

On the secant method

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Abstract. We apply the Secant method to solve nonlinear operator equations in a Banach space. We assume that the operator has Hölder continuous derivatives. When the operator has a bounded second Fréchet-derivative, our results reduce to the one's obtained by J.E. DENNIS, T.J. YPMA and others.

1. Introduction

Consider an equation

$$(1) \quad F(x) = 0$$

where F is a nonlinear operator between two Banach spaces $E, \hat{E}$. A Newton-like method can be defined as any iterative method of the form

$$(2) \quad x_{n+1} = x_n - L_n^{-1}F(x_n), \quad n = 0, 1, 2, \dots; \quad x_0 \text{ pre-chosen}$$

for generating approximate solutions to (1). The $\{L_n\}$ denotes a sequence of invertible linear operators. This is plainly too general and what is really implicit in the title is that L_n should be a conscious approximation to $F'(x_n)$, since when $L_n = F'(x_n)$, the method reduces to the Newton-Kantorovich method. The convergence of (2) to a solution of (1) has been described already by DENNIS in [4] and the references there. The basic assumption made is that the Fréchet-derivative F' of F is Lipschitz continuous in some ball around the initial iterate. We relax this requirement to operators that are only Hölder (c, p) continuous, $c > 0$, $0 < p \leq 1$. Moreover the Secant method is being examined as in a special case of (2). An error analysis is also provided.

Relevant work has been done by T.J. YPMA [10], [11].

1980 *Mathematics Subject Classification* (1985 *Revision*): 65J15, 65B05, 65L50, 65M50, 47H15, 47H17.

Keywords: Hölder continuity, Fréchet-derivative, Newton's method, Banach space.

Our results can be compared favorably with the ones obtained in [4], [10], [11], [8]. In particular, they reduce to the ones in [4] for $p = 1$.

I. Preliminaries

From now on we assume that F is once Fréchet-differentiable at every point $x \in E$ and note that $F'(x) \in L(E, \hat{E})$, the space of bounded linear operators from E to $\hat{E}$.

Definition 1. We say that the Fréchet-derivative $F'(x)$ is Hölder continuous over a domain D if for some $c > 0$, $p \in [0, 1]$

$$(3) \quad \|F'(x) - F'(y)\| \leq c\|x - y\|^p, \quad \text{for all } x, y \in D.$$

We then say that $F'(\cdot) \in H_D(c, p)$.

Definition 2. Let t_0 and t' be non-negative real numbers and let g be a continuously differentiable real function on $[t_0, t_0 + t']$ and p be a continuously Fréchet-differentiable operator on

$$\bar{U}(x_0, t') = \{x \in E \mid \|x - x_0\| \leq t'\} \subset E$$

into $\hat{E}$. Then the equation

$$t = g(t)$$

will be said to majorize the equation

$$x = P(x) \quad \text{on} \quad U(x_0, t')$$

if

$$\|P(x_0) - x_0\| \leq g(t_0) - t_0$$

and

$$\|P'(x)\| \leq g'(t) \quad \text{for} \quad \|x - x_0\| \leq t - t_0 < t'.$$

We will need the following results whose proofs can be found in [5] and [8] respectively.

Lemma 1. Let $\{x_n\}$, $n = 0, 1, 2, \dots$ be a sequence in E and $\{t_n\}$, $n = 0, 1, 2, \dots$ a sequence of non-negative real numbers such that

$$(4) \quad \|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n = 0, 1, 2, \dots$$

and

$$t_n \rightarrow t^* < \infty \quad \text{as} \quad n \rightarrow \infty.$$

Then there exists a unique point $x^* \in E$ such that

$$x_n \rightarrow x^* \quad \text{as} \quad n \rightarrow \infty$$

and

$$\|x^* - x_n\| \leq t^* - t_n, \quad n = 0, 1, 2, \dots$$

If inequality (4) holds then we say that iteration $\{t_n\}$ majorizes iteration $\{x_n\}$, $n = 0, 1, 2, \dots$.

Lemma 2. Let $F : E \rightarrow E$ and $D \subseteq E$. Assume D is open and that $F'(\cdot)$ exists for every $x \in D$. Let D_0 be a convex set with $D_0 \subseteq D$ such that $F'(\cdot) \in H_{D_0}(c, p)$, then

$$\|F(x) - F(y) - F'(x)(x - y)\| \leq \frac{c}{1+p} \|x - y\|^{p+1} \quad \text{for all } x, y \in D_0.$$

We can now prove a lemma which reduces to lemma 3.5 in [5] for $p = 1$.

