

The extensibility of $D(-1)$ -triples $\{1, b, c\}$

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Abstract. Let $b = 5, 10, 17, 26, 37$ or 50 . In this paper, we show that for those integers c greater than 1 such that both $c - 1$ and $bc - 1$ are squares, the system of simultaneous Pell equations

$$z^2 - cx^2 = c - 1, \quad bz^2 - cy^2 = c - b$$

has only the trivial solutions $(x, y, z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$. This implies that there do not exist integers $c, d (> 1)$ such that the product of any two distinct elements of the set $\{1, b, c, d\}$ diminished by 1 is a square. We also show that in case b is a positive integer with $\sqrt{b-1}$ a prime, if it is true for the smallest eight c 's with $c > 1$ for which the set $\{1, b, c\}$ has the property above, then the same is true for all such c 's.

1. Introduction

Diophantus raised the problem of finding four (positive rational) numbers a_1, a_2, a_3, a_4 such that $a_i a_j + 1$ is a square for each $1 \leq i < j \leq 4$ and gave a solution $\{1/16, 33/16, 68/16, 105/16\}$. The first set of four positive integers $\{1, 3, 8, 120\}$ with the property above was found by Fermat. Replacing “+1” by “+ n ” leads to the following definition.

Definition 1.1. Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \dots, a_m\}$ is called a $D(n)$ - m -tuple (or a Diophantine m -tuple with the property $D(n)$, or a P_n -set of size m) if $a_i a_j + n$ is a square for each $1 \leq i < j \leq m$.

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In 1993, DUJELLA showed that if $n \not\equiv 2 \pmod{4}$ and $n \notin S := \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists a $D(n)$ -quadruple ([7]), and conjectured that if $n \in S$, then there does not exist a $D(n)$ -quadruple ([8]). In the case of $n = -1$, there are several results supporting the validity of this conjecture. It is known that the following $D(-1)$ -triples can not be extended to $D(-1)$ -quadruples:

$\{1, 5, 10\}$ (by MOHANTY and RAMASAMY [14]); $\{1, 2, 5\}$ (by BROWN [5]);
 $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, $\{1, 26, 37\}$
 (by KEDLAYA [13]); $\{1, 5, 422\}$ (by ABU MURIEFAH and AL-RASHED [2]).

DUJELLA, in 1998, proved that the pair $\{1, 2\}$ can not be extended to a $D(-1)$ -quadruple ([9, Corollary 1]), and recently DUJELLA and FUCHS showed the following.

Theorem 1.2 ([10, Theorem 1b]). *There does not exist a $D(-1)$ -quadruple $\{a, b, c, d\}$ with $2 \leq a < b < c < d$.*

Thus, it is enough to examine the extensibility of $D(-1)$ -triples $\{1, b, c\}$ with $b \geq 5$. In this paper, we show the following.

Theorem 1.3. *There does not exist a $D(-1)$ -quadruple $\{1, b, c, d\}$ with $b = 5, 10, 17, 26, 37$ or 50 .*

Our proof of this theorem goes along the lines of that of [9, Corollary 1] or of Theorem 1.2. For $b = 5$, ABU MURIEFAH and AL-RASHED have already had a partial result on the non-extensibility of the $D(-1)$ -triples $\{1, 5, c\}$ ([1, Theorem 2.1]). We also obtained a similar but stronger result on the non-extensibility of $D(-1)$ -triples $\{1, b, c\}$ with $\sqrt{b-1}$ a prime (or with $b = 17$ or 37) (see Corollary 3.10), together with which applying the reduction method due to DUJELLA and PETHŐ ([11]) to each case of $b = 5, 10, 17, 26, 37$ and 50 completes the proof of Theorem 1.3.

Remark 1.4. Dujella told us that FILIPIN [12] filled a gap in [1] to show the non-extensibility of the triples $\{1, 5, c\}$, and he further showed that of the triples $\{1, 10, c\}$; FILIPIN sent us the manuscript [12]. We would like to thank them.

