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A characterization of the identity function with an equation of Hosszú type

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Abstract. The functional equation

$$f : \mathbb{R} \to \mathbb{R},\ f(x+y-xy)+f(xy)=f(x)+f(y) \quad \text{for all } x,y \in \mathbb{R}$$

was first presented by M. Hosszú (1967) and now it is referred to as the Hosszú equation. The aim of this note is to consider an equation of Hosszú type on the domain $\mathbb{N}$. We prove that if a completely multiplicative function f satisfies the equation

$$f(p+q+pq)=f(p)+f(q)+f(pq)$$

for all primes p, q and $f(p_0) \neq 0$ for some prime number p_0, then $f(n)=n$ for all positive integers n.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathcal{P}$ denote the set of all positive integers, all real numbers and all prime numbers, respectively. Let $\mathcal{M}$ ($\mathcal{M}^*$) be the set of all complex-valued multiplicative (completely multiplicative) functions.

The functional equation

$$f : \mathbb{R} \to \mathbb{R},\ f(x+y-xy)+f(xy)=f(x)+f(y) \quad \text{for all } x,y \in \mathbb{R} \quad (1)$$

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was first presented by M. Hosszú (1967) and now it is referred to as the Hosszú equation. It follows from a result of Z. Daróczy [3] that if f is a solution of (1), then there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x) + f(0)$ $(x \in \mathbb{R})$ and

$$F(x + y) = F(x) + F(y), \tag{2}$$

for all $x, y \in \mathbb{R}$. The equation (2) is the Cauchy functional equation and it is well-known that continuity along with the functional equation (2) characterizes F in the form $F(x) = cx$ for all $x \in \mathbb{R}$, where c is an arbitrary real number.

In 1992, C. Spiro [7] considered the equation (2) in the case $F \in \mathcal{M}$ and restricted the domain from $\mathbb{R}$ to $\mathbb{N}$. She showed that if a function $F \in \mathcal{M}$ satisfying (2) for all primes x and y, then $F(n) = n$ for all $n \in \mathbb{N}$. In [4] the identity function was characterized as the multiplicative function F for which $F(p + n^2) = F(p) + F(n^2)$ holds for all primes p and for all $n \in \mathbb{N}$. For other results in this topic we refer to works [1], [2], [5] and [6].

The aim of this note is to consider an equation of Hosszú type on the domain $\mathbb{N}$. We prove the following

Theorem 1. *If the function $f \in \mathcal{M}^*$ satisfies the equation*

$$f(p + q + pq) = f(p) + f(q) + f(pq) \quad \text{for all} \quad p, q \in \mathcal{P} \tag{3}$$

and

$$f(p_0) \neq 0 \quad \text{for some prime } p_0, \tag{4}$$

then $f(n) = n$ for all $n \in \mathbb{N}$.

2. Lemmas

The proof of our theorem is based on Lemmas 1–2.

Lemma 1. *If $f \in \mathcal{M}^*$ satisfies* (3) *and* (4)*, then $f(2) \neq 0$.*

Proof. In order to prove Lemma 1, we proceed by contradiction. Assume that $f(2) = 0$.

Let

$$\mathcal{P}_0 := \{p \in \mathcal{P} \mid f(p) = 0\} \quad \text{and} \quad \mathcal{P}_1 := \{p \in \mathcal{P} \mid f(p) \neq 0\}.$$

We note from our assumptions that

$$\text{if} \quad p \in \mathcal{P}_1 \text{ then } f(p+2) = f(p) + 2. \tag{5}$$

Indeed, using (3) and the fact that $f \in \mathcal{M}^*$, we have

$$f(p)f(p+2) = f(p+p+p^2) = f(p) + f(p) + f(p)^2 = f(p)\left[f(p) + 2\right],$$

from which (5) follows immediately.