Lemma 3. Assume that for any $x, y \in D_0 \subset D$ there is a divided difference operator $\delta F(x, y) \in L(E, \hat{E})$ such that

$$(5) \quad \delta F(x, y)(x - y) = F(x) - F(y)$$

and if $u \in D_0$,

$$(6) \quad \|\delta F(x, y) - \delta F(y, u)\| \leq \ell_1 \|x - u\|^p + \ell_2 \|x - y\|^p + \ell_2 \|y - u\|^p,$$

where $\ell_1, \ell_2 \geq 0$ are independent of x, y and u .

Then the following hold:

- (a) $\delta F(x, x) = F'(x)$, $x \in \text{Int } D_0$; and
- (b) $F'(\cdot) \in H_{D_0}[2(\ell_1 + \ell_2), p]$ for any fixed $p \in (0, 1]$.

PROOF. (a) Let us choose $x \in \text{Int } D_0$ and $\delta > 0$ such that $U(x, \delta) \subset D_0$.

For $\|\Delta x\| < \delta$, we have

$$\begin{aligned} \|F(x + \Delta x) - F(x) - \delta F(x, x)(\Delta x)\| &= \|[\delta F(x + \Delta x, x) - \delta F(x, x)](\Delta x)\| \\ &\leq \|\delta F(x + \Delta x, x) - \delta F(x, x)\| \|\Delta x\| \leq (\ell_1 + \ell_2) \|\Delta x\|^p \|\Delta x\|. \end{aligned}$$

The above inequality proves (a) when $\ell_1 + \ell_2 \neq 0$ and $\|\Delta x\| \rightarrow 0$.

To cover the case when $\ell_1 = \ell_2 = 0$, note that by (6) there is an $L \in L(E, \hat{E})$ such that $\delta F(x, y) = L$ for every $x, y \in D_0$. Therefore, by (5), we can choose δ arbitrarily above and set $F'(x) = L$.

(b) For part (b), let $x, y \in D_0$ then by (6)

$$\begin{aligned} \|F'(x) - F'(y)\| &\leq \|\delta F(x, x) - \delta F(x, y)\| + \|\delta F(x, y) - \delta F(y, y)\| \\ &\leq \ell_1 \|x - y\|^p + \ell_2 \|x - y\|^p + \ell_1 \|x - y\|^p + \ell_2 \|x - y\|^p \\ &= 2(\ell_1 + \ell_2) \|x - y\|^p. \end{aligned}$$

That completes the proof of the lemma.

Note that conditions of the form (6) have been considered in [4, p.444] for $p = 1$. Moreover, conditions (6) can be reduced to the ones considered by T. J. YPMA in [10, p.242] if we choose the divided difference operator δf to be the Fréchet derivative F' of F .

From now on we will assume that $p \in (0, 1)$ and let $D_0 = \bar{U}(x_0, R) \subset D$ for some fixed $x_0 \in D$ and sufficiently small $R > 0$.

II. Main convergence results

We can now prove the following theorem on the convergence of iteration (2) to a locally unique solution x^* of equation (1).

Theorem 1. *Let $F'(\cdot) \in H_{D_0}(c, p)$ and assume that:*

(i) *for every n with $\{x_k\} \subset D_0$, $k = 0, 1, 2, \dots, n$, there exists an invertible operator $L_n \in L(E, \hat{E})$ and positive real numbers $\bar{d}$, d , d_n such that:*

$$(7) \quad \|L_n^{-1}\| \leq d_n^{-1}, \quad \text{with } \bar{d} \leq d_n \leq d.$$

(ii) *For $a, b > 0$, with $b \leq \bar{d}$, both independent of n the following estimate holds:*

$$(8) \quad \|F'(x_n) - L_n\| \leq d_n + ac \left(\sum_{j=1}^n \|x_j - x_{j-1}\| \right)^p - b, \quad n = 0, 1, 2, \dots$$

(the convention $\sum_{j=1}^0 \|x_j - x_{j-1}\| = 0$ is understood).

(iii) *Let us define the real function f and the iteration $\{t_n\}$ by*

$$(9) \quad f(t) = \frac{ca}{p+1} t^{p+1} - bt + d_0 \|L_0^{-1} F(x_0)\|, \quad t \in [0, \infty),$$

$$(10) \quad t_{n+1} = t_n + \frac{f(t_n)}{d_n}; \quad t_0 = 0$$

and assume that the function f has a minimum positive zero r_0 such that

$$(11) \quad 0 < \|x_1 - x_0\| < r_0 < M,$$

where

$$(12) \quad M = \min \left[\left(\frac{b}{ca} \right)^{1/p}, \frac{1}{a(p+1)}, \frac{t_1}{d} (d_0 + ac t_1^p), \left(\frac{\bar{d} + b - d}{c(2^p + a(1+p))} \right)^{1/p} \right]$$

provided that

$$(13) \quad \bar{d} + b - d > 0$$

and

$$(14) \quad \bar{U}(x_0, r_0) \subset D_0.$$

Then

(a) *the sequence $\{t_n\}$ is increasing, bounded above by its limit r_0 and majorizes the sequence $\{x_n\}$ given by (2) that remains in $\bar{U}(x_0, r_0)$ for all $n = 0, 1, 2, \dots$*