2. The simultaneous Pell equations

Let $\{1, b, c\}$ be a $D(-1)$ -triple. Then there exist positive integers r, s, t such that $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$, and we have

$$t^2 - bs^2 = r^2. \quad (1)$$

Let (t_k, s_k) ($s_k < s_{k+1}$, $k = 0, 1, 2, \dots$) denote the positive solutions of (1). Then there exists an integer k such that $c = c_k := s_k^2 + 1$. Suppose $\{1, b, c, d\}$ is a $D(-1)$ -quadruple. Then there exist integers x, y, z such that

$$d - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2.$$

Eliminating d , we obtain the simultaneous Pell equations

$$\begin{cases} z^2 - cx^2 = c - 1, \\ bz^2 - cy^2 = c - b. \end{cases} \quad (2)$$

$$(3)$$

Thus, Theorem 1.3 is a corollary of the following.

Theorem 2.1. *Let $b = 5, 10, 17, 26, 37$ or 50 and $c = c_k$ as above. Then the simultaneous Pell equations (2) and (3) have only the trivial solutions $(x, y, z) = (0, \pm\sqrt{b-1}, \pm\sqrt{c-1})$.*

We will prove Theorem 2.1 in the forthcoming sections.

3. An upper bound for c

Throughout this section, let $b \geq 5$ and assume that $c = c_k$ is minimal for which the equations (2) and (3) have a nontrivial solution.

By Lemma 1 in [10], the positive solutions (z, x) of (2) and (z, y) of (3) are respectively given by

$$z + x\sqrt{c} = (z_0^{(i)} + x_0^{(i)}\sqrt{c})(s + \sqrt{c})^{2m} \quad (i = 1, \dots, i_0, m \geq 0), \quad (4)$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^{2n} \quad (j = 1, \dots, j_0, n \geq 0), \quad (5)$$

where

$$0 < z_0^{(i)} < c, \quad 0 < z_1^{(j)} < c \quad (6)$$

for all i and j . (We call $(z_0^{(i)}, x_0^{(i)})$ ($1 \leq i \leq i_0$) and $(z_1^{(j)}, y_1^{(j)})$ ($1 \leq j \leq j_0$) the fundamental solutions of (2) and (3), respectively.) By (4), there exist i and m such that

$$z = v_m^{(i)}, \quad \text{where} \quad (7)$$

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = (2c-1)z_0^{(i)} + 2sx_0^{(i)}, \quad v_{m+2}^{(i)} = 2(2c-1)v_{m+1}^{(i)} - v_m^{(i)},$$

and by (5), there exist j and n such that

$$z = w_n^{(j)}, \quad \text{where} \quad (8)$$

$$w_0^{(j)} = z_1^{(j)}, \quad w_1^{(j)} = (2bc - 1)z_1^{(j)} + 2tcy_1^{(j)}, \quad w_{n+2}^{(j)} = 2(2bc - 1)w_{n+1}^{(j)} - w_n^{(j)}.$$

By induction we have

$$v_m^{(i)} \equiv (-1)^m z_0^{(i)} \pmod{2c}, \quad w_n^{(j)} \equiv (-1)^n z_1^{(j)} \pmod{2c}. \quad (9)$$

Further, the sequences $\{v_m^{(i)}\}$ and $\{w_n^{(j)}\}$ satisfy the following relations.

Lemma 3.1 (cf. [10, Lemma 2]).

$$v_m^{(i)} \equiv (-1)^m (z_0^{(i)} - 2cm^2 z_0^{(i)} - 2csmx_0^{(i)}) \pmod{8c^2},$$

$$w_n^{(j)} \equiv (-1)^n (z_1^{(j)} - 2bcn^2 z_1^{(j)} - 2ctny_1^{(j)}) \pmod{8c^2}.$$

In what follows, we will assume $v_m^{(i)} = w_n^{(j)}$ (and omit the superscripts (i) and (j)). Putting $d_0 := (z_0^2 + 1)/c$, we see from (6) that $d_0 \leq c$ and from (2) and (3) that

$$d_0 - 1 = x_0^2, \quad bd_0 - 1 = y_1^2, \quad cd_0 - 1 = z_0^2$$

(see the proof of Lemma 3 in [10]). Since $d_0 = x_0^2 + 1$ is a positive integer, the minimality of $c = c_k$ implies the following.

Lemma 3.2 (cf. [10, Lemma 4]). $z_0 = z_1 = \sqrt{c-1} (= s)$ and $x_0 = 0$, $y_1 = \pm\sqrt{b-1} (= \pm r)$.

Lemma 3.3. (i) $m \equiv n \pmod{2}$.

(ii) $n \leq m \leq 2n$.

(iii) If $n \neq 0$ and $b < \sqrt{c}$, then we have $\sqrt[4]{c}/\sqrt{b-1} < n$.

