

Strong quasi-metric spaces and countable paracompactness of bispaces

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Abstract. T. G. RAGHAVAN and I. L. REILLY proved that if (X, d) is a quasi-metric space such that $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$, then (X, d) is strong (i.e. $\tau(d) \subseteq \tau(d^{-1})$). Here, we show that the converse of this result is also true. We obtain, in this way, a characterization of strong quasi-metric spaces in terms of bitopological countable paracompactness.

1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the sets of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

Our basic references for quasi-metric spaces are [3] and [7] and for general topology it is [2].

Let us recall that a *quasi-metric* on a (nonempty) set X is a function $d : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = 0 \Leftrightarrow x = y$; and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X .

If d is a quasi-metric on X , then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X called *the conjugate*

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of d . The function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$, is a metric on X .

Each quasi-metric d on X generates a T_1 -topology $\tau(d)$ on X which has as a base the family of open d -balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-metric d on X is said to be *strong* (*point symmetric* in [3]) if $\tau(d) \subseteq \tau(d^{-1})$. By a *strong quasi-metric space* we mean a quasi-metric space (X, d) such that the quasi-metric d is strong. A topological space (X, τ) is called (*strongly*) *quasi-metrizable* if there is a (strong) quasi-metric d on X such that $\tau(d) = \tau$ on X . In this case, we say that d is *compatible* with τ .

Strong quasi-metric spaces were introduced by R. A. STOLTENBERG [11], who proved, among other results, that every strongly quasi-metrizable space is developable and, hence, countably metacompact. Conversely, each quasi-metrizable developable space is strongly quasi-metrizable [3], [6]. Furthermore, a quasi-metrizable space (X, τ) is compact if and only if every quasi-metric d on X compatible with τ is strong [6]. The Niemytzki plane, the Pixley–Roy space and the Dieudonné example of a Tychonoff locally compact non-normal space are paradigmatic examples of non-metrizable strongly quasi-metrizable spaces [3], [11]. Recently, separable completely metrizable spaces have been characterized [1] in terms of quasi-metrics d such that $\tau(d^{-1})$ is compact, and hence, d^{-1} is strong (see [8] for a generalization of such a characterization to completely metrizable spaces).

The notion of a bitopological space appears in a natural way when one considers the topologies $\tau(d)$ and $\tau(d^{-1})$ generated by a quasi-metric d and its conjugate. Recall that a *bitopological space* is an ordered triple (X, P, Q) such that X is a (nonempty) set and P and Q are topologies on X [5]. In the sequel we shall use the term *bispace* instead of bitopological space.

It is well known that paracompactness is one of the most intractable notions in the setting of bispaces (see [10], and pages 910–912 and the bibliography of [7]). In particular, T. G. RAGHAVAN and I. L. REILLY introduced in [9] the following concept in their study on metrizability of quasi-metric spaces: In the bispace (X, P, Q) P is (*countably*) *paracompact with respect to Q* provided that each (countable) P -open cover of X has a P -open refinement which is Q -locally finite.

Then, they proved, among other results, that if (X, d) is a quasi-metric space such that $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$, then d is strong, and thus, $(X, \tau(d^{-1}))$ is metrizable (Proposition 7 and Theorem 2 of [9]).

The main purpose of this note is to prove that the converse of this result is also

true. Thus, we obtain a somewhat unexpected characterization of strong quasi-metric spaces in terms of Raghavan–Reilly’s notion of bitopological countable paracompactness.

2. The results

In order to obtain our main result we first characterize those bispaces (X, P, Q) such that P is countably paracompact with respect to Q in the style of the following classical characterization of countable metacompactness due to F. ISHIKAWA [4]: A topological space X is countably metacompact if and only if for any decreasing sequence $(F_n)_{n \in \mathbb{N}}$ of nonempty closed sets such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ there is a decreasing sequence $(G_n)_{n \in \mathbb{N}}$ of open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

Proposition 1. *Let (X, P, Q) be a bispace. Then P is countably paracompact with respect to Q if and only if for any decreasing sequence $(F_n)_{n \in \mathbb{N}}$ of nonempty P -closed sets such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ there is a decreasing sequence $(G_n)_{n \in \mathbb{N}}$ of P -open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} \text{cl}_Q G_n = \emptyset$.*

PROOF. If P is countably paracompact with respect to Q and $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty P -closed sets satisfying $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then $\{X \setminus F_n : n \in \mathbb{N}\}$ is a countable P -open cover of X and, hence, it has a P -open refinement $\mathcal{W}$ which is Q -locally finite. For each $W \in \mathcal{W}$ let $f(W)$ be the first $n \in \mathbb{N}$ such that $W \subseteq X \setminus F_n$. Put, then,

$$V_n = \bigcup \{W \in \mathcal{W} : f(W) = n\}$$

for all $n \in \mathbb{N}$. Therefore $\{V_n : n \in \mathbb{N}\}$ is a P -open cover of X which is Q -locally finite and verifies $V_n \subseteq X \setminus F_n$ for all $n \in \mathbb{N}$. Now let

$$G_n = \bigcup \{V_k : k \geq n+1\},$$

for all $n \in \mathbb{N}$. Clearly $(G_n)_{n \in \mathbb{N}}$ is a decreasing sequence of P -open sets. Since $(F_n)_{n \in \mathbb{N}}$ is decreasing and $F_n \subseteq X \setminus V_n$, it follows that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$. Furthermore, given $x \in X$ there exists a Q -neighborhood H of x which meets only a finite number of V_n ’s. Consequently there is $m \in \mathbb{N}$ with $H \cap G_m = \emptyset$, and, hence, $\bigcap_{n=1}^{\infty} \text{cl}_Q G_n = \emptyset$.

