

Flow-invariant structures on unit tangent bundles

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Abstract. We study unit tangent bundles T_1M for which the structural operator $h = \frac{1}{2} \mathfrak{L}_\xi \phi$, its characteristic derivative $h' = \nabla_\xi h$ or the characteristic Jacobi operator $\ell = R(\cdot, \xi)\xi$ is invariant under the geodesic flow generated by the characteristic vector field ξ . Also, we prove that the operator ℓ on T_1M is η -parallel if and only if the base manifold is of constant curvature.

1. Introduction

One way to study the geometry of a Riemannian manifold (M, G) is to investigate the interaction of the manifold with its unit tangent sphere bundle T_1M endowed with its standard contact Riemannian structure (η, g, ϕ, ξ) . Special properties for the geometry of (M, G) will be reflected in special properties for the contact structure on T_1M and vice versa. In particular, the characteristic vector field ξ on T_1M contains crucial information about M . In fact, all the geodesics in M are controlled by the geodesic flow on T_1M which is precisely given by ξ .

Apart from the defining structure tensors η, g, ϕ and ξ , two other operators play a fundamental role in contact Riemannian geometry, namely the structural operator $h = \frac{1}{2} \mathfrak{L}_\xi \phi$ and the characteristic Jacobi operator $\ell = R(\cdot, \xi)\xi$, where $\mathfrak{L}_\xi$ denotes Lie differentiation in the characteristic direction ξ . An important topic in the study of the contact metric structure on unit tangent bundles has been

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to determine those Riemannian manifolds (M, G) for which the corresponding contact structure on T_1M enjoys symmetry properties related to the geodesic flow.

A first symmetry type for the contact metric structure occurs when the geodesic flow, generated by ξ , leaves some structure tensors invariant. This is always the case for ξ and η since $\mathfrak{L}_\xi \xi = 0$ and $\mathfrak{L}_\xi \eta = 0$. The metric g is left invariant by the flow of ξ (or equivalently, the flow consists of local isometries or ξ is a Killing vector field) if and only if the structural operator h vanishes. By definition, this corresponds precisely to $\mathfrak{L}_\xi \phi = 0$, i.e., also ϕ is preserved under the geodesic flow. Y. TASHIRO proved in [14] that this happens for a unit tangent bundle $(T_1M; \eta, g, \phi, \xi)$ if and only if (M, G) has constant curvature $c = 1$. In this paper, we investigate when the operators h , ℓ and $h' = \nabla_\xi h$ on T_1M are preserved by the geodesic flow. Namely, we prove in Section 4:

Theorem 1. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ) . Then T_1M satisfies $\mathfrak{L}_\xi h = 0$ if and only if (M, G) is of constant curvature $c = 1$.*

Theorem 2. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ) . Then T_1M satisfies $\mathfrak{L}_\xi \ell = 0$ if and only if (M, G) is of constant curvature $c = 0$ or $c = 1$.*

Theorem 3. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ) . Then T_1M satisfies $\mathfrak{L}_\xi h' = 0$ if and only if (M, G) is of constant curvature $c = -1$, $c = 0$ or $c = 1$.*

A second type of symmetry occurs when some structure tensors are covariantly parallel along the integral curves of ξ . On a contact metric space, it always holds $\nabla_\xi \xi = \nabla_\xi \eta = \nabla_\xi g = \nabla_\xi \phi = 0$, but the other structure tensors need not be parallel in the ξ -direction. Recently, it was proved that T_1M satisfies the condition $\nabla_\xi h = 0$ or, equivalently, $\nabla_\xi \ell = 0$, if and only if (M, G) is of constant curvature $c = 0$ or $c = 1$ ([12], [13]). This result can easily be verified from the formulas in this paper.