We now prove that

$$\{p \in \mathcal{P} \mid p \leq 23\} \subset \mathcal{P}_0. \tag{6}$$

By the assumption $f(2) = 0$ and by (3), we have

$$f(3p+2) = f(p) \quad \text{for all } p \in \mathcal{P}. \tag{7}$$

Repeated use of (7) gives $f(7) = f(23)$, $f(5) = f(17) = f(53) = f(161) = f(7)f(23) = f(7)^2$, $f(3) = f(11) = f(35) = f(5)f(7) = f(7)^3$, $f(13) = f(41) = f(125) = f(5)^3$ and $f(19) = f(59) = f(179) = f(539) = f(7)^2 f(11)$. If $f(7) \neq 0$, then $f(3)f(5) \neq 0$, and so one can deduce from (5) that $f(7)^3 = f(3) = f(5) - 2 = f(7)^2 - 2$, $f(7)^2 = f(5) = f(7) - 2$, which is impossible.

We must therefore have that $\{p \in \mathcal{P} \mid p \leq 23\} \subset \mathcal{P}_0$ and the proof of (6) is completed.

From (4) we have $p_0 \in \mathcal{P}_1$, $\mathcal{P}_1 \neq \emptyset$. If $2 \in \mathcal{P}_1$, then we are done. Otherwise, let $T := \min \mathcal{P}_1$, i.e.

$$T \text{ is a prime such that } f(T) \neq 0 \text{ and } f(n) = 0 \quad \text{for all } 1 < n < T. \tag{8}$$

We prove the following assertions:

$$f(T) = -2, \tag{9}$$

$$|f(\pi)| \geq 2 \quad \text{for all } \pi \in \mathcal{P}_1, \tag{10}$$

and

$$3^nT + 3^n - 1 \in \mathcal{P}_1,\ f\left(3^nT + 3^n - 1\right) = -2 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

Since $f(T) \neq 0$, we get from (5) that $f(T) = f(T+2) - 2$ and (9) is true if $f(T+2) = 0$. If $f(T+2) \neq 0$, then $T+2 \in \mathcal{P}$, $T \equiv 2 \pmod 3$, therefore $f(T+4) = f(3)f(\frac{T+4}{3}) = 0$. Hence it follows from (5) that $f(T+2) = f(T+4) - 2 = -2$, $f(T) = f(T+2) - 2 = -4$. In this case

$$7 \cdot (T+3) + T + 2 \equiv 7 \cdot (2+3) + 2 + 2 = 39 \equiv 0 \pmod 3,$$

consequently we infer from (3) and (6) that

$$f\left(7 \cdot (T+3) + T + 2\right) = f(3)f\left(\frac{7 \cdot (T+3) + T + 2}{3}\right) = 0$$

and

$$\begin{aligned} f(7 \cdot (T+3) + T + 2) &= f(7) + f(T+2) + f(7 \cdot (T+2)) \\ &= f(T+2) = -2, \end{aligned}$$

which are impossible. Thus $f(T+2) = 0$, and (9) is proved.

By (9), we have $|f(T)| = 2$. Assume that $q \in \mathcal{P}_1$, $f(q) \neq 0$ and (10) is true for all $\pi \in \mathcal{P}_1$, $\pi < q$. Since $q \in \mathcal{P}_1$, $f(q) \neq 0$, we get from (5) that $f(q) = f(q+2) - 2 = -2$ if $f(q+2) = 0$. Assume that $f(q+2) \neq 0$ and $q+2 \notin \mathcal{P}$. Then

$$q + 2 = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r},$$

where $\pi_1 < \ldots < \pi_r < q$, $\pi_i \in \mathcal{P}_1$ $(i = 1, \ldots, r)$ and $\alpha_1, \ldots, \alpha_r$ are non-negative integers with $\alpha_1 + \cdots + \alpha_r \geq 2$. Therefore we have

$$|f(q)| = |f(q+2) - 2| \geq |f(q+2)| - 2 \geq 2^{\alpha_1 + \cdots + \alpha_r} - 2 \geq 2.$$

Assume now that $f(q+2) \neq 0$ and $q+2 \in \mathcal{P}$. Then $3 \mid q+4$ and

$$|f(q)| = |f(q+2) - 2| = |f(q+4) - 4| = \left|f(3)f\left(\frac{q+4}{3}\right) - 4\right| = 4.$$