(b) The sequence $\{x_n\}$ converges to a unique solution x^* in $\bar{U}(x_0, r_0)$ of equation (1) and

$$(15) \quad \|x_{n+1} - x^*\| \leq r_0 - t_{n+1}; \quad n = 0, 1, 2, \dots$$

PROOF. (a) Using (2), (9), (10), and (11) we have

$$\|x_1 - x_0\| = \|L_0^{-1}F(x_0)\| = t_1 - t_0 = t_1 \leq r_0.$$

That is $x_1 \in \bar{U}(x_0, r_0)$.

Let us assume that:

$$(16) \quad \begin{aligned} &\{x_k\} \subset \bar{U}(x_0, r_0), \\ &\|x_k - x_{k-1}\| \leq t_k - t_{k-1} \quad \text{for } k = 1, 2, \dots, n \end{aligned}$$

and $t_k \leq r_0$. We shall show that (16) holds for $k = n + 1$.

The iterate x_{n+1} is well defined since $F(x_n)$ and L_n^{-1} are. By (2), (7), (8), (9) and (16) we get

$$(17) \quad \begin{aligned} &\|x_{n+1} - x_n\| \leq \|L_n^{-1}\| \cdot \|F(x_n)\| \\ &\leq d_n^{-1} [\|F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})\| \\ &\quad + \|L_{n-1} - F'(x_{n-1})\| \|x_n - x_{n-1}\|] \\ &\leq d_n^{-1} \left[\frac{c}{p+1} \|x_n - x_{n-1}\|^{p+1} + (d_{n-1} + act_{n-1}^p - b) \|x_n - x_{n-1}\| \right] \\ &\leq d_n^{-1} \left[\frac{c}{p+1} (t_n - t_{n-1})^{p+1} + (d_{n-1} + act_{n-1}^p - b)(t_n - t_{n-1}) \right]. \end{aligned}$$

By (17) to show,

$$\|x_{n+1} - x_n\| \leq d_n^{-1} f(t_n) = t_{n+1} - t_n,$$

we must have

$$(18) \quad \begin{aligned} &\frac{c}{p+1} (t_n - t_{n-1})^{p+1} + (d_{n-1} + act_{n-1}^p - b)(t_n - t_{n-1}) \\ &\leq \frac{ca}{p+1} t_n^{p+1} - bt_n + d_0 t_1, \end{aligned}$$

for all $n = 1, 2, \dots$.

It can easily be seen that

$$(t_n - t_{n-1})^{p+1} \leq t_n^{p+1} - t_{n-1}^{p+1}.$$

Therefore, inequality (18) is certainly true for $n \geq 1$ if

$$(19) \quad \begin{aligned} &\frac{c(1-a)}{p+1} t_n^{p+1} + ct_{n-1}^p \left(at_n - \frac{1}{p+1} \right) \\ &+ t_{n-1}(b - d_{n-1}) + \left(d_{n-1}t_n - d_0t_1 - act_{n-1}^{p+1} \right) \leq 0. \end{aligned}$$

By the choice of a , b , $\bar{d}$ and r_0 each one of the parentheses in (19) is nonpositive.

That is, inequality (18) is true for all $n = 1, 2, \dots$.

By the mean value theorem there is some $\xi_n \in (t_n, r_0)$ if $t_n \neq r_0$, such that

$$(20) \quad \begin{aligned} r_0 - t_{n+1} &= g_n(r_0) - g_n(t_n) = g'_n(\xi_n)(r_0 - t_n) \\ &= d_n^{-1}[d_n + ca\xi_n^p - b](r_0 - t_n), \end{aligned}$$

where we have denoted

$$g_n(t) = t + \frac{f(t)}{d_n}.$$

Also by (8),

$$\begin{aligned} 0 &\leq d_n + ca \left(\sum_{j=1}^n \|x_j - x_{j-1}\| \right)^p - b \leq d_n + ca t_n^p - b \\ &< d_n + ca \xi_n^p - b = \frac{(r_0 - t_{n+1})d_n}{r_0 - t_n}. \end{aligned}$$

If $t_n = r_0$, then $t_{n+1} = r_0$. That is

$$(21) \quad t_{n+1} \leq r_0.$$

Furthermore

$$(22) \quad \|x_{n+1} - x_0\| \leq \sum_{j=1}^{n+1} \|x_j - x_{j-1}\| \leq t_{n+1} < r_0.$$

That is $x_{n+1} \in \bar{U}(x_0, r_0)$.