PROOF. (i) It is obvious from (9).

(ii) It is exactly Lemma 6 in [10].

(iii) By (i), Lemmas 3.1 and 3.2, we have

$$s - 2cm^2 s \equiv s - 2bcn^2 s \pm 2\sqrt{b-1}ctn \pmod{8c^2},$$

which implies

$$s(m^2 - bn^2) \equiv \pm\sqrt{b-1}tn \pmod{4c}. \quad (10)$$

Since $s = \sqrt{c-1}$ and $t = \sqrt{bc-1}$, we have

$$(c-1)(m^2 - bn^2)^2 \equiv (b-1)(bc-1)n^2 \pmod{4c},$$

which implies

$$(m^2 - bn^2)^2 \equiv (b-1)n^2 \pmod{c}. \quad (11)$$

Now, suppose $n \leq \sqrt[4]{c}/\sqrt{b-1}$. By (ii), we have

$$|s(m^2 - bn^2)| \leq \sqrt{c-1}(b-1)n^2 < c, \quad (m^2 - bn^2)^2 \leq (b-1)^2 n^4 \leq c.$$

On the other hand, by the assumption $b < \sqrt{c}$ we know that

$$\sqrt{b-1}tn \leq \sqrt{bc-1}\sqrt[4]{c} < c, \quad (b-1)n^2 \leq \sqrt{c} < c.$$

It follows from (10) and (11) that

$$s(m^2 - bn^2) = -\sqrt{b-1}tn, \quad (m^2 - bn^2)^2 = (b-1)n^2.$$

Hence we have

$$s^2(m^2 - bn^2)^2 = (b-1)t^2n^2 = t^2(m^2 - bn^2)^2,$$

which together with $n \neq 0$ implies $t^2 = s^2$, which is a contradiction. $\square$

From this lemma, it is easy to see the following.

Proposition 3.4. *Let x, y, z be positive integer solutions of the simultaneous Pell equations (2) and (3). If $b < \sqrt{c}$, then we have*

$$\left(\frac{\sqrt[4]{c}}{\sqrt{b-1}} - 1 \right) \log(4c-3) < \log y.$$

PROOF. Let $z = v_m = w_n$. $x > 0$ implies $m > 0$. It follows from (4) and Lemma 3.2 that

$$x = \frac{s}{2\sqrt{c}} \{ (s + \sqrt{c})^{2m} - (s - \sqrt{c})^{2m} \},$$

which together with $y^2 - bx^2 = b-1 > 0$ implies

$$\begin{aligned} y &> x\sqrt{b} = \frac{s\sqrt{b}}{2\sqrt{c}} \{ (s + \sqrt{c})^{2m} - (s - \sqrt{c})^{2m} \} \\ &> (s + \sqrt{c})^{2(m-1)} \{ (s + \sqrt{c})^2 - (s - \sqrt{c})^2 \} \\ &> (s + \sqrt{c})^{2(m-1)} > (4c-3)^{m-1}. \end{aligned}$$

Hence the proposition follows from Lemma 3.3 (iii). $\square$

In order to get an upper bound for $\log y$, we need the following theorem, which is a slightly modified version of Rickert's theorem (or of a special case of Bennett's theorem).

Theorem 3.5 (cf. [5, Theorem 3.2], [15, Theorem] or [17, Theorem]). *Let b and N be integers with $b \geq 5$ and $N \geq 2.39b^7$. Then the numbers*

$$\theta_1 := \sqrt{1 + \frac{1-b}{N}} \quad \text{and} \quad \theta_2 := \sqrt{1 + \frac{1}{N}}$$

satisfy

$$\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > \left\{ 32.1 \frac{b^2(b-1)^2}{2b-1} N \right\}^{-1} q^{-1-\lambda} \quad (12)$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda := \frac{\log \frac{16.1b^2(b-1)^2 N}{2b-1}}{\log \frac{3.37N^2}{b^2(b-1)^2}} < 1.$$

PROOF. Note that the condition $N \geq 2.39b^7$ implies $\lambda < 1$. All we have to do is find those real numbers satisfying the assumption in the following lemma.