The proof of (10) is finished. □

Now we prove (11). By (7), we have $-2 = f(T) = f(3T+2)$ and so (11) holds for $n = 1$. Assume that $-2 = f(T) = f(3^nT + 3^n - 1)$ with some $n \in \mathbb{N}$, $n \geq 1$ and $3^nT + 3^n - 1 \in \mathcal{P}$, i.e. (11) is true for n. Then by (5), (7) and (9) we have

$$-2 = f(T) = f(3^nT + 3^n - 1) = f(3^{n+1}T + 3^{n+1} - 1).$$

Assume that $Q_{n+1} := 3^{n+1}T + 3^{n+1} - 1 \notin \mathcal{P}$. Then

$$Q_{n+1} = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r}, \quad (\pi_1, \ldots, \pi_r \in \mathcal{P}_1),$$

where $\alpha_1, \ldots, \alpha_r$ are non-negative integers, $\alpha_1 + \cdots + \alpha_r \geq 2$. This contradicts (10), because

$$2 = |f(Q_{n+1})| = |f(\pi_1^{\alpha_1})| \cdots |f(\pi_r^{\alpha_r})| \geq 2^{\alpha_1 + \ldots + \alpha_r} \geq 4.$$

Thus we have $Q_{n+1} = 3^{n+1}T + 3^{n+1} - 1 \in \mathcal{P}$ and so (11) is proved.

Now we complete the proof of Lemma 1. If $T - 2$ has a prime divisor $\pi \neq 3$, then

$$3^{\pi-2}T + 3^{\pi-2} - 1 \equiv 3^{\pi-1} - 1 \equiv 0 \pmod{\pi}.$$

This contradicts (11), since $3^{\pi-2}T + 3^{\pi-2} - 1 \in \mathcal{P}$ and $3^{\pi-1} - 1 > \pi$.

If $T - 2 = 3^m$, $m \in \mathbb{N}$, then $(T+1, 11) = (3^m + 3, 11) = 1$. Hence we can choose a prime $p \leq 23$ such that

$$p + T + pT = (T+1)p + T \equiv 0 \pmod{11}.$$

Indeed, since $\{T + 1 \equiv 3^m + 3 \pmod{11}\} = \{4, 6, 1, 8, 7\}$, we can choose p as $p = 2, 23, 11, 17, 7$, respectively. Thus we infer from (6) that

$$\begin{aligned} 0 &= f(11) f\left(\frac{p + T + pT}{11}\right) \\ &= f(p + T + pT) = f(p) + f(T) + f(p)f(T) = f(T) = -2, \end{aligned}$$

which is a contradiction.

Hence we must have $f(2) \neq 0$, and Lemma 1 is proved.

Lemma 2. *If $f \in \mathcal{M}^*$ satisfies* (3) *and* (4), *then $f(n) = n$ for all $n \in \{1, 2, 3, \ldots, 31\}$.*

PROOF. By using Lemma 1 and (5), we have

$$f(2)^2 = f(2) + 2.$$

Thus, we have either $f(2) = -1$ or $f(2) = 2$. If $f(2) = -1$, then repeated use of (3), (5) and (6) gives $f(11) = f(2)+f(3)+f(2)f(3) = -1$, $f(53) = f(2) + f(17) + f(2)f(17) = -1$ and $f(5)f(11) = f(55) = f(53) + 2 = 1$. Therefore we have $f(5) = -1$, $f(7) = f(5)+2 = 1$ and $f(9) = f(7)+2 = 3$. The last relation shows that $f(3) \neq 0$ and so $f(3) = f(5) - 2 = -3$, which contradicts to $f(9) = f(3)^2 = 3$. We must therefore have that $f(2) = 2$.

We note that $f(2) = 2$ implies $f(3)f(5) \neq 0$. Indeed, if $f(3) = 0$, then we infer from (3) and (5) that $f(11) = f(2) + f(3) + f(2)f(3) = 2$, $f(13) = f(11) + 2 = 4$ and $f(15) = f(13) + 2 = 6$. This is impossible, because $f(15) = f(3)f(5) = 0$. Now assume that $f(3) \neq 0$ and $f(5) = 0$. Then we get from (3) and (5) that $f(3) = -2$, $f(11) = f(2) + f(3) + f(2)f(3) = -4$. Hence $0 = f(35) = f(2) + f(11) + f(2)f(11) = -10$, which is a contradiction.