Therefore, the assertions (16) are true for all $n = 1, 2, \dots$.

To complete the proof of (a) we must show that

$$(23) \quad \lim_{n \rightarrow \infty} t_n = r_0.$$

The sequence $\{t_n\}$ is increasing and bounded above by r_0 and as such it converges to some $r^* \leq r_0$. But,

$$0 = \lim_{n \rightarrow \infty} (t_{n+1} - t_n) = \lim_{n \rightarrow \infty} \frac{f(t_n)}{d_n} \geq \lim_{n \rightarrow \infty} \frac{f(t_n)}{d} = \frac{f(r^*)}{d}.$$

But this implies $f(r^*) = 0$, that is $r^* = r_0$.

(b) By part (a) there exists $x^* \in \bar{U}(x_0, r_0)$ such that $x^* = \lim_{n \rightarrow \infty} x_n$ and inequality (15) holds. We must show that x^* is a root of F .

But iteration (2) gives

$$(24) \quad \begin{aligned} \|F(x_n)\| &\leq \|L_n\| \cdot \|x_{n+1} - x_n\| \\ \text{and } \|x_{n+1} - x_n\| &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore it suffices to show that the sequence $\|L_n\|$, $n = 0, 1, 2, \dots$ is uniformly bounded. This follows readily from (8) and (23) since

$$\begin{aligned} \|L_n\| &\leq \|F'(x_n)\| + d_n - b + ac \left(\sum_{j=1}^n \|x_j - x_{j-1}\| \right)^p \\ &\leq \|F'(x_0)\| + c\|x_n - x_0\|^p + d_n - b + acr_0^p \\ &\leq \|F'(x_0)\| + (c + ac)r_0^p - b + d \equiv B. \end{aligned}$$

Therefore the inequality (24) gives

$$\|F(x_n)\| \leq B\|x_n - x_{n+1}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

which implies that $F(x^*) = 0$.

To show uniqueness let us assume that there exists a second solution $z^* \in \bar{U}(x_0, r_0)$. Then from the identity

$$\begin{aligned} &x_{n+1} - z^* \\ &= L_n^{-1} [(L_n - F'(x_n))(x_n - z^*) + (F(z^*) - F(x_n) - F'(x_n)(z^* - x_n))] \end{aligned}$$

we obtain using (8) and lemma 2

$$\begin{aligned} \|x_{n+1} - z^*\| &\leq d_n^{-1} \left[(d_n + acr_0^p - b) + \frac{c}{p+1} \|x_n - z^*\|^p \right] \|x_n - z^*\| \\ &\leq \bar{d}^{-1} \left[car_0^p + \frac{c}{p+1} (2r_0)^p + (d - b) \right] \|x_n - z^*\| \\ &= A\|x_n - z^*\| \leq \dots \leq A^{n+1}\|x_0 - z^*\| \leq A^{n+1}r_0, \end{aligned}$$

where we have denoted $A = A(r_0) = [cr_0^p(a + \frac{2^p}{p+1}) + d - b]\bar{d}^{-1}$. By the choice of r_0 , $\lim_{n \rightarrow \infty} A^n = 0$. Therefore $x^* = \lim_{n \rightarrow \infty} x_n = z^*$.

That completes the proof of the theorem.

We now state and prove a proposition that will enable us to show the uniqueness of x^* in a larger ball.

Proposition 1. *Let $F'(\cdot) \in H_{D_0}(c, p)$, $D_0 \subset D$.*

Assume:

(i) Inequality (8) holds for $n = 0$; and

(ii) the function $\bar{f}(t)$ defined by

$$\bar{f}(t) = \frac{c}{p+1}t^{p+1} + (\delta^1 - 1)d_0t + d_0\|L_0^{-1}F(x_0)\|,$$

$$\text{with } \delta^1 \equiv \frac{\|F'(x_0) - L_0\|}{d_0}, \quad t \in [0, \infty)$$

has two zeros r'_0 and r'_1 , $r'_0 < r'_1$ such that $U(x_0, r'_1) \subset D_0$.

Then, equation (1) has a unique solution x^* in $\bar{U}(x_0, r'_1)$.

Moreover:

(a) Iteration $x'_{n+1} = x'_n - L_0^{-1}F(x'_n)$ converges to x^* for $\|x'_0 - x_0\| < r_2 \leq r'_1$ and $U(x_0, r_2) \subset D_0$.