A final symmetry notion on contact Riemannian manifolds is the notion of η -parallelity. We call a $(1, 1)$ -tensor T η -parallel if $g((\nabla_X T)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ . In particular, the tensor ϕ is η -parallel if and only if the contact structure is CR-integrable, $(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. On a unit tangent bundle T_1M , this occurs if and only if (M, G) is of constant curvature, as can easily be verified from the formulas further on. Contact metric spaces with η -parallel structural operator h were completely classified by the first

two authors in [6]. As a corollary of that result, we obtain also that the standard contact Riemannian structure (η, g, ϕ, ξ) of a unit tangent sphere bundle T_1M has η -parallel h if and only if (M, G) is of constant curvature. Now, we consider the case when the characteristic Jacobi operator ℓ on T_1M is η -parallel. We show in Section 5:

Theorem 4. *Let T_1M be the unit tangent sphere bundle with the standard contact Riemannian structure (η, g, ϕ, ξ) . Then the characteristic Jacobi operator ℓ of T_1M is η -parallel if and only if (M, G) is of constant curvature.*

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞ . We start by collecting some fundamental material about contact metric geometry. We refer to [2] for further details. A $(2n + 1)$ -dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η , we have a unique vector field ξ , the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X . It is well-known that there exists a Riemannian metric g and a $(1, 1)$ -tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi \quad (1)$$

where X and Y are vector fields on M . From (1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2)$$

A Riemannian manifold M equipped with structure tensors (η, g, ϕ, ξ) satisfying (1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g, \phi, \xi)$. Given a contact Riemannian manifold M , we define the *structural operator* h by $h = \frac{1}{2}\mathfrak{L}_\xi\phi$, where $\mathfrak{L}_\xi$ denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \quad (3)$$

$$\nabla_X \xi = -\phi X - \phi hX \quad (4)$$

where ∇ is the Levi-Civita connection. From (3) and (4) we see that each trajectory of ξ is a geodesic. We denote by R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$$

for all vector fields X, Y, Z . Along a trajectory of ξ , the Jacobi operator $\ell = R(\cdot, \xi)\xi$ is a symmetric $(1, 1)$ -tensor field. We call it the *characteristic Jacobi operator*. We have

$$\ell = \phi\ell\phi - 2(h^2 + \phi^2), \quad (5)$$

$$\nabla_\xi h = \phi - \phi\ell - \phi h^2. \quad (6)$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold*. It is easy to see that a contact Riemannian manifold is *K-contact* if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$.

3. The contact metric structure of the unit tangent bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [9], [11], [16]). We only briefly review some notations and definitions. Let $M = (M, G)$ be an n -dimensional Riemannian manifold and let TM denote its tangent bundle with the projection $\pi : TM \rightarrow M$, $\pi(x, u) = x$. For a vector $X \in T_x M$, we denote by X^H and X^V , the *horizontal lift* and the *vertical lift*, respectively. Then we can define a Riemannian metric $\tilde{g}$, the *Sasaki metric* on TM , in a natural way by

$$\tilde{g}(X^H, Y^H) = \tilde{g}(X^V, Y^V) = G(X, Y) \circ \pi, \quad \tilde{g}(X^H, Y^V) = 0$$

for all vector fields X and Y on M . Also, a natural almost complex structure tensor J of TM is defined by $JX^H = X^V$ and $JX^V = -X^H$. Then we easily see that $(TM; \tilde{g}, J)$ is an almost Hermitian manifold. We note that J is integrable if and only if (M, G) is locally flat ([9]). Now we consider the unit tangent sphere bundle $(T_1 M, \bar{g})$, which is an isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^V$. For $X \in T_x M$, we define the *tangential lift* of X to $(x, u) \in T_1 M$ by

$$X_{(x, u)}^T = X_{(x, u)}^V - G(X, u)N_{(x, u)}.$$

Clearly, the tangent space $T_{(x, u)} T_1 M$ is spanned by vectors of the form X^H and X^T where $X \in T_x M$. We put

$$\bar{\xi} = -JN, \quad \bar{\phi} = J - \bar{\eta} \otimes N.$$

Then we find $\bar{g}(X, \bar{\phi}Y) = 2d\bar{\eta}(X, Y)$. By taking $\xi = 2\bar{\xi}$, $\eta = \frac{1}{2}\bar{\eta}$, $\phi = \bar{\phi}$, and $g = \frac{1}{4}\bar{g}$, we get the standard contact Riemannian structure (ϕ, ξ, η, g) . Indeed, we easily check that these tensors satisfy (1). Here we notice that ξ determines

the geodesic flow. The tensors ξ and ϕ are explicitly given by

$$\xi = 2u^H, \quad \phi X^T = -X^H + \frac{1}{2} G(X, u)\xi, \quad \phi X^H = X^T \quad (7)$$

