

## On skew 2-groups

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*To the memory of Edith Szabó*

**Abstract.** We study 2-groups whose non-linear irreducible characters are of the third kind, i.e. real but not afforded by a real representation.

The purpose of this short note is to draw attention to an interesting class of finite 2-groups, and to make a start in studying them. Our results are far from definitive. First, let us recall the definition of the *Frobenius–Schur indicator*  $\nu(\chi)$  of an irreducible character  $\chi$  of a finite group  $G$ .  $\nu(\chi) = 1$  if  $\chi$  is afforded by a real representation,  $\nu(\chi) = -1$  if  $\chi$  is real, but is not afforded by a real representation, and  $\nu(\chi) = 0$  if  $\chi$  is not real-valued.  $\chi$  is said to be of the *first, second, or third kind*, if  $\nu(\chi) = 1, 0$ , or  $-1$ , respectively. For the theory of this indicator, see, e.g., [JL, chapter 23]. Here we denote by  $\text{Irr}(G)$  the set of irreducible characters of  $G$ , by  $X = X(G)$  the set of non-linear irreducible characters, and by  $t(x)$ , for  $x \in G$ , the number of elements  $y$  such that  $x = y^2$ . In particular  $t(1) = t + 1$ , where  $t$  is the number of involutions of  $G$ . We need the

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following two formulae:

$$\nu(\chi) = \frac{\sum_{x \in G} \chi(x^2)}{|G|}, \quad (1)$$

$$t(x) = \sum_{\chi \in \text{Irr}(G)} \nu(\chi) \chi(x). \quad (2)$$

We also need the fact that irreducible characters of third kind have even degree.

Recall that  $G$  is *real*, if all its irreducible characters are real. This is equivalent to all elements being *real*, i.e. conjugate to their inverses. In [CM], D. CHILLAG and the present author considered groups in which all non-linear characters are real. Here we consider a more restricted class.

*Definition.* A group is termed *skew*, if it is non-abelian, and all its non-linear characters have Frobenius–Schur indicator  $-1$ .

Thus in a skew group all non-linear characters are real. Among the linear characters, the real ones are the ones of order 2 (or 1), i.e. the characters of  $G/G^2$ , and they are of the first kind.

**Theorem 1** (W. Willems). *A skew group  $G$  is a 2-group.*

PROOF. This is proved by W. WILLEMS in [W], under the extra hypothesis that  $G$  is real. However, he relies on a paper (his reference [3]) which seems not to have been published. We will indicate here a different argument, which avoids the reality assumption, and also avoids the application of the Feit–Thompson theorem, which was used in [W]. That argument embodies a considerable simplification, suggested by the referee, of my original argument. Let  $G$  be a minimal counter example to the theorem. First, since all non-linear irreducible characters have even degree, a theorem of J. G. THOMPSON [T] shows that  $G$  has a normal 2-complement  $K$ . Let  $T$  be a Sylow 2-subgroup of  $G$ . Then  $T \cong G/K$ , therefore  $G$  and  $T$  have the same number of linear characters of order 2, and all irreducible characters of  $T$  can be considered as characters of  $G$ . Formula (2), for  $x = 1$ , shows that  $T$  has at least as many involutions as  $G$  has. This is possible only if  $G$  and  $T$  have the same number of involutions, and then (2) implies that all non-linear irreducible characters of  $G$

are characters of  $T$ , which means that  $K$  is contained in the kernels of all these characters. But then  $K = 1$ , and  $G$  is a 2-group.  $\square$

A similar argument establishes the following variation:

**Theorem 2.** *Let  $G$  be a finite group, in which all non-linear irreducible characters have even degree, and they are not of the first kind. Then  $G$  is a direct product of a 2-group and an abelian group.*

