int_K\int_K\int_G\int_G f(ztk\cdot xs(kk')\cdot y)dkdk'd\mu(t)d\mu(s)\\
&\quad=\int_K\int_G\int_G f(ztk\cdot xsk'\cdot y)dkdk'd\mu(t)d\mu(s)
\end{aligned}$$

we obtain

$$\Big|f(z)\int_K\int_G g(xtk\cdot y)dkd\mu(t)-g(y)\int_K\int_G f(ztk\cdot x)dkd\mu(t)\Big|\le 2\delta\|\mu\|,$$

and finally

$$|f(z)|\,\Big|\int_K\int_G g(xtk\cdot y)dkd\mu(t)-g(x)g(y)\Big|\le 2\delta\|\mu\|+|g(y)|\delta.$$

Since f is unbounded, it follows that

$$\int_K\int_G g(xtk\cdot y)dkd\mu(t)=g(x)g(y),\quad \text{for all } x,y\in G,$$

which ends the proof in this case.

If g is unbounded, equation (6.1) holds if $f=0$. Let us assume now that $f\neq 0$. Then there exists $z\in G$ such that $f(z)\neq 0$. From inequality (6.7), we obtain

$$\Big|\frac{\int_K\int_G f(ztk\cdot x)dkd\mu(t)}{f(z)}-g(x)\Big|\le\frac{\delta}{|f(z)|},\quad \text{for all } x\in G.$$

Since g is unbounded, the function defined by

$$h(x)=\frac{\int_K\int_G f(ztk\cdot x)dkd\mu(t)}{f(z)}$$

is also unbounded.

On the other hand h satisfies the following inequality:

$$\Big|\int_K\int_G h(xtk\cdot y)dkd\mu(t)-h(x)g(y)\Big|\le\frac{\delta\|\mu\|}{|f(z)|},\quad \text{for all } x,y\in G. \tag{6.9}$$

Now, by the preceding discussion, we conclude that g satisfies the equation (6.8). To see that f, g satisfy (6.1), let $x,y,z\in G$. Using inequality (6.7) and the fact that g satisfies the equation (6.8), we get

$$\begin{aligned}&|g(z)|\,\Big|\int_K\int_G f(xtk\cdot y)dkd\mu(t)-f(x)g(y)\Big|\\&\quad\le\Big|\int_K\int_K\int_G\int_G f(xtk\cdot ysk'\cdot z)dkdk'd\mu(t)d\mu(s)\end{aligned}$$

$$
\begin{aligned}
&- g(z)\int_K\int_G f(xtk\cdot y)dkd\mu(t)\Big|\\
&+\Big|\int_K\int_K\int_G\int_G f(xtk\cdot ysk'\cdot z)dkdk'd\mu(s)d\mu(t)\\
&- f(x)\int_K\int_G g(ysk'\cdot z)dk'd\mu(t)\Big| \le 2\delta\|\mu\|.
\end{aligned}
$$

Hence f, g satisfy the equation (6.1) and the proof of the theorem is complete. $\square$

As a consequence, we have the superstability of the equation (6.8).

Corollary 6.2 ([17] Theorem 2.1)**.** *Let $f : G \longrightarrow \mathbb{C}$ be a continuous function. Assume that there exists $\delta \ge 0$ such that*

$$
\Big|\int_K\int_G f(xtk\cdot y)dkd\mu(t) - f(x)f(y)\Big| \le \delta, \quad x, y \in G. \tag{6.10}
$$

Then either

$$
|f(x)| \le \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 4\delta}}{2}, \quad x \in G, \tag{6.11}
$$

or f is a solution of the equation (6.8).

Remark 6.3. In Theorem 6.1 it is not necessary to assume that f satisfies the condition $K(\mu)$ if K is a compact subgroup of homomorphisms of G.