Conversely, let $\{W_n : n \in \mathbb{N}\}$ be a countable P -open cover of X . Assume, without loss of generality that, for each $n \in \mathbb{N}$, $W_n \neq X$. Now put

$$F_n = X \setminus \bigcup \{W_k : 1 \leq k \leq n\},$$

for all $n \in \mathbb{N}$. Then $(F_n)_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty P -closed sets with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. So, there is a decreasing sequence $(G_n)_{n \in \mathbb{N}}$ of P -open sets such that $F_n \subseteq G_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} \text{cl}_Q G_n = \emptyset$.

Define $V_1 = W_1$, and $V_n = W_n \cap G_{n-1}$ for all $n \geq 2$. Thus, it is clear that $\{V_n : n \in \mathbb{N}\}$ is a countable P -open cover of X such that $V_n \subseteq W_n$ for all $n \in \mathbb{N}$. It remains to show that $\{V_n : n \in \mathbb{N}\}$ is Q -locally finite. To this end, given $x \in X$ we choose the first $n \in \mathbb{N}$ such that $x \in X \setminus \text{cl}_Q G_n$. Then, there exists a Q -neighborhood H of x satisfying $H \cap G_n = \emptyset$. Thus $H \cap G_{n+k} = \emptyset$ for all $k \in \mathbb{N} \cup \{0\}$, and, therefore, $H \cap V_{n+k} = \emptyset$ for all $k \in \mathbb{N}$. Hence, the collection $\{V_n : n \in \mathbb{N}\}$ is Q -locally finite. We conclude that P is countably paracompact with respect to Q . $\square$

Corollary. *Let (X, P, Q) be a bispace. If P is countably paracompact with respect to Q , then (X, P) is countably metacompact.*

The following result provides a partial converse to the above corollary. Recall that a bispace (X, P, Q) is said to be pairwise normal [5] if given a P -closed set A and a disjoint Q -closed set B , there exist a P -open set G and a disjoint Q -open set H such that $A \subseteq G$ and $B \subseteq H$.

It is well known [5] that if (X, d) is a quasi-metric space, then the bispace $(X, \tau(d), \tau(d^{-1}))$ is pairwise normal.

Proposition 2. *Let (X, P, Q) be a pairwise normal bispace such that $P \subseteq Q$. If (X, P) is countably metacompact then P is countably paracompact with respect to Q .*

PROOF. Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing sequence of nonempty P -closed sets with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. By countable metacompactness of (X, P) , there is a decreasing sequence $(W_n)_{n \in \mathbb{N}}$ of P -open sets such that $F_n \subseteq W_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} W_n = \emptyset$. Since $P \subseteq Q$, each F_n is Q -closed, so, by pairwise normality of (X, P, Q) , there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of P -open sets satisfying $F_n \subseteq G_n \subseteq \text{cl}_Q G_n \subseteq W_n$ for all $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} \text{cl}_Q G_n = \emptyset$. By Proposition 1, we conclude that P is countably paracompact with respect to Q . $\square$

Theorem. *A quasi-metric space (X, d) is strong if and only if $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$.*

PROOF. Suppose that (X, d) is strong. Then $\tau(d) \subseteq \tau(d^{-1})$. Moreover, the bispace $(X, \tau(d), \tau(d^{-1}))$ is pairwise normal and $(X, \tau(d))$ is countably metacompact (see Section 1). So, by Proposition 2, $\tau(d)$ is countably paracompact with respect to $\tau(d^{-1})$. The converse was proved by RAGHAVAN and REILLY in [9], as we indicated in Section 1. $\square$

The above result (compare Proposition 4 of [9]) suggests the following natural question: Let (X, d) be a strong quasi-metric space. Is it $\tau(d)$ paracompact with respect to $\tau(d^{-1})$?

We conclude the paper with an example which shows that this question has a negative answer.

Example. Let d be the quasi-metric on $\mathbb{R}$ given by:

$$\begin{aligned} d(x, x) &= 0 \quad \text{for all } x \in X, \\ d(x, y) &= 1/(y + 1) \quad \text{if } x \text{ is irrational and } y \in \mathbb{N}, \\ d(x, y) &= 1 \quad \text{otherwise.} \end{aligned}$$

Clearly $\tau(d^{-1})$ is the discrete topology on $\mathbb{R}$, so (X, d) is strong. Consider the following $\tau(d)$ -open cover of X :

$$\mathcal{W} = \{B_d(x, 1) : x \text{ is irrational}\} \cup \{\{x\} : x \text{ is rational with } x \notin \mathbb{N}\}.$$

Suppose that $\mathcal{W}$ has a $\tau(d)$ -open refinement $\mathcal{V}$ which is $\tau(d^{-1})$ -locally finite. Then each $x \in \mathbb{R}$, and thus each $x \in \mathbb{N}$, is only in finitely many elements of $\mathcal{V}$, which, obviously, is not possible. We conclude that $\tau(d)$ is not paracompact with respect to $\tau(d^{-1})$.

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