Lemma 3.6 (cf. [5, Lemma 3.1], [15, Lemma 2.1]). *Let $\theta_1, \dots, \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer k , we can find rational numbers p_{ijk} ($0 \leq i, j \leq m$) with nonzero determinant such that $f^{-1}D^k p_{ijk}$ ($0 \leq i, j \leq m$) are integers and*

$$|p_{ijk}| \leq pP^k \quad (0 \leq i, j \leq m), \quad \left| \sum_{j=0}^m p_{ijk} \theta_j \right| \leq lL^{-k} \quad (0 \leq i \leq m).$$

Then

$$\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \dots, \left| \theta_m - \frac{p_m}{q} \right| \right\} > cq^{-1-\lambda}$$

holds for all integers $p_1, \dots, p_m, q$ with $q > 0$, where

$$\lambda = \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad c^{-1} = 2mf^{-1}pDP (\max\{1, 2f^{-1}l\})^\lambda.$$

Note that l, p, L, P, p_{ijk} in [5, Lemma 3.1] denote $f^{-1}l, f^{-1}p, L/D, DP, f^{-1}D^k p_{ijk}$ in the lemma above (or in [15, Lemma 2.1]), respectively. In our situation, we take $m = 2$ and θ_1, θ_2 as in Theorem 3.5. The only difference from Theorem 3.2 in [5] is that we may take $f = 2$ and $D = 2b^2(b-1)^2N$, whereas in [5] $f = 1$ and $D = 4b^2(b-1)^2N$ are taken (note that C_k in [5] denotes $f^{-1}D^k$ in our notation). The validity of this substitution follows from the fact that $b(b-1)$ is even. Indeed, let $p_{ij}(x)$ be those polynomials appearing in [15, Lemma 3.3], which have rational coefficients of degree at most k ([15, (3.7)]). Following [15], we take $p_{ijk} = p_{ij}(1/N)$ for varying values of k . Denoting $b(b-1) = 2b'$ with an integer b' , we see from the expression (3.7) in [15] of $p_{ij}(1/N)$ that

$$2^{l_1}(b')^{l_2}N^k p_{ij}(1/N) \in \mathbb{Z}$$

for some integers l_1, l_2 ; further, we see $l_1 \leq 3k - 1$ in the same way just as the proof of Lemma 4.3 in [15] and $l_2 \leq 2k$ is easy to find. Hence we obtain

$$2^{-1} \cdot 2^k \{b(b-1)\}^{2k} N^k p_{ij}(1/N) \in \mathbb{Z}.$$

Thus, by exactly the same arguments as the ones following Lemma 3.1 in [5] (with $a_0 = 1 - b, a_1 = 0, a_2 = 1$), the numbers

$$\begin{aligned} p &= \left(1 + \frac{1}{2N}\right)^{1/2}, & P &= \frac{8}{2b-1} \left(1 + \frac{3}{2N}\right), \\ l &= \frac{27}{64} \left(1 - \frac{b-1}{N}\right)^{-1}, & L &= \frac{27}{4} \left(1 - \frac{b-1}{N}\right)^2 N^3 \end{aligned}$$

and $f = 2, D = 2b^2(b-1)^2N, p_{ijk} = p_{ij}(1/N)$ satisfy the assumption in Lemma 3.6. Since $N \geq 2.39$ and $b \geq 5$, we have

$$DP \leq \frac{16.1b^2(b-1)^2N}{2b-1}, \quad 2pDP \leq \frac{32.1b^2(b-1)^2N}{2b-1}, \quad \frac{L}{D} \geq \frac{3.37N^2}{b^2(b-1)^2}.$$

Therefore, Theorem 3.5 immediately follows from Lemma 3.6. $\square$

The following is essentially the same as Lemma 6 in [9].

Lemma 3.7. *Let $N = t^2$ and θ_1, θ_2 be as in Theorem 3.5. Then all positive integer solutions x, y, z of the simultaneous Pell equations (2) and (3) satisfy*

$$\max \left\{ \left| \theta_1 - \frac{bsx}{ty} \right|, \left| \theta_2 - \frac{bz}{ty} \right| \right\} < \frac{b-1}{y^2}.$$