(b) The following estimate is true:

$$\|x'_n - x^*\| \leq |r'_0 - t'_n|$$

where $\{t'_n\}$ is generated by $t'_{n+1} = t'_n + \frac{\bar{f}(t'_n)}{d_0}$.

PROOF. Let us first note that inequality (8) for $n = 0$ gives $0 < b/d_0 \leq 1 - \delta^1$. That is $\delta^1 < 1$. Define the nonlinear operator P on D_0 by

$$P(x) = x - L_0^{-1}F(x).$$

We will show that if $t' \in [r'_0, r'_1)$, then $g(t) = t + \bar{f}(t)/d_0$ majorizes $P(x)$ on $\bar{U}(x_0, t') \subset D_0$.

We have

$$\|P(x_0) - x_0\| = \|L_0^{-1}F(x_0)\| = g(0) - 0.$$

Let x, t be such that $x \in \bar{U}(x_0, t') \cap D_0$ and $\|x - x_0\| \leq t < t'$. Then

$$\begin{aligned} \|P'(x)\| &= \|I - L_0^{-1}F'(x)\| = \|L_0^{-1}((L_0 - F'(x_0)) + (F'(x_0) - F'(x)))\| \\ &\leq \|L_0^{-1}\|(\|F'(x) - F'(x_0)\| + \|F'(x_0) - L_0\|) \leq \delta^1 + c\frac{t^p}{d_0} = g'(t). \end{aligned}$$

By hypothesis r'_0 is the unique fixed point of $g(t)$ in $[0, t']$ and $g(t') \leq t'$ with equality holding if and only if $t' = r'_0$.

The results now follows from the well known classical theorem on the existence and uniqueness of solutions of equation (1) via majorizing sequences given in KANTOROVICH [[5], page 697].

The following Corollary is an immediate consequence of Theorem 1 and Proposition 1.

Corollary. Let $F'(\cdot) \in H_{D_0}(c, p)$, $D_0 \subset D$. Assume that the hypotheses of Theorem 1 and Proposition 1 are satisfied. Then equation (1) has a unique solution x^* in $D_0 \cap \bar{U}(x_0, r'_1)$ and the iteration $\{x_n\}$ given by (2), $n = 0, 1, 2, \dots$ converges to x^* with

$$\|x_{n+1} - x^*\| \leq r_0 - t_{n+1}, \quad n = 0, 1, 2, \dots$$

We will now study the convergence of the following secant iterations as special cases of iteration (2):

$$(25) \quad x_{n+1} = x_n - \delta F(x_n, x_{n-1})^{-1} F(x_n)$$

or

$$(26) \quad x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1} F(x_n)$$

where x_0, x_{-1} are given.

We can prove a theorem concerning the convergence of iteration (26) to a locally unique solution x^* of equation (1). A similar theorem can be proved for iteration (25).

Theorem 2. Under the assumptions of Lemma 3, $F'(\cdot) \in H_{D_0}(c_1, p)$ with $c_1 = 2(\ell_1 + \ell_2)$. Let $D_0 \subset D$ with $x_{-1}, x_0 \in \text{int} D_0$. Assume:

(i) the linear operator $L_0 = \delta F(x_{-1}, x_0)$ is invertible and

$$\|L_0^{-1}\| \leq \beta; \quad \|x_{-1} - x_0\| \leq \eta_{-1}; \quad \|L_0^{-1} F(x_0)\| \leq \eta.$$

(ii) Let us define the real function f_1 and the iteration $\{s_n\}$ by

$$f_1(s) = \frac{c_1 a_1}{p+1} s^{p+1} - b_1 s + \bar{d}_0 \|L_0^{-1} F(x_0)\|, \quad s \in [0, \infty),$$

$$s_{n+1} = s_n + \frac{f_1(s_n)}{\bar{d}_n}, \quad s_0 = 0, \quad n = 0, 1, 2, \dots,$$

where we have denoted

$$a_1 = \frac{3(p+1)}{4}, \quad b_1 = \frac{1 - \beta(\ell_1 + \ell_2)\eta_{-1}^p}{2\beta}, \quad \bar{d}_0 = \beta^{-1},$$

$$\bar{d}_n = \beta^{-1} [1 - \beta(\ell_1 + \ell_2) \|x_n - x_{n-1}\|^p - c_1 \beta \|x_n - x_0\|^p - \beta(\ell_1 + \ell_2) \eta_{-1}^p],$$

$$n = 1, 2, \dots$$

The function f_1 has a minimum positive zero $\bar{r}_0$ such that

$$(27) \quad 0 < \max(\eta_{-1}, \eta) < \bar{r}_0 < \bar{M} =$$

$$= \min \left(\left(\frac{2b_1}{3c_1} \right)^{1/p}, M, \left[\frac{1 - 3\ell_0(\ell_1 + \ell_2)\eta_{-1}^p}{2\beta(\ell_1 + \ell_2 + c_1)} \right] \right),$$