PROOF. As in the previous proof,  $G$  has a normal 2-complement  $K$ , and if  $T$  is a Sylow 2-subgroup, then  $G$  and  $T$  have the same number of involutions. This means that all involutions of  $G$  lie in  $T$ , and they generate a normal 2-subgroup  $S$  of  $G$ . By induction  $G/S$  is a direct product of a 2-group and an abelian group. We may assume that  $K \neq 1$ . Since  $K \cap S = 1$ , and  $G/K$  is a 2-group,  $G$  is also a direct product of a 2-group and an abelian group.  $\square$

*Examples.* The quaternion group  $\mathbf{Q}$  (of order 8) is skew, while in the dihedral 2-groups  $\mathbf{D}_n$  of order  $2^n$  all characters are of the first kind. More generally, for each  $n$  one of the two extraspecial groups of order  $2^{2n+1}$  is skew, namely the one that is a central product of  $\mathbf{Q}$  and several dihedral groups of order 8. The other extraspecial group has all its characters of the first kind. A direct product of a skew group and an elementary abelian 2-group is skew.

Two further examples, of order 64, will be noted below.

Recall that in a 2-group  $G^2 = \Phi(G)$ , the Frattini subgroup.

**Proposition 3.** *Let  $G$  be a skew group, and write  $|G : G^2| = 2^d$ . Then  $|G| \leq 2^{2d-1}$ . Equality holds only for  $G \cong \mathbf{Q}$ .*

PROOF. Write  $|G : G'| = 2^k$ , recall that  $X$  is the set of non-linear irreducible characters of  $G$ , and let  $A = \sum_{\chi \in X} \chi(1)$ . Then  $t(1) = 2^d - A$ , implying  $A < 2^d$ . Let  $m = \max_{\chi \in X} \chi(1)$ . Then  $m \leq A$ , therefore  $m \leq 2^{d-1}$ . Thus  $|G| = 2^k + \sum_{\chi \in X} \chi(1)^2 \leq 2^k + Am < 2^k + 2^{2d-1}$ . If  $k \geq 2d-1$ , we obtain  $|G| \leq 2^k$ , which is impossible. Thus  $k < 2d-1$  and  $|G| \leq 2^{2d-1}$ .

Suppose that equality holds. Then  $m = 2^{d-1}$ , therefore  $|G : Z(G)| \geq 2^{2d-2}$ , so  $|Z(G)| = 2$ . Moreover, there is only one character of degree  $m$ , and  $A - m < 2^d - 2^{d-1} = 2^{d-1}$ . Thus  $|G| = 2^{2d-1} \leq 2^k + (A - m)2^{d-2} + m^2 <$

$2^k + 2^{2d-3} + 2^{2d-2}$ , which does not hold for  $k \leq 2d - 3$ . Thus  $k = 2d - 2$  i.e.  $G' = Z(G)$ . Since  $|Z(G)| = 2$ , that means that  $G$  is extraspecial, and then its order is  $2^{d+1}$ . Thus  $d = 2$ ,  $|G| = 8$ , and  $G \cong \mathbf{Q}$ .  $\square$

**Proposition 4.** *Let  $G$  be a skew group, and  $1 \neq z \in G^2$ . Then  $t(z) > t(1)$ . In particular,  $z$  is a square.*

This follows immediately from the formula  $t(z) = 2^d - \sum_{\chi \in X} \chi(z)$ . On the other hand, if all non-linear characters are of the first kind, we have  $t(z) < t(1)$ , while if all non-linear characters are of the second kind, then  $t(z)$  is constant on  $G^2$ . The last property actually characterizes 2-groups in which all non-linear characters are of the second kind, by [CM, Proposition 4.1].

We quote some further results from [CM].

**Proposition 5.** *Let  $G$  be a non-real 2-group, in which all non-linear characters are real. Then  $G/G'$  has exponent 4, while all other factors of the lower central series, and also all factors of the upper central series, have exponent 2. Let  $R/G'$  be the subgroup consisting of the elements of order at most 2 in  $G/G'$ . Then  $R$  is the set of real elements of  $G$ , all non-linear characters of  $G$  vanish off  $R$ , and if  $x \notin R$ , then the conjugacy class of  $x$  is the coset  $xG'$ .*

This follows by specializing to 2-groups Theorems 1.3, 1.4, and Proposition 4.9 of [CM].