where X and Y are vector fields on M . From now on, we consider $T_1M = (T_1M; \eta, g)$ with the standard contact Riemannian structure. We list the fundamental formulae which we need for the proof of our theorems. They are derived in, e.g., [3], [4], [5], [8], [12], [14]. The Levi-Civita connection ∇ of (T_1M, g) is given by

$$\begin{aligned} \nabla_{X^T} Y^T &= -G(Y, u)X^T, & \nabla_{X^T} Y^H &= \frac{1}{2} (K(u, X)Y)^H, \\ \nabla_{X^H} Y^T &= (D_X Y)^T + \frac{1}{2} (K(u, Y)X)^H, \\ \nabla_{X^H} Y^H &= (D_X Y)^H - \frac{1}{2} (K(X, Y)u)^T. \end{aligned} \quad (8)$$

For the Riemann curvature tensor R , we give only the two expressions we need for the characteristic Jacobi operator ℓ :

$$\begin{aligned} R(X^T, Y^H)Z^H &= -\frac{1}{2} \{K(Y, Z)(X - G(X, u)u)\}^T \\ &\quad + \frac{1}{4} \{K(Y, K(u, X)Z)u\}^T - \frac{1}{2} \{(D_Y K)(u, X)Z\}^H, \\ R(X^H, Y^H)Z^H &= (K(X, Y)Z)^H + \frac{1}{2} \{K(u, K(X, Y)u)Z\}^H \\ &\quad - \frac{1}{4} \{K(u, K(Y, Z)u)X - K(u, K(X, Z)u)Y\}^H \\ &\quad + \frac{1}{2} \{(D_Z K)(X, Y)u\}^T \end{aligned} \quad (9)$$

for all vector fields X , Y and Z on M . In the above, we denote by D the Levi-Civita connection and by K the Riemannian curvature tensor associated with G . From (7) and (8), it follows

$$\nabla_{X^T} \xi = -2\phi X^T - (K_u X)^H, \quad \nabla_{X^H} \xi = -(K_u X)^T \quad (10)$$

where $K_u = K(\cdot, u)u$ is the Jacobi operator associated with the unit vector u . From (4) and (10), it follows that

$$hX^T = X^T - (K_u X)^T, \quad hX^H = -X^H + \frac{1}{2} G(X, u)\xi + (K_u X)^H. \quad (11)$$

Using the formulae (9), we get

$$\ell X^T = (K_u^2 X)^T + 2(K'_u X)^H, \quad \ell X^H = 4(K_u X)^H - 3(K_u^2 X)^H + 2(K'_u X)^T \quad (12)$$

where $K'_u = (D_u K)(\cdot, u)u$ and $K_u^2 = K(K(\cdot, u)u, u)u$. By using (6), (7) and (9) we obtain

$$\begin{aligned} h' X^T &= -2(K_u X)^H + 2(K_u^2 X)^H - 2(K'_u X)^T, \\ h' X^H &= -2(K_u X)^T + 2(K_u^2 X)^T + 2(K'_u X)^H \end{aligned} \quad (13)$$

where we put $h' = \nabla_\xi h$.

The above formulae (10)–(13) are also found in [5]. Finally, from (8) and (12) we compute

$$\begin{aligned} \ell' X^T &= 4(K'_u K_u X + K_u K'_u X)^T + 4(K''_u X + K_u^2 X - K_u^3 X)^H, \\ \ell' X^H &= 8(K'_u X - K'_u K_u X - K_u K'_u X)^H + 4(K''_u X + K_u^2 X - K_u^3 X)^T \end{aligned} \quad (14)$$

where $\ell' = (\nabla_\xi R)(\cdot, \xi)\xi$.

4. Invariance under the geodesic flow

In this section, we prove Theorems 1, 2 and 3. We start with Theorem 1. Suppose that $T_1 M = (T_1 M; \eta, g)$ satisfies

$$\mathfrak{L}_\xi h = 0. \quad (15)$$

The definition of the Lie differential yields

$$\begin{aligned} (\mathfrak{L}_\xi h)X &= \mathfrak{L}_\xi(hX) - h(\mathfrak{L}_\xi X) \\ &= [\xi, hX] - h[\xi, X] = (\nabla_\xi h)X - \nabla_{hX}\xi + h\nabla_X\xi. \end{aligned} \quad (16)$$

Together with (4), we see that the condition (15) is equivalent to

$$h' = 2(h\phi - h^2\phi). \quad (17)$$

Since $h' = \nabla_\xi h$ is a self-adjoint operator, from (17) we see that $h^2 = 0$, which implies that $T_1 M$ is Sasakian and $c = 1$ (cf. [14]).