$$\bar{U}(x_0, \bar{r}_0) \subset D_0,$$

provided that $1 - 3\beta(\ell_1 + \ell_2)\eta_{-1}^p > 0$ where M is as defined in (12) with

$$\begin{aligned}\bar{d} &= \beta^{-1}[1 - \beta(\ell_1 + \ell_2 + c_1)\bar{r}_0^p - \beta(\ell_1 + \ell_2)\eta_{-1}^p], \\ d &= \beta^{-1}, \quad c = c_1, \quad a = a_1, \quad b = b_1, \quad \text{and } t_1 = s_1.\end{aligned}$$

Then

(a) the sequence $\{s_n\}$ is increasing, bounded above by its limit $\bar{r}_0$ and majorizes the sequence $\{x_n\}$ given by (26) that remains in $\bar{U}(x_0, \bar{r}_0)$ for all $n = 0, 1, 2, \dots$.

(b) The sequence $\{x_n\}$ given by (26) converges to a unique solution x^* in $\bar{U}(x_0, \bar{r}_0)$ of equation (1) with

$$\|x_{n+1} - x^*\| \leq \bar{r}_0 - s_{n+1}, \quad n = 0, 1, 2, \dots$$

PROOF. The proof will be accomplished by finding the analog of Theorem 1 with $L_n = \delta F(x_{n-1}, x_n)$.

By hypothesis

$$\|x_{-1} - x_0\| < \bar{r}_0 \quad \text{and} \quad \|x_0 - x_1\| < \bar{r}_0$$

and

$$\|F'(x_0) - L_0\| \leq \bar{d}_0 - b_1.$$

Let us assume

$$\sum_{j=1}^k \|x_j - x_{j-1}\| \leq s_k < \bar{r}_0,$$

the linear operators L_k are invertible and (8) holds for all $k = 1, 2, \dots, n-1$ with $c = c_1$, $a = a_1$, $b = b_1$ and $d_k = \bar{d}_k$.

As in Theorem 1 we can show that

$$\sum_{j=1}^n \|x_j - x_{j-1}\| \leq s_n < \bar{r}_0.$$

Using (6) we can easily obtain

$$(28) \quad \|L_k - F'(x_k)\| \leq (\ell_1 + \ell_2)\|x_{k-1} - x_k\|^p, \quad k = 0, 1, 2, \dots, n.$$

Using (3) and (28) we get

$$\begin{aligned}\|L_n - L_0\| &\leq \|L_n - F'(x_n)\| + \|F'(x_n) - F'(x_0)\| + \|F'(x_0) - L_0\| \\ &\leq (\ell_1 + \ell_2)\|x_n - x_{n-1}\|^p + c_1\|x_n - x_0\|^p + (\ell_1 + \ell_2)\eta_{-1}^p.\end{aligned}$$

Then

$$\|L_0^{-1}L_n - I\| \leq \|L_0^{-1}\|\|L_n - L_0\| \leq \beta[(\ell_1 + \ell_2 + c_1)\bar{r}_0^p + (\ell_1 + \ell_2)\eta_{-1}^p] < 1,$$

which is true since $\bar{r}_0$ is a minimum positive zero of the equation $f_1(s) = 0$. That is the linear operator L_n is invertible and

$$\|L_n^{-1}\| \leq \bar{d}_n^{-1}.$$

Moreover, inequality (8) holds for $k = n$ by (27), (28) and the estimate $\bar{d}_n \geq b_1$.

It can easily be seen by the choice of $\bar{r}_0$ that

$$\bar{d} \leq \bar{d}_n \leq d, \quad n = 0, 1, 2, \dots, \quad b_1 \leq \bar{d} \quad \text{and} \quad \bar{d} + b_1 \geq d.$$

The hypotheses (i), (ii) and (iii) of Theorem 1 are now satisfied. Therefore the results follow immediately from Theorem 1.

Note that the above theorem gives us a way of choosing linear operators L_n , $n \geq 0$ in such a way that condition (8) is satisfied.

III. Error analysis and applications

Here we look at iteration (26) in a way different than before which enables us to find the order of convergence of (26) to a solution x^* of (1).