In the following theorem we shall investigate the stability of the functional equation (6.2), under the additional condition that f satisfies the Kannappan type condition $K_1(\mu)$:

$$
\begin{cases}
\displaystyle\int_G\int_G f(ztxsy)d\mu(t)d\mu(s) = \int_G\int_G f(ztysx)d\mu(t)d\mu(s),\\[2ex]
\displaystyle\int_G f(xsy)d\mu(s) = \int_G f(ysx)d\mu(s), \quad \text{for all } x, y, z \in G.
\end{cases}
$$

Theorem 6.4. *Let $f, g : G \longrightarrow \mathbb{C}$ be continuous functions. Assume that there exists $\delta \ge 0$ such that*

$$
\Big|\int_K\int_G f(xtk\cdot y)\overline{\chi(k)}dkd\mu(t) - f(y)g(x)\Big| \le \delta, \quad x, y \in G, \tag{6.12}
$$

and f fulfills $K_1(\mu)$. Then either

i) *f, g are bounded or*

ii) *f is unbounded and g satisfies*

$$\int_K \int_G \check{g}(xtk \cdot y) dk d\check{\mu}(t) = \check{g}(x)\check{g}(y), \quad x, y \in G, \tag{6.13}$$

or

iii) *g is unbounded and f, g satisfy the equation* (6.2)*.*

Proof. In the proof, we use ideas and methods that are analogous to those used in [5].

In order to apply Theorem 6.1, we recall the following formula proved for $\mu = \delta_e$ by Badora (see [5]).

$$\begin{aligned}\chi(k) \int_K \int_G & f(xsk' \cdot y)\overline{\chi(k')} dk' d\mu(s) - \chi(k) f(y) g(x) \\ & - \int_K \int_G f(xsk' \cdot (k \cdot y))\overline{\chi(k')} dk' d\mu(s) + g(x) f(k \cdot y) \\ & = g(x)(f(k \cdot y) - \chi(k) f(y)),\end{aligned} \tag{6.14}$$

for all $x, y \in G$.

On the other hand, by using the condition $K_1(\mu)$, the K-invariance of μ and some computations used in [5], we prove that

$$\begin{aligned} g(z) & \Big[\int_K \int_G f(ytk \cdot x)\overline{\chi(k)} dk d\mu(t) - g(y) f(x) \Big] \\ & + \int_K \Big[\int_K \int_G \int_G f(zsk' \cdot (ytk \cdot x))\overline{\chi(k')} dk' d\mu(t) d\mu(s) \\ & - f(z) \int_G g(ytk \cdot x) d\mu(t) \Big] \overline{\chi(k)} dk \\ & - \int_K \Big[\int_K \int_G \int_G f(zsk' \cdot (k^{-1} \cdot ytx))\overline{\chi(k')} dk' d\mu(t) d\mu(s) \\ & - f(z) \int_G g(k^{-1} \cdot ytx) d\mu(t) \Big] dk \\ = g(z) & \Big[\int_K \int_G f(k^{-1} \cdot ytx) dk d\mu(t) - g(y) f(x) \Big]. \end{aligned} \tag{6.15}$$

Now we are ready to prove the theorem. If f is unbounded, $f = 0$ satisfies the equation (6.13). If $g \neq 0$, then in view of (6.15), there exists some constant $\delta' \geq 0$ such that

$$\Big|\int_K \int_G f(k^{-1} \cdot ytx) dk d\mu(t) - g(y)f(x)\Big| \leq \delta', \quad x, y \in G, \tag{6.16}$$

which can be written

$$\Big|\int_K \int_G \check{f}(xtk \cdot y) dk d\check{\mu}(t) - \check{f}(x)\check{g}(y)\Big| \leq \delta', \quad x, y \in G. \tag{6.17}$$

It follows from Theorem 6.1, that g satisfies the equation (6.13). If g is unbounded, then by (6.14)

$$f(k \cdot x) = \chi(k)f(x), \quad \text{for all } x \in G. \tag{6.18}$$

By using the equation (6.15), we obtain that f, g satisfy some inequality like (6.17) and hence by Theorem 6.1 we deduce that f, g are solutions of the equation

$$\int_K \int_G \check{f}(xtk \cdot y) dk d\check{\mu}(t) = \check{f}(x)\check{g}(y), \quad x, y \in G. \tag{6.19}$$

Now from Theorem 4.1 of the section 4, we deduce that f, g are solutions of (6.2) and the proof is completes. □

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