Note that if  $G$  is a real group, then all factors of either the lower or upper central series have exponent 2.

**Lemma 6.** *A faithful character of a group  $G$  vanishes on  $Z_2(G) - Z(G)$ .*

This is well known. See [I, proof of (2.31)].

**Proposition 7.** *Let  $G$  be a non-abelian 2-group such that each factor group  $H$  of  $G$  satisfies: if  $1 \neq z \in H^2$ , then  $t(z) > t(1)$ . If  $\text{cl}(G) \leq 3$ , then  $G$  is a skew group. If we assume that  $G$  is real, we can relax the inequality to  $t(z) \geq t(1)$ . Dually, if we assume the reverse inequality,  $t(z) < t(1)$ , or that  $G$  is real and  $t(z) \leq t(1)$ , then all non-linear characters of  $G$  are of the first kind.*

PROOF. Let  $\chi$  be a non-linear character of  $G$ . We wish to prove that  $\chi$  is of the third kind. We may assume that  $\chi$  is faithful. Then  $Z(G)$  is cyclic. Suppose that it has order 4 at least. Then the restriction of  $\chi$  to  $Z(G)$  is a multiple of a faithful linear character, and it is not real. If we assume that  $G$  is real, this is a contradiction. If we do not assume reality, then we obtain that all faithful characters are of the second kind. Let  $N$  be the subgroup of order 2 in  $Z(G)$ , and let  $N = \{1, z\}$ . Then  $N$  is the unique minimal normal subgroup of  $G$ , and thus lies in the kernels of all non-faithful characters. Since  $\nu(\chi) = 0$  for the faithful characters, equation (2) shows that  $t(z) = t(1)$ , contradicting our assumptions. Thus  $Z(G) = N$  has order 2. By induction,  $G/N$  is either abelian or a skew group. If it is abelian, then  $N = G' = Z(G)$ , and thus  $G$  is an extraspecial group. Then  $G$  has a unique non-linear irreducible character, which is real, and our claims follow easily by counting involutions. Now assume that  $G/N$  is a skew group. We have  $\chi(z) = -\chi(1)$ . If  $cl(G) = 2$ , then  $\chi$  vanishes off  $Z(G)$ . If  $cl(G) = 3$ , then  $\chi$  vanishes on  $Z_2(G) - Z(G)$ , and in particular on  $G' - Z(G)$ . If  $G$  is real, all squares are in  $G'$ , by the remark following Proposition 5. If  $G$  is not real, let  $K = G/\gamma_3(G)$ . Then  $K$  is a skew group by induction, and so  $\exp(K') = 2$ , by Proposition 5 and the remark following it. Therefore  $K^2 \leq Z(K)$ . That means that in  $G$  the squares are in  $Z_2(G)$ . Thus in either case  $\chi$  vanishes on non-central squares. Thus  $|G|\nu(\chi) = \sum_{x \in G} \chi(x^2) = (t(1) - t(z))\chi(1)$ , and this number is, by assumption, non-positive, and either it is strictly negative, or  $\chi$  is real, so in either case  $\chi$  is of the third kind.  $\square$

A similar proof establishes the dual statement.

As a rule, skewness is not inherited by subgroups, but there are exceptions.

**Proposition 8.** *Let  $G$  be a non-real skew 2-group, and write  $G/G' = K \times L$ , where  $K$  is cyclic of order 4. Let  $M = K^2$ , and write  $M \times L = H/G'$ . Then  $H$  is a skew group. Dually, if all non-linear irreducible characters of  $G$  are of the first kind, the same applies to  $H$ .*

PROOF. Let  $G$  be a non-real skew group, let  $\lambda$  be a character of  $G/G'$  with kernel  $L$ , considered as a character of  $G$ , and let  $\chi$  be a non-linear character of  $G$ . If  $x \in H$ , then  $\lambda(x^2) = 1$ , and if  $x \notin H$ , then  $\lambda(x^2) = -1$ . We have  $\sum \chi(x^2) = -|G| = \sum_{x \notin H} \chi(x^2) + \sum_{x \in H} \chi(x^2) = A + B$ , say.