Next, we prove Theorem 2. From the definition of Lie differentiation, we see that the condition $\mathfrak{L}_\xi \ell = 0$ is equivalent to

$$\ell' = \ell\phi - \phi\ell + \ell\phi h - \phi h\ell. \quad (18)$$

From (18), by using (7), (11), (12) and (14) a straightforward computation yields

$$0 = (2K'_u X - 5K'_u K_u X - 3K_u K'_u X)^H + (2K''_u X + 4K_u^2 X - 4K_u^3 X)^T,$$

$$0 = (2K'_u X + K'_u K_u X + 3K_u K'_u X)^T + (2K''_u X + 4K_u X - 4K_u^2 X)^H$$

for all vector fields X on M .

These equations are equivalent to the conditions

$$0 = 2K'_u X - 5K'_u K_u X - 3K_u K'_u X, \quad 0 = 2K'_u X + K'_u K_u X + 3K_u K'_u X,$$

$$0 = 2K''_u X + 4K_u^2 X - 4K_u^3 X, \quad 0 = 2K''_u X + 4K_u X - 4K_u^2 X$$

for all tangent vectors X to M . The first two of these are equivalent to

$$0 = K'_u K_u X + K_u K'_u X, \quad (19)$$

$$0 = K'_u K_u X - K'_u X \quad (20)$$

and the last two imply

$$0 = K_u^3 X - 2K_u^2 X + K_u X. \quad (21)$$

Now we replace X by $K'_u X$ in (21) and use first (19) and then (20) to compute

$$\begin{aligned} 0 &= K_u^3 K'_u X - 2K_u^2 K'_u X + K_u K'_u X \\ &= -K'_u K_u^3 X - 2K'_u K_u^2 X - K'_u K_u X = -6K'_u X. \end{aligned}$$

This implies that (M, G) is a locally symmetric space ([10], [15]). Further, we see from (21) that the eigenvalues of K_u are constant and equal to 0 or 1, i.e., (M, G) is a globally Osserman space (i.e., the eigenvalues of K_u do not depend on the point p and not on the choice of unit vector u at p). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([1], [8]). Therefore, we conclude that M is a space of constant curvature $c = 0$ or $c = 1$.

Conversely, when (M, G) is of constant curvature c , we find the following explicit expressions for h , ℓ , h' and ℓ' from (11)–(14):

$$\begin{aligned} hX^T &= (1 - c)X^T, \quad hX^H = (c - 1)\left(X^H - \frac{1}{2}G(X, u)\xi\right), \\ \ell X^T &= c^2 X^T, \quad \ell X^H = (4c - 3c^2)\left(X^H - \frac{1}{2}G(X, u)\xi\right), \\ h'X^T &= 2(c^2 - c)\left(X^H - \frac{1}{2}G(X, u)\xi\right), \quad h'X^H = 2(c^2 - c)X^T, \end{aligned}$$

$$\ell' X^T = 4(c^2 - c^3) \left(X^H - \frac{1}{2} G(X, u) \xi \right), \quad \ell' X^H = 4(c^2 - c^3) X^T \quad (22)$$

for vector fields X on M . From these, we easily check that $T_1 M$ satisfies (18) when $c = 0$ or $c = 1$.

Finally, we prove Theorem 3. Suppose that $T_1 M$ satisfies $\mathfrak{L}_\xi h' = 0$. Then we see that the condition is equivalent to

$$\nabla_\xi h' = 2h'\phi - \phi(hh' + h'h). \quad (23)$$

(We use the commutation relation $\phi h' + h'\phi = 0$ here. It follows from the second equation of (3) and by using $\nabla_\xi \phi = 0$.) If we take the skew-symmetric part of (23), we obtain $\phi(hh' + h'h) = 0$, which implies

$$hh' + h'h = 0. \quad (24)$$

So, (23) reduces to

$$\nabla_\xi h' = 2h'\phi.$$

Next, we start from (6), written in the equivalent form $\ell = \phi h' - \phi^2 - h^2$ to compute

$$\ell' = \phi \nabla_\xi h' - (hh' + h'h) = 2\phi h'\phi = 2h'. \quad (25)$$

Using the expressions (13) and (14) for h' and ℓ' , this condition is equivalent to the system

$$\begin{aligned} 0 &= K'_u X + K'_u K_u X + K_u K'_u X, & 0 &= K'_u X - 2K'_u K_u X - 2K_u K'_u X, \\ 0 &= K''_u X + K_u X - K_u^3 X \end{aligned}$$

for all tangent vectors X to M . In a similar way as before, we can conclude that the manifold (M, G) must be locally symmetric ($K'_u = 0$) and of constant curvature c equal to -1 , 0 or 1 ($K_u^3 - K_u X = 0$).