PROOF. Since $\theta_1 = s\sqrt{b}/t$ and $\theta_2 = \sqrt{bc}/t$, we have

$$\begin{aligned} \left| \theta_1 - \frac{bsx}{ty} \right| &= \frac{s\sqrt{b}}{t} \left| 1 - \frac{x\sqrt{b}}{y} \right| \\ &= \frac{s\sqrt{b}}{t} \left| 1 - \frac{bx^2}{y^2} \right| \cdot \left| 1 + \frac{x\sqrt{b}}{y} \right|^{-1} < \frac{b-1}{y^2} \end{aligned}$$

and

$$\begin{aligned} \left| \theta_2 - \frac{bz}{ty} \right| &= \frac{1}{t} \left| \sqrt{bc} - \frac{bz}{y} \right| = \frac{b}{t} \left| c - \frac{bz^2}{y^2} \right| \cdot \left| \sqrt{bc} + \frac{bz}{y} \right|^{-1} \\ &< \frac{b}{t} \cdot \frac{c-b}{y^2} \cdot \frac{1}{2\sqrt{bc}} < \frac{1}{2y^2} \cdot \frac{bc-1}{\sqrt{bc}(bc-1)} < \frac{1}{2y^2}. \end{aligned}$$

These complete the proof of Lemma 3.7. $\square$

Proposition 3.8. *If $b \geq 5$ and $c \geq 2.4b^6$, then we have*

$$\log y < \frac{2 \log \frac{1.9c}{b-1} \cdot \log(17b^6c^2)}{\log \frac{0.41c}{b^6}}.$$

PROOF. We apply Theorem 3.5 with $N = t^2$, $p_1 = bsx$, $p_2 = bz$ and $q = ty$. Note that $c \geq 2.4b^6$ implies $N = t^2 = bc-1 > 2.39b^7$. Theorem 3.5 and Lemma 3.7 together imply

$$\left\{ 32.1 \frac{b^2(b-1)^2}{2b-1} t^2 \right\}^{-1} (ty)^{-1-\lambda} < \frac{b-1}{y^2}.$$

Noting $\lambda < 1$, we have

$$y^{1-\lambda} < 32.1 \frac{b^2(b-1)^3}{2b-1} t^{3+\lambda} < \frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}.$$

It follows from

$$\frac{1}{1-\lambda} = \frac{\log \frac{3.37(bc-1)^2}{b^2(b-1)^2}}{\log \frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}$$

that

$$\log y < \frac{\log \frac{3.37(bc-1)^2}{b^2(b-1)^2} \cdot \log \frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1}}{\log \frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4}}.$$

Since

$$\begin{aligned} \log \frac{3.37(bc-1)^2}{b^2(b-1)^2} &< 2 \log \frac{1.9c}{b-1}, \\ \log \frac{32.1b^2(b-1)^3(bc-1)^2}{2b-1} &< \log(17b^6c^2), \\ \log \frac{3.37(2b-1)(bc-1)}{16.1b^4(b-1)^4} &> \log \frac{0.418c}{b^6} \left(1 - \frac{1}{bc}\right) \geq \log \frac{0.41c}{b^6}, \end{aligned}$$

we obtain the proposition. $\square$

We are now ready to bound above for c .

Theorem 3.9. *Let $b \geq 5$. If c is minimal for which the equations (2) and (3) have a nontrivial solution, then we have $c < 200b^6$.*

PROOF. Suppose $c \geq 200b^6$. Propositions 3.4 implies

$$\begin{aligned} \log y &> \left(\frac{200^{1/12}c^{1/4}}{c^{1/12}} - 1 \right) \log(4c-3) \\ &> 200^{1/12}c^{1/6} \log(4c-3) \left(1 - \frac{1}{200^{1/12}c^{1/6}} \right) > 1.47c^{1/6} \log(4c-3) \end{aligned}$$

and Proposition 3.8 implies

$$\begin{aligned} \log y &< \frac{2 \log(1.9c/4) \cdot \log(17c^3/200)}{\log(0.41 \cdot 200)} \\ &< 1.37 \log(0.48c) \log(0.44c). \end{aligned}$$

Hence we have

$$f(c) := 1.37 \log(0.48c) \log(0.44c) - 1.47c^{1/6} \log(4c-3) > 0.$$

However, $f(c)$ is a decreasing function for $c(\geq 200b^6) \geq 200 \cdot 5^6$ with $f(200 \cdot 5^6) < 0$, which is a contradiction. $\square$

It is to be noted that in case $r = \sqrt{b-1}$ is an odd prime, the Pell equation (1) has exactly the three fundamental solutions

$$(t, s) = (r, 0), (b-r, \pm(r-1)). \quad (13)$$

To see this, let $(t, s) = (\beta, \alpha)$ be a solution of (1) with $\gcd(\alpha, \beta) = 1$, which is called primitive. Then Theorem 22 in [16] implies that there exists an integer j such that

$$\beta \equiv j\alpha \pmod{r^2} \quad \text{and} \quad j^2 \equiv b \equiv 1 \pmod{r^2}.$$