Similarly  $\sum(\chi\lambda)(x^2) = -|G| = -A + B$ . It follows that  $A = 0$ , and  $\sum_{x \in H} \chi(x^2) = -2|H|$ . Since  $|G : H| = 2$ , the character  $\chi|_H$  is either irreducible or the sum of two irreducible characters of  $H$ , and the above equality shows that the only possibility is that  $\chi|_H$  is the sum of two irreducible characters of the third kind. This shows in particular that  $H$  is not abelian, since abelian groups do not have characters of the third kind. Since each non-linear character of  $H$  occurs in  $\chi|_H$ , for some  $\chi$ , we see that  $H$  is a skew group.  $\square$

The dual statement is proved in the same way. Note that in that case  $H$  may be abelian.

**Proposition 9.** *Let  $G$  be a non-real skew group. Suppose that  $G/G'$  is the direct product of  $r$  cyclic subgroups of order 4 and  $s$  subgroups of order 2. Then  $s \geq r + 2 \geq 3$ , and all non-linear irreducible characters of  $G$  have degree at least  $2^{r+1}$ . If  $H$  is a subgroup of  $G$  such that  $|G : H| \leq 2^r$ , then  $H' = G'$ , and  $G$  contains a real skew subgroup  $S$  of index  $2^r$  such that  $S' = G'$ .*

PROOF. Let  $H$  and  $\chi$  be as in the previous proposition, and let  $\eta$  be one of the irreducible characters of  $H$  that occur in  $\chi|_H$ . Then  $\chi(1) = 2\eta(1)$ , and  $\eta$  is not linear, because it is of the third kind. Thus the claim about the degrees follows by induction on  $r$ , and then all subgroups of small index have derived subgroup  $G'$ , by Theorem 1 of [M]. Also, repeatedly applying the process of passing from  $G$  to  $H$  shows that the subgroup  $S$  consisting of all elements of order 2 (or 1) (*modulo*  $G'$ ) is a skew group satisfying  $S' = G'$ , which has index  $2^r$ .  $S$  is real, because  $\exp(S/S') = 2$ .

Let  $N$  be a normal subgroup of  $G$  which is maximal in  $G'$ , and write  $T = G/N$ . Then  $|T| = 2^{2r+s+1}$ , and the non-linear characters of  $T$  have degree at least  $2^{r+1}$ . Therefore  $|T : Z(T)| \geq 2^{2r+2}$ . On the other hand Proposition 5 shows that  $\exp(T/Z(T)) = 2$ , and therefore  $T^2 \leq Z(T)$  and  $|T : Z(T)| \leq 2^{r+s}$ . Combining the two inequalities yields  $s \geq r + 2$ .  $\square$

**Proposition 10.** *Let  $G$  be a skew group, in which  $|G : G^2| = 2^d$  and  $|G| = 2^{2d-2}$ . Then  $d \leq 4$  and  $|G| \leq 2^6$ . There are three such groups.*

PROOF. We use the notations  $X$ ,  $A$ , and  $k$ , as in the proof of Proposition 3, and recall the inequalities  $A < 2^d$  and  $|G| \leq 2^k + Am$ . Obviously  $m \leq 2^{d-2}$  and  $k \leq 2d - 3$ . If  $m < 2^{d-2}$  we get  $|G| < 2^k + 2^{2d-3} \leq$

$2^{2d-2}$ . Therefore  $m = 2^{d-2}$ . This implies that  $|Z(G)| \leq 4$ . Let  $r$  be the number of irreducible characters of degree  $m$ . Then  $r \leq 3$  and  $|G| < 2^k + (A - rm)2^{d-3} + r \cdot 2^{2d-4}$ .