Conversely, when (M, G) has constant curvature c equal to -1 , 0 or 1 , we use again the expressions (22) to show that (23) holds. This proves Theorem 3.

5. η -parallel characteristic Jacobi operator on $T_1 M$

If the characteristic Jacobi operator ℓ of a given contact Riemannian manifold satisfies $g((\nabla_{\bar{X}} \ell) \bar{Y}, \bar{Z}) = 0$ for all vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ orthogonal to ξ , then we say that ℓ is η -parallel. In this section, we determine the unit tangent sphere bundles with η -parallel characteristic Jacobi operator by proving Theorem 4.

From (8) and (12), we compute the covariant derivatives of the characteristic Jacobi operator ℓ :

$$\begin{aligned} (\nabla_{X^H} \ell) Y^H &= \left(4(D_X K)(Y, u)u - 3(D_X K)(K_u Y, u)u \right. \\ &\quad \left. - 3K_u((D_X K)(Y, u)u) + K(u, K'_u Y)X + K'_u(K(X, Y)u) \right)^H \\ &\quad + \left(-2K(X, K_u Y)u + \frac{3}{2}K(X, K_u^2 Y)u \right. \\ &\quad \left. + 2(D_{X_u}^2 K)(Y, u)u + \frac{1}{2}K_u^2(K(X, Y)u) \right)^T, \end{aligned} \quad (26)$$

$$\begin{aligned} (\nabla_{X^H} \ell) Y^T &= \left(\frac{1}{2}K(u, K_u^2 Y)X + 2(D_{X_u}^2 K)(Y, u)u \right. \\ &\quad \left. - 2K_u(K(u, Y)X) + \frac{3}{2}K_u^2(K(u, Y)X) \right)^H \\ &\quad + \left((D_X K)(K_u Y, u)u + K_u((D_X K)(Y, u)u) \right. \\ &\quad \left. - K(X, K'_u Y)u - K'_u(K(u, Y)X) \right)^T, \end{aligned} \quad (27)$$

$$\begin{aligned} (\nabla_{X^T} \ell) Y^H &= \left(4K(Y, X)u + 4K(Y, u)X + 2K(u, X)K_u Y - 3K_u(K(Y, X)u) \right. \\ &\quad \left. - 3K_u(K(Y, u)X) - 3K(K_u Y, X)u \right. \\ &\quad \left. - 3K(K_u Y, u)X - \frac{3}{2}K(u, X)K_u^2 Y \right. \\ &\quad \left. - 2K_u(K(u, X)Y) + \frac{3}{2}K_u^2(K(u, X)Y) \right)^H \\ &\quad + \left(2(D_X K)(Y, u)u + 2(D_u K)(Y, X)u \right. \\ &\quad \left. + 2(D_u K)(Y, u)X - K'_u(K(u, X)Y) \right)^T \\ &\quad + G(X, u) \left(12(K_u^2 Y)^H - 8(K_u Y)^H - 6(K'_u Y)^T \right), \end{aligned} \quad (28)$$

$$\begin{aligned} (\nabla_{X^T} \ell) Y^T &= \left(2(D_X K)(Y, u)u + 2(D_u K)(Y, X)u \right. \\ &\quad \left. + 2(D_u K)(Y, u)X + K(u, X)K'_u Y \right)^H \\ &\quad + \left(K_u(K(Y, X)u) + K_u(K(Y, u)X) \right. \\ &\quad \left. + K(K_u Y, X)u + K(K_u Y, u)X \right)^T \\ &\quad - G(X, u) \left(4(K_u^2 Y)^T + 6(K'_u Y)^H \right) \\ &\quad + G(Y, u) \left((K_u^2 X)^T + 2(K'_u X)^H \right). \end{aligned} \quad (29)$$