(In this case, we say that (β, α) belongs to j .) Since $\gcd(j+1, j-1)$ divides 2 and r is an odd prime, the latter relation implies $j \equiv \pm 1 \pmod{r^2}$. On the other hand, by Theorem 23 in [16] we see that the fundamental solutions of (1) which are primitive are the ones that belong to the values ± 1 of $j \pmod{r^2}$; $(\beta, \alpha) = (b-r, \pm(r-1))$ belong to ∓ 1 respectively. Since r is a prime, the fundamental solution of (1) independent of the ones above is only the trivial one $(r, 0)$. Therefore, the fundamental solutions of (1) are given by (13).

Since we may take the fundamental solutions of (1) with $0 < t < b$ (cf. (2) and (6)), it is easy to check in each case that the same is true for $b = 5, 17$ or 37 . Hence for these b 's, the positive solutions (t, s) of (1) are given by $t + s\sqrt{b} =$

$$\begin{aligned} & \left\{ b - r + (r-1)\sqrt{b} \right\} \left(2b - 1 + 2r\sqrt{b} \right)^\nu \quad (\nu = 0, 1, 2, \dots), \\ & \left\{ b - r - (r-1)\sqrt{b} \right\} \left(2b - 1 + 2r\sqrt{b} \right)^\nu \quad (\nu = 1, 2, 3, \dots), \\ & r \left(2b - 1 + 2r\sqrt{b} \right)^\nu \quad (\nu = 1, 2, 3, \dots). \end{aligned} \quad (14)$$

Since our numbering for the solutions takes as (t_0, s_0) the first solution above with $\nu = 0$, we have

$$t_8 + s_8\sqrt{b} = r(2b - 1 + 2r\sqrt{b})^3,$$

which implies

$$\begin{aligned} c_8 &= s_8^2 + 1 = [(b-1)\{6(2b-1)^2 + 8b(b-1)\}]^2 + 1 \\ &= 4(b^2 - 2b + 1)(256b^4 - 512b^3 + 352b^2 - 96b + 9) + 1 \\ &> b^2 \left(2 - \frac{4}{b} \right) \cdot 256b^4 \left(2 - \frac{4}{b} \right) > 256b^6 > 200b^6. \end{aligned}$$

Consequently we obtain

Corollary 3.10. *Assume that $\sqrt{b-1}$ is a prime or that $b = 17$ or 37 . If they have only the trivial solutions for $c = c_k$ with $0 \leq k \leq 7$, then the equations (2) and (3) have only the trivial solutions for $c = c_k$ with $k \geq 0$.*

Remark 3.11. In case $b = 65$, the Pell equation $t^2 - 65s^2 = 64$ of (1) has exactly the five fundamental solutions

$$(t, s) = (8, 0), (18, \pm 2), (57, \pm 7).$$

Since $t_{14} + s_{14}\sqrt{65} = 8(129 + 16\sqrt{65})^3$ and $c_{14} = s_{14}^2 + 1 > 200 \cdot 65^6$, if they have only the trivial solutions for $c = c_k$ with $0 \leq k \leq 13$, then the equations (2) and (3) with $b = 65$ have only the trivial solutions for $c = c_k$ with $k \geq 0$.

Since $(t, s) = (b - r, r - 1)$ is a positive solution of (1) and the attached $c = b - (2r - 1) < b$, c_0 is always less than b . Hence, the non-extensibility of $D(-1)$ -triples $\{1, b, c\}$ for $b = 2, 5, 10, 17, 26, 37$ and for the attached c 's with $c \geq c_1$ implies that of $\{1, b, c_0\}$ for $b = 5, 10, 17, 26, 37, 50$ and for the attached c_0 's. Therefore, it is enough to show Theorem 2.1 for $k \geq 1$.