Proposition 2. *Under the hypotheses of Theorem 2 the solution x^* of equation (1) obtained via iteration (26) is such that*

$$\|x_{n+1} - x^*\| \leq \gamma_1 \|x_n - x^*\| (\|x_n - x^*\| + \|x_{n-1} - x^*\|)^p + \gamma_2 \|x_n - x^*\|^{p+1},$$

$$n = 0, 1, 2, \dots$$

where,

$$\gamma_1(n) = \gamma_1 = \bar{d}_n^{-1}(\ell_1 + \ell_2) \quad \text{and} \quad \gamma_2(n) = \gamma_2 = \frac{\bar{d}_n^{-1}c_1}{1+p}.$$

PROOF. Using (26) we have

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - \delta F(x_{n-1}, x_n)^{-1} F(x_n) \\ &= \delta F(x_{n-1}, x_n)^{-1} [(\delta F(x_{n-1}, x_n) - F'(x_n) + F'(x_n))(x_n - x^*) - F(x_n)] \\ &= \delta F(x_{n-1}, x_n)^{-1} [(\delta F(x_{n-1}, x_n) - F'(x_n)(x_n - x^*)) \\ &\quad + (F'(x_n)(x_n - x^*) - F(x_n) + F(x^*))]. \end{aligned}$$

By taking norms above we obtain

$$\|x_{n+1} - x^*\| \leq \bar{d}_n^{-1} \left[(\ell_1 \|x_n - x_{n-1}\|^p + \ell_2 \|x_n - x_{n-1}\|^p) \|x_n - x^*\| \right]$$

$$\begin{aligned}
+\frac{c_1}{1+p}\|x_n - x^*\|^{p+1} \Big] &\leq \bar{d}_n^{-1} \left[(\ell_1 + \ell_2) (\|x_n - x^*\| + \|x_{n-1} - x^*\|)^p \right. \\
&\quad \left. \times \|x_n - x^*\| + \frac{c_1}{1+p}\|x_n - x^*\|^{p+1} \right].
\end{aligned}$$

The result now follows from the above inequality.

We now give two examples as possible applications of the theory introduced above for finding solutions x^* of (1), for illustrational purposes. The motivated reader can fill the computational details.

Example 1. Consider the function G defined on $[0, b]$ by

$$G(t) = At^{1+\bar{p}} + Bt$$

where, $A, B \in \mathbb{R}$, $\bar{p} \in [0, 1]$ and $b > 0$.

Let $\|\cdot\|$ denote the max norm on $\mathbb{R}$, then

$$\|G''(t)\| = \max_{t \in [0, b]} |A(1 + \bar{p})\bar{p}t^{\bar{p}-1}| = \infty,$$

which implies that the Newton-Kantorovich hypotheses are not satisfied [4].

However, it can easily be seen that $G'(t)$ is Hölder continuous on $[0, b]$ with

$$c = A(1 + \bar{p}) \text{ and } p = \bar{p}.$$

Therefore, under the assumptions of theorem 2, iteration (26) can be used to find a solution t^* of the equation $G(t) = 0$.

We can further apply our results by modifying an example considered also by ROKNE [9].

Example 2. Consider the differential equation

$$\begin{aligned}
(29) \quad &y'' + y^{1+p} = 0, \quad p \in (0, 1) \\
&y(0) = y(1) = 0.
\end{aligned}$$

We divide the interval $[0, 1]$ into n subintervals and we set $h = 1/n$. Let $\{v_k\}$ be the points of subdivision with

$$0 = v_0 < v_1 < \cdots < v_n = 1.$$

A standard approximation for the second derivative is given by

$$y''_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad y_i = y(v_i), \quad i = 1, 2, \dots, n-1.$$

Take $y_0 = y_n = 0$ and define the operator $F : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}^{n-1}$ by

$$(30) \quad F(y) = H(y) + h^2\varphi(y),$$

$$H = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \varphi(y) = \begin{bmatrix} y_1^{1+p} \\ y_2^{1+p} \\ \vdots \\ y_{n-1}^{1+p} \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Then

$$F'(y) = H + h^2(p+1) \begin{bmatrix} y_1^p & & & \\ & y_2^p & & \\ & & \ddots & \\ & & & y_{n-1}^p \end{bmatrix}.$$

The Newton-Kantorovich hypotheses for the solution of the equation

$$(31) \quad F(y) = 0$$

may not be satisfied. We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form y_i^{p-1} and they may not exist.

The secant hypotheses [4, p. 445] for $p \neq 1$ are not satisfied.