Now, we suppose that the Jacobi operator ℓ of T_1M is η -parallel. Then from (26)–(29), we obtain for tangent vectors X, Y, Z orthogonal to u :

$$\begin{aligned} 0 &= 4G((D_X K)(Y, u)u, Z) - 3G((D_X K)(K_u Y, u)u, Z) \\ &\quad - 3G(K_u((D_X K)(Y, u)u), Z) + G(K(u, K'_u Y)X, Z) \\ &\quad + G(K'_u(K(X, Y)u), Z), \end{aligned} \quad (30)$$

$$\begin{aligned} 0 &= -4G(K(X, K_u Y)u, Z) + 3G(K(X, K_u^2 Y)u, Z) \\ &\quad + 4G((D_{X_u}^2 K)(Y, u)u, Z) + G(K_u^2(K(X, Y)u), Z), \end{aligned} \quad (31)$$

$$\begin{aligned} 0 &= G(K(u, K_u^2 Y)X, Z) + 4G((D_{X_u}^2 K)(Y, u)u, Z) \\ &\quad - 4G(K_u(K(u, Y)X), Z) + 3G(K_u^2(K(u, Y)X), Z), \end{aligned} \quad (32)$$

$$\begin{aligned} 0 &= G((D_X K)(K_u Y, u)u, Z) + G(K_u((D_X K)(Y, u)u), Z) \\ &\quad - G(K(X, K'_u Y)u, Z) - G(K'_u(K(u, Y)X), Z), \end{aligned} \quad (33)$$

$$\begin{aligned} 0 &= 8G(K(Y, X)u, Z) + 8G(K(Y, u)X, Z) \\ &\quad + 4G(K(u, X)K_u Y, Z) - 6G(K_u(K(Y, X)u), Z) \\ &\quad - 6G(K_u(K(Y, u)X), Z) - 6G(K(K_u Y, X)u, Z) \\ &\quad - 6G(K(K_u Y, u)X, Z) - 3G(K(u, X)K_u^2 Y, Z) \\ &\quad - 4G(K_u(K(u, X)Y), Z) + 3G(K_u^2(K(u, X)Y), Z), \end{aligned} \quad (34)$$

$$\begin{aligned} 0 &= 2G((D_X K)(Y, u)u, Z) + 2G((D_u K)(Y, X)u, Z) \\ &\quad + 2G((D_u K)(Y, u)X, Z) - G(K'_u(K(u, X)Y), Z), \end{aligned} \quad (35)$$

$$\begin{aligned} 0 &= 2G((D_X K)(Y, u)u, Z) + 2G((D_u K)(Y, X)u, Z) \\ &\quad + 2G((D_u K)(Y, u)X, Z) + G(K(u, X)K'_u Y, Z), \end{aligned} \quad (36)$$

$$\begin{aligned} 0 &= G(K_u(K(Y, X)u), Z) + G(K_u(K(Y, u)X), Z) \\ &\quad + G(K(K_u Y, X)u, Z) + G(K(K_u Y, u)X, Z). \end{aligned} \quad (37)$$

If we multiply (33) by 3 and sum with (30), we get

$$\begin{aligned} 0 &= 4G((D_X K)(Y, u)u, Z) + G(K(u, K'_u Y)X, Z) \\ &\quad + G(K'_u(K(X, Y)u), Z) - 3G(K(X, K'_u Y)u, Z) \\ &\quad - 3G(K'_u(K(u, Y)X), Z). \end{aligned} \quad (38)$$

By using the first Bianchi identity, (38) is rewritten as follows.

$$0 = 4G((D_X K)(Y, u)u, Z) + 3G(K(u, X)K'_u Y, Z)$$

$$\begin{aligned}
& -G(K'_u(K(u, X)Y), Z) + 2G(K(K'_u Y, u)X, Z) \\
& - 2G(K'_u(K(u, Y)X), Z).
\end{aligned} \tag{39}$$

If we apply (35) and (36) in (39), we obtain

$$\begin{aligned}
0 &= 2G((D_X K)(Y, u)u, Z) + 2G((D_u K)(Y, X)u, Z) \\
&+ 4G((D_u K)(Y, u)X, Z) - G(K(K'_u Y, u)X, Z) \\
&+ 2G((D_Y K)(X, u)u, Z) + 2G((D_u K)(X, u)Y, Z).
\end{aligned} \tag{40}$$

We suppose that $X = Y = Z$ are orthogonal to u . Then from (40), we find that $(D_X K)(\cdot, X)X = 0$ for all tangent vectors X . From this, we conclude that the base manifold is locally symmetric. In the case when $\dim M = 2$, we at once see that M is of constant curvature.