Since $f(3) \neq 0$ and $f(5) \neq 0$, we have $f(5) = f(3) + 2$, $f(7) = f(5) + 2 = f(3) + 4$, $f(11) = f(2) + f(3) + f(2)f(3) = 3f(3) + 2$ and $f(35) = f(2) + f(11) + f(2)f(11) = 3f(11) + 2 = 9f(3) + 8$. These imply that

$$9f(3) + 8 = (f(3) + 2)(f(3) + 4) = f(3)^2 + 6f(3) + 8.$$

Therefore we get by using the fact $f(3) \neq 0$ that $f(3) = 3$, consequently $f(5) = 5$, $f(7) = f(5) + 2 = 7$, $f(11) = 3f(3) + 2 = 11$, and $f(13) = f(11) + 2 = 13$, $f(17) = f(2 + 5 + 2 \cdot 5) = 17$, $f(19) = f(17) + 2 = 19$ and $f(23) = f(2 + 7 + 2 \cdot 7) = 23$.

Finally, we infer from (3) and (5) that

$$7 \cdot 17 = f(7)f(17) = f(119) = f(3 + 29 + 3 \cdot 29) = 4f(29) + 3,$$

consequently $f(29) = 29$ and $f(31) = f(29) + 2 = 31$. Lemma 2 is proved. □

3. The proof of the theorem

Assume that $f(n) = n$ for all positive integers $n < N$. By Lemma 2 we may suppose that $N \geq 37$. We shall prove that $f(N) = N$, which proves our theorem.

If $N \notin \mathcal{P}$, then it is obvious from our assumption that $f(N) = N$. Assume that $N \in \mathcal{P}$.

Case I: $f(N) \neq 0$. Since $N \in \mathcal{P}$ and $f(N) \neq 0$, we have by (5) that $f(N+2) = f(N) + 2$, consequently $f(N) = N$ if $N + 2 \notin \mathcal{P}$.

Assume that $N + 2 \in \mathcal{P}$ and $f(N+2) \neq 0$. Then $N \equiv 2 \pmod 3$, therefore from (5) we get $f(N+2) = f(N+4) - 2 = N + 4 - 2 = N + 2$, which implies $f(N) = N$.

Assume now that $N + 2 \in \mathcal{P}$ and $f(N+2) = 0$. In this case we have $N \equiv 2 \pmod 3$ and $f(N) = -2$. Hence

$$f\left[(N+3)p + N + 2\right] = f\left[p + (N+2) + p(N+2)\right]) = f(p)$$

holds for all primes p. This with $p = 7$ gives

$$f\left(\frac{8N+23}{3}\right) = \frac{7}{3}.$$

Therefore, by using the facts $(8N + 23, 2) = 1$, $N \geq 37$, we have $Q := \frac{8N+23}{3} \in \mathcal{P}$. This implies that

$$f(Q+2) = f(Q) + 2 = \frac{7}{3} + 2 = \frac{13}{3},$$

consequently $Q + 2 \in \mathcal{P}$ and $Q \equiv 2 \pmod 3$. This is impossible, because

$$f(Q+4) = 3f\left(\frac{Q+4}{3}\right) = 3f\left(\frac{8N+35}{9}\right) = \frac{8N+35}{3}$$

and

$$f(Q+4) = f(Q+2) + 2 = \frac{19}{3}.$$

Case II: $f(N) = 0$. In this case we have

$$f\left[(N+1)p + N\right] = f(p + N + pN) = f(p)$$

holds for all primes p. Hence

$$f(3N+2) = f(2) = 2,$$

which shows that $3N + 2 \in \mathcal{P}$ and $f(3N+4) = f(3N+2) + 2 = 4$. Hence we infer that $3N + 4 \in \mathcal{P}$ and

$$f(3N+6) = f\left((3N+4) + 2\right) = 6 \qquad \text{and}$$

$$f(3N+6) = f(3)f(N+2) = 3f(N+2).$$

Thus we have $f(N+2) = 2$, $N+2 \in \mathcal{P}$ and $f(N+4) = f(N+2)+2 = 4$. This is impossible, because $N, N+2 \in \mathcal{P}$ imply that $3 \mid N+4$ and

$$f(N+4) = f(3)f\left(\frac{N+4}{3}\right) = 3 \cdot \left(\frac{N+4}{3}\right) = N+4.$$

Hence the theorem is proved.

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