Corollary 3.12. *If it holds for $c = c_k$ with $1 \leq k \leq 7$, then Theorem 2.1 holds for $c = c_k$ with $k \geq 0$.*

4. The reduction method

Throughout this section, let $b = 5, 10, 17, 26, 37$ or 50 and assume that $(m \geq) n \geq 1$ (that is, Theorem 2.1 is not valid) and that $c = c_k$ is minimal for which Theorem 2.1 is not valid. By the non-extensibility of those $D(-1)$ -triples listed in Section 1, we may assume that $c \geq 26$. Moreover, since $t_1 + s_1\sqrt{b}$ equals the middle expression of (14) with $\nu = 1$, we may assume that

$$c \geq (c_1 =) b + 2r + 1. \quad (15)$$

We will complete the proof of Theorem 2.1 by combining Corollary 3.12 with the reduction method ([11]) of DUJELLA and PETHŐ (based on that of Baker and Davenport).

By (7) and (8), we have

$$\begin{aligned} v_m &= \frac{s}{2} \{ (s + \sqrt{c})^{2m} + (s - \sqrt{c})^{2m} \}, \\ w_n &= \frac{1}{2\sqrt{b}} \{ (s\sqrt{b} \pm r\sqrt{c})(t + \sqrt{bc})^{2n} + (s\sqrt{b} \mp r\sqrt{c})(t - \sqrt{bc})^{2n} \}. \end{aligned}$$

Putting

$$P = s(s + \sqrt{c})^{2m}, \quad Q = \frac{1}{\sqrt{b}}(s\sqrt{b} \pm r\sqrt{c})(t + \sqrt{bc})^{2n},$$

we have

$$P^{-1} = \frac{1}{s}(s - \sqrt{c})^{2m}, \quad Q^{-1} = \frac{\sqrt{b}}{c - b}(s\sqrt{b} \mp r\sqrt{c})(t - \sqrt{bc})^{2n}. \quad (16)$$

It follows from $v_m = w_n$ that

$$P + (c - 1)P^{-1} = Q + \frac{c - b}{b}Q^{-1}.$$

Let us bound the linear form “ $\log(Q/P)$ ” in logarithms.

First, we have

$$\begin{aligned} Q - P &= (c - 1)P^{-1} - \frac{c - b}{b}Q^{-1} \\ &> (c - 1)(P^{-1} - Q^{-1}) = (c - 1)(Q - P)P^{-1}Q^{-1}. \end{aligned}$$

Since we see from $b \geq 5$, $c \geq 26$, $m \geq n \geq 1$ and (15) that

$$\begin{aligned} P - (c - 1) &= (c - 1) \left\{ \frac{(s + \sqrt{c})^{2m}}{s} - 1 \right\} \\ &\geq (c - 1) \left(s + 2\sqrt{c} + \frac{c}{s} - 1 \right) > 0, \\ Q &\geq \frac{1}{\sqrt{b}}(s\sqrt{b} - r\sqrt{c})(t + \sqrt{bc})^2 \\ &> \frac{c - b}{2b\sqrt{c}}(4bc - 3) = 2 \left(1 - \frac{3}{4bc} \right) (c - b)\sqrt{c} \\ &> 1.9(2\sqrt{b - 1} + 1)\sqrt{c} \geq 9.5\sqrt{c} > 1, \end{aligned} \tag{17}$$

we have $Q > P$.

Secondly, since (16) and (17) imply $P > Q - (c - 1)P^{-1} > Q - 1$, we have

$$\frac{Q - P}{Q} < Q^{-1} < \frac{1}{9.5\sqrt{c}} < \frac{1}{2}. \tag{18}$$

On the other hand, by $b \geq 5$ and (15) we have

$$\begin{aligned} Q^{-1} &\leq \frac{\sqrt{b}}{c - b}(s\sqrt{b} + r\sqrt{c})(t + \sqrt{bc})^{-2n} \leq \frac{2b\sqrt{c}}{c - b}(t + \sqrt{bc})^{-2n} \\ &< \frac{b(\sqrt{b} + 1)}{\sqrt{b} - 1}(t + \sqrt{bc})^{-2n} < 1.62b(t + \sqrt{bc})^{-2n}. \end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned} 0 &< \log \frac{Q}{P} = -\log \left(1 - \frac{Q - P}{Q} \right) < -\log(1 - Q^{-1}) \\ &< Q^{-1} + Q^{-2} = (1 + Q^{-1})Q^{-1} \\ &< \left(1 + \frac{1}{9.5\sqrt{c}} \right) 1.62b(t + \sqrt{bc})^{-2n} \\ &< 1.7b(t + \sqrt{bc})^{-2n}. \end{aligned}$$

Therefore, we have

$$0 < (\Lambda :=) n \log \alpha_1 - m \log \alpha_2 + \log \alpha_3 < 1.7b\alpha_1^{-n}, \quad (20)$$

where

$$\begin{aligned} \alpha_1 &:= 2bc - 1 + 2\sqrt{bc(bc-1)}, & \alpha_2 &:= 2c - 1 + 2\sqrt{c(c-1)}, \\ \alpha_3 &:= \frac{\sqrt{b(c-1)} \pm \sqrt{(b-1)c}}{\sqrt{b(c-1)}}. \end{aligned}$$

The following theorem of Baker and Wüstholz gives a lower bound for $\log \Lambda$.