Let $y \in \mathbb{R}^{n-1}$, $H \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and define the norms of y and H by

$$\|y\| = \max_{1 \leq j \leq n-1} |y_j|, \quad \|M\| = \max_{1 \leq j \leq n-1} \sum_{k=1}^{n-1} |m_{jk}|.$$

For all $y, z \in \mathbb{R}^{n-1}$ for which $|y_i| > 0$, $|z_i| > 0$, $i = 1, 2, \dots, n-1$ we obtain for $p = \frac{1}{2}$, say

$$\begin{aligned} \|F'(y) - F'(z)\| &= \left\| \text{diag} \left\{ \left(1 + \frac{1}{2} \right) h^2 \left(y_j^{1/2} - z_j^{1/2} \right) \right\} \right\| \\ &= \frac{3}{2} h^2 \max_{1 \leq j \leq n-1} \left| y_j^{1/2} - z_j^{1/2} \right| \leq \frac{3}{2} h^2 [\max |y_j - z_j|]^{1/2} = \frac{3}{2} h^2 \|y - z\|^p. \end{aligned}$$

Therefore under the assumptions of theorem 2, iteration (26) can be used to find solutions y^* of (31) as follows:

A linear operator $L \in L(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ can be represented by a matrix with entries q_{ij} and

$$\|L\| = \max \left\{ \sum_{j=1}^{n-1} |q_{ij}| : 1 \leq i \leq n-1 \right\}.$$

Let F be an operator defined on $\mathbb{R}^{n-1}$ with values in $\mathbb{R}^{n-1}$. Let us denote by $F_1, \dots, F_{n-1}$ the components of F . For each $v \in \mathbb{R}^{n-1}$ we can write

$$F(v) = (F_1(v), \dots, f_{n-1}(v))^{tr}.$$

Let $v, w \in \mathbb{R}^{n-1}$ and define $\delta F(v, w)$ by the matrix with entries

$$(32) \quad \delta F(v, w)_{ij} = \frac{1}{v_j - w_j} (F_i(v_1, \dots, v_j, w_{j+1}, \dots, w_m) - F_i(v_1, \dots, v_{j-1}, w_j, \dots, w_m)), \quad m = n - 1.$$

It can easily be seen that the operator defined by (32) satisfies (5) and $\delta F(v, w) \in L(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$.

Denote by

$$P_j F_i(v) = \frac{\partial F_i(v)}{\partial v_j}, \quad i, j = 1, 2, \dots, n - 1.$$

We can choose $n = 10$ which gives (9) equations for iteration (26), if we look at it as a system of linear equations given $z_{-1}, z_0 \in \mathbb{R}^9$. As in [9], since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin x$. This gives us the following vector

$$z_{-1} = \begin{bmatrix} 4.015241E + 01 \\ 7.637852E + 01 \\ 1.051351E + 02 \\ 1.236112E + 02 \\ 1.299991E + 02 \\ 1.236752E + 02 \\ 1.052571E + 02 \\ 7.654622E + 01 \\ 4.034951E + 01 \end{bmatrix}$$

Choose z_0 by setting

$$z_0(v_i) = z_{-1}(v_i) - 10^{-5}, \quad i = 1, 2, \dots, 9.$$

Using iteration (26) with the above values and (32), after seven iter-

ations we get

$$z_6 = \begin{bmatrix} 3.357455E+01 \\ 6.520294E+01 \\ 9.156631E+01 \\ 1.091680E+02 \\ 1.153630E+02 \\ 1.091680E+02 \\ 9.156663E+02 \\ 6.520294E+01 \\ 3.357455E+01 \end{bmatrix} \quad \text{and} \quad z_7 = \begin{bmatrix} 3.357450E+01 \\ 6.520290E+01 \\ 9.156660E+01 \\ 1.091680E+02 \\ 1.536301E+02 \\ 1.091680E+02 \\ 9.156660E+02 \\ 6.520290E+01 \\ 3.357450E+01 \end{bmatrix}.$$

We choose $z_6 = x_{-1}$ and $z_7 = x_0$ for our Theorem 2. From now on we assume that F is restricted on $\bar{U}(x_0, .1)$. With the notation of Theorem 2 we can easily obtain the following results: $\beta \leq 25.5882$, $\eta_{-1} \leq 5E-05$, $\ell_1 = \ell_2 = .03$, $c_1 = .12$, $a_1 = 1.125$, $s_1 = t_1 = \eta \leq 9.15311E-05$, $d_0 = d = .039080513$, $b_1 = .01932812$ and

$$f_1(s) = 9E-02 s^{3/2} - .01932812 s + 3.577082405E-06 = 0.$$

The above equation has a minimum positive solution R such that

$$\bar{r}_0 \doteq R = 9.18E-05 \quad \text{and} \quad |R - \bar{r}_0| \leq 5E-06.$$

With the above values and using (27) we get

$$\begin{aligned} \bar{d} &= .03693415, \\ \max(\eta - 1, \eta) &= 9.15311E-05, \end{aligned}$$

and

$$\bar{M} = 9.45561085E-05.$$

All the hypotheses of Theorem 2 are now satisfied with the above values.

Therefore, the iteration generated by (26) converges to a unique solution x^* in $\bar{U}(x_0, R)$ of equation (31).

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(Received September 15, 1991)