Next, multiplying (37) by 6 and summing with (34), we get

$$\begin{aligned}
0 &= 8G(K(Y, X)u, Z) + 8G(K(Y, u)X, Z) \\
&+ 4G(K(u, X)K_u Y, Z) - 3G(K(u, X)K_u^2 Y, Z) \\
&- 4G(K_u(K(u, X)Y), Z) + 3G(K_u^2(K(u, X)Y), Z).
\end{aligned} \tag{41}$$

From (31) and (32), and using the fact that the base manifold M is locally symmetric, we obtain

$$\begin{aligned}
0 &= 8G(K(Y, X)u, Z) + 8G(K(Y, u)X, Z) \\
&- G(K_u^2(K(Z, Y)u), X) - G(K(u, K_u^2 X)Y, Z).
\end{aligned} \tag{42}$$

In (42), we put $Y = Z$. Then $G(K(Y, X)Y, u) = 0$ for any orthogonal triple u, X, Y . By CARTAN's theorem ([7]), the base manifold (M, G) must have constant curvature if $\dim M \geq 3$. We conclude that M is of constant curvature for all dimensions.

Conversely, we can use the expressions (22) to show that $T_1 M$ has η -parallel characteristic Jacobi-operator ℓ when the manifold M is of constant curvature c .

References

- [1] J. BERNDT and L. VANHECKE, Geodesic spheres and two-point homogeneous spaces, *Israel J. Math.* **93** (1996), 373–385.
- [2] D. E. BLAIR, Riemannian geometry of contact and symplectic manifolds, Progress in Math., 203, *Birkhäuser, Boston, Basel, Berlin*, 2002.

- [3] D. E. BLAIR, When is the tangent sphere bundle locally symmetric?, *Geometry and Topology*, World Scientific, Singapore **509** (1989), 15–30.
- [4] E. BOECKX and L. VANHECKE, Characteristic reflections on unit tangent sphere bundles, *Houston J. Math.* **23** (1997), 427–448.
- [5] E. BOECKX, D. PERRONE and L. VANHECKE, Unit tangent sphere bundles and two-point homogeneous spaces, *Periodica Math. Hungarica* **36** (1998), 79–95.
- [6] E. BOECKX and J. T. CHO, η -parallel contact metric spaces, *Diff. Geom. Appl.* **22** (2005), 275–285.
- [7] E. CARTAN, Leçons sur la géométrie des espaces de Riemann, *Gauthier-Villars, Paris*, 1946.
- [8] J. T. CHO and S. H. CHUN, On the classification of contact Riemannian manifolds satisfying the condition (C), *Glasgow Math. J.* **45** (2003), 99–113.
- [9] P. DOMBROWSKI, On the geometry of the tangent bundle, *J. Reine Angew. Math.* **210** (1962), 73–88.
- [10] A. GRAY, Classification des variétés approximativement kählériennes de courbure sectionnelle holomorphe constante, *J. Reine Angew. Math.* **279** (1974), 797–800.
- [11] O. KOWALSKI, Curvature of the induced Riemannian metric of the tangent bundle of a Riemannian manifold, *J. Reine Angew. Math.* **250** (1971), 124–129.
- [12] D. PERRONE, Tangent sphere bundles satisfying $\nabla_{\xi}\tau = 0$, *J. Geom.* **49** (1994), 178–188.
- [13] D. PERRONE, Torsion tensor and critical metrics on contact $(2n + 1)$ -manifolds, *Monatsh. Math.* **114** (1992), 245–259.
- [14] Y. TASHIRO, On contact structures of unit tangent sphere bundles, *Tôhoku Math. J.* **21** (1969), 117–143.
- [15] L. VANHECKE and T. J. WILLMORE, Interactions of tubes and spheres, *Math. Anal.* **21** (1983), 31–42.
- [16] K. YANO and S. ISHIHARA, Tangent and cotangent bundles, *M. Dekker Inc.*, 1973.

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