Let  $r = 1$ . Then the inequality  $|G| = 2^{2d-2} \leq 2^k + (A - m)2^{d-3} + m^2 < 2^k + (2^d - 2^{d-2})2^{d-3} + 2^{2d-4} = 2^k + 2^{2d-3} + 2^{2d-5}$  implies  $k = 2d - 3$ , i.e.  $|G'| = 2$ , and then  $G' \leq Z(G)$ . But  $G$  is not extraspecial, because its order is an even power of 2, and so we have  $|Z(G)| = 4$ , and since  $|G'| = 2$ , the non-central elements of  $G$  have two conjugates each. Writing  $k(G)$  for the class number of  $|G|$ , we obtain  $k(G) = 4 + (2^{2d-2} - 4)/2 = 2^{2d-3} + 2$ . That means that  $G$  has just two non-linear irreducible characters, and writing  $|G| = \sum_{\text{Irr}(G)} \chi(1)^2$  shows that both non-linear characters have the same degree  $2^{d-2}$ , a contradiction.

Now assume that  $r = 2$ . Then the inequality for  $|G|$  becomes  $2^{2d-2} < 2^k + 2^{2d-3} + 2^{2d-4}$ , and this again implies  $k = 2d - 3$ ,  $|G'| = 2$ , and  $|Z(G)| = 4$ . Since  $cl(G) = 2$ , we have  $G^2 \leq Z(G)$ . But  $|G^2| = 2^{d-2}$ , so that  $d - 2 \leq 2$ ,  $d \leq 4$ , and  $|G| \leq 2^6$ .

Finally, let  $r = 3$ . In this case we get that  $k \geq 2d - 4$ . If  $k = 2d - 3$ , then  $k(G)$  is as above, and there are only two non-linear characters, contradicting  $r = 3$ . Thus  $k = 2d - 4$ . Since  $2^k + 3 \cdot 2^{2d-4} = 2^{2d-2}$ , we see that the three characters of degree  $m$  are all the non-linear characters of  $G$ , and  $k(G) = 2^{2d-4} + 3$ . On the other hand, since  $|G'| = |Z(G)| = 4$ , we have  $k(G) \geq 4 + (2^{2d-2} - 4)/4 = 2^{2d-4} + 3$ . But we know already that this inequality is an equality, and that means that each non-central element  $x$  has exactly four conjugates, which are the elements of  $xG'$ . Taking  $x \in Z_2(G)$ , we get  $G' = [x, G] \leq Z(G)$ . Thus again  $cl(G) = 2$ . Since  $\exp(Z(G)) = 2$ , by Proposition 5 and its remark, we have  $G^2 \leq Z(G)$ , yielding  $|G| \leq 2^6$  as in the previous case.

Thus we have either  $d = 3$  or  $d = 4$ . In the first case it is easy to see that the only possibility is  $\mathbf{Q} \times C_2$ . In the second case we have  $|G| = 64$ . Using the information gathered so far in the proof, and also the previous propositions and the HALL–SENIOR tables [HS], one can determine that the only possibilities are the groups numbered 187 and 108 in the tables. Of these the first one is real, the second one not.  $\square$

*Remark.* It is easy to see that among the groups of order 64 at most, the only other skew groups are the direct products of  $\mathbf{Q}$  by two or three

copies of  $C_2$ , one extraspecial group of order 32, and the direct product of the latter group and  $C_2$ .

**Corollary 11.** *Let  $G$  be as in Proposition 9, and assume that  $s = 3$ . Then  $G$  is the group number 108 in the Hall–Senior list.*

PROOF. If  $s = 3$ , then Proposition 9 shows that  $r = 1$ , and thus  $d = 4$ . Since  $G \not\cong \mathbf{Q}$ , Proposition 3 shows that  $|G| \leq 64$ , and the previous proposition, and the remark following it, apply.  $\square$

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