Theorem 4.1 ([4, Theorem]). *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational coefficients $\beta_1, \dots, \beta_l$, we have*

$$\log |\Lambda| \geq -18(l+1)! l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log \beta,$$

where $\beta := \max\{|\beta_1|, \dots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \dots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) := \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α .

Applying Theorem 4.1 with $l = 3$, $d = 4$, $\beta = m \leq 2n$ and

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2} \log \alpha_1 < \frac{1}{2} \log(4bc), & h'(\alpha_2) &= \frac{1}{2} \log \alpha_2 < \frac{1}{2} \log(4c), \\ h'(\alpha_3) &\leq \frac{1}{4} \{\log(b(c-1))^2 + \log(\alpha_3^2)\} \\ &= \frac{1}{2} \log \left\{ b(c-1) + \sqrt{bc(b-1)(c-1)} \right\} < \frac{1}{2} \log(2bc), \end{aligned}$$

we have

$$\begin{aligned} \log \Lambda &> -18 \cdot 4! \cdot 3^4 (32 \cdot 4)^5 \cdot \frac{1}{2} \log(4bc) \cdot \frac{1}{2} \log(4c) \cdot \frac{1}{2} \log(2bc) \\ &\quad \times \log 24 \cdot \log(2n), \end{aligned}$$

which together with (20) implies

$$\frac{n-1}{\log(2n)} < 4.8 \cdot 10^{14} \log(4bc) \cdot \log(4c). \quad (21)$$

Corollary 3.12 and (21) give upper bounds for n :

if $b = 5$, then $c \leq c_7 = 974170$ and $n < 6 \cdot 10^{18}$;
 if $b = 10$, then $c \leq c_7 = 34199105$ and $n < 9 \cdot 10^{18}$;
 if $b = 17$, then $c \leq c_7 = 482812730$ and $n < 2 \cdot 10^{19}$;
 if $b = 26$, then $c \leq c_7 = 3947106277$ and $n < 2 \cdot 10^{19}$;
 if $b = 37$, then $c \leq c_7 = 22480504226$ and $n < 2 \cdot 10^{19}$;
 if $b = 50$, then $c \leq c_7 = 99106595345$ and $n < 2 \cdot 10^{19}$.

Now, dividing (20) by $\log \alpha_2$ leads to the inequality

$$0 < n\kappa - m + \mu < AB^{-n}, \quad (22)$$

where

$$\kappa := \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu := \frac{\log \alpha_3}{\log \alpha_2}, \quad A := \frac{1.7b}{\log \alpha_2}, \quad B := \alpha_1.$$

The following is based on the BAKER–DAVENPORT lemma ([3, Lemma]).

Lemma 4.2 ([11, Lemma 5 a]). *Let N be a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6N$. Put $\epsilon := \|\mu q\| - N \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then the inequality (22) has no solution in the range*

$$\frac{\log(Aq/\epsilon)}{\log B} \leq n < N.$$

We apply Lemma 4.2 with N the upper bound for n in each case. In the first step of reduction, we have to examine $36 \cdot 2 = 72$ cases (the doubling comes from the signs “ $\pm$ ” in α_3), of which the second convergent of κ such that $q > 6N$ is needed only in three cases. Thus we obtain a new bound $(n \leq)N_1$ with $N_1 \leq 6$ in each case. The second step of reduction (with $N = N_1$) requires the second convergent of κ such that $q > 6N_1$ only in four cases. Thus we obtain a new bound $(n \leq)N_2$ with $N_2 = 0$ or 1 . The former contradicts the assumption. The latter occurs only in five cases. Then the third step of reduction (with $N = 1$) requires the second convergent of κ such that $q > 6$ only in one case and gives $n < 1$, which contradicts the assumption. This completes the proof of Theorem 2.1.

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