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A problem of Galambos on Oppenheim series expansions

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Abstract. In this paper, we investigate the Hausdorff dimension of exceptional sets in the metric properties of digits of Oppenheim series expansions and answer a question posed by Galambos.

1. Introduction

For any $x \in (0,1]$, the algorithm

$$x = x_1, \quad d_n = [1/x_n] + 1, \quad x_n = 1/d_n + a_n/b_n \cdot x_{n+1}, \tag{1}$$

where $a_n = a_n(d_1, \dots, d_n)$ and $b_n = b_n(d_1, \dots, d_n)$ are positive integer valued functions and $[y]$ denotes the integer part of y, leads to the Oppenheim expansion [12]

$$x \sim \frac{1}{d_1} + \frac{a_1}{b_1}\frac{1}{d_2} + \cdots + \frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n}\frac{1}{d_{n+1}} + \cdots. \tag{2}$$

By (1),

$$\frac{1}{d_n} < x_n \le \frac{1}{d_n - 1}, \tag{3}$$

and hence by the last equality in (1),

$$d_{n+1} > \frac{a_n}{b_n} d_n (d_n - 1). \tag{4}$$

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The expansion defined by (1) and (2) is convergent and its sum is equal to x. A sufficient condition for a series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is (see [12])

$$d_{n+1} \geq \frac{a_n}{b_n} d_n(d_n - 1) + 1 \quad \text{for all } n \geq 1. \tag{5}$$

Definition 1.1. We call the expansion (2) (obtained by the algorithm (1)) restricted Oppenheim expansion of x if a_n and b_n depend on the last denominator d_n only and if the function

$$h_n(j) = \frac{a_n(j)}{b_n(j)} j(j-1) \tag{6}$$

is integer-valued, for all $n \geq 1$ and $j \geq 2$.

In the present paper, we deal with restricted Oppenheim expansions only. In this case, (4) and (5) are equivalent.

The representation (2) under (1) was first studied by Oppenheim [12], including Lüroth ([11]), Engel, Sylvester expansions ([2]) and Cantor infinite product ([13]) as special cases. Oppenheim established the arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory of such expansions were laid down by Galambos [5], [6], [7], [9], see also the monographs of Galambos [8], Schweiger [14], Vervaat [15], Dajani and Kraaikamp [1]. From [8], Chapter 6, it can be seen that the integer approximations $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_{n-1}(x))$ defined by

$$T_n(x) < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq T_n(x) + 1, \quad n \geq 1, \tag{7}$$

where $h_0(x) \equiv 1$, plays an important role in the metric theory of Oppenheim expansions, see Galambos [8] Chapter VI. Moreover, they are stochastically independent and are distributed as the denominators in the Lüroth expansion. Galambos, see [8] Page 132, posed the question to calculate the Hausdorff dimension of the set

$$B_m = \{x \in (0,1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}, \quad m \geq 2,$$

and compare this with the Lüroth case. In [16], the second author concerned this problem under the condition $h_n(j)$ is of order t $(t \geq 1)$, see [16] for the definition. In this paper, we continue to consider this problem. Under more natural conditions, we obtain the Hausdorff dimension of B_m and thus answer the question of Galambos. To obtain the lower bound of the Huasdorff dimension of a fractal set, a mass distribution is needed, which is a necessary (and sufficient) tool for this. The mass distribution constructed here is quite technical and subtle.

We use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_H$ to denote the Hausdorff dimension and 'cl' the closure of a subset of $(0,1]$ respectively.

2. Hausdorff dimension of B_m

For any $m \geq 2$, let

$$B_m = \{x \in (0,1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}.$$

By (7), it is easy to check that

$$B_m = \left\{x \in (0,1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m+1 \text{ for all } n \geq 1\right\}, \tag{8}$$

where $h_0(n) \equiv 1$. Thus in order to calculate the Hausdorff dimensions of B_m, $m \geq 2$, it is sufficient to consider the following sets

$$C_m = \left\{x \in (0,1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m \text{ for all } n \geq 1\right\}, \quad m \geq 3.$$

From now on, we fix $m \geq 3$ be a positive integer.

Lemma 2.1. *For any integer $a \geq 1$, let $S(a)$ be determined by the following equation*

$$\sum_{a<b\leq ma} \left(\frac{a}{b(b-1)}\right)^{S(a)} = 1. \tag{9}$$

Then

$$\lim_{a\to+\infty} S(a) = 1.$$

Proof. Since

$$\sum_{a<b\leq ma} \left(\frac{a}{b(b-1)}\right) = 1 - \frac{1}{m} < 1,$$

we have $S(a) \leq 1$ for all $a \geq 1$.

On the other hand, for any $1/2 < s < 1$,

$$\sum_{a<b\leq ma} \left(\frac{a}{b(b-1)}\right)^s \geq \sum_{a\leq b\leq ma} \left(\frac{a}{b(b-1)}\right)^s - \left(\frac{1}{a-1}\right)^s$$

$$\geq \int_a^{ma} \frac{a^s}{x^{2s}} dx - \left(\frac{1}{a-1}\right)^s$$

$$= \frac{1}{1-2s}((ma)^{1-2s} - a^{1-2s}) \cdot a^s - \left(\frac{1}{a-1}\right)^s$$

$$= \frac{(1-m^{1-2s}) \cdot a^{1-s}}{2s-1} - \left(\frac{1}{a-1}\right)^s > 1, \ a \text{ is large enough.}$$

Thus when a is large enough, $S(a) > s$. The proof of Lemma 2.1 is finished. $\square$

We now state the mass distribution principle, see [4] Proposition 2.3, that will be used later.

Lemma 2.2. *Let $E \subset (0,1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for any $x \in E$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then $\dim_H E \geq s$.

Now we are in the position to prove the main result of this paper.

Theorem 2.3. *Suppose $h_j(d) \geq d-1$ for all $j \geq 1$ and $d \geq 2$, then for each $m \geq 3$,*

$$\dim_H C_m = 1.$$

Proof. For any $j \geq 1$ and $d \geq 2$, define

$$G_j(d) = m \cdot h_j(d);$$

$$M_j(m) = G_{j-1} \circ G_{j-2} \circ \cdots \circ G_1(m), \quad M_1(m) := m.$$

From the assumption on $h_j(d)$, it is easy to check that

$$M_j(m) \geq m^j - m^{j-1} - \cdots - m^2 - m \quad \text{for each } j \geq 1,$$

thus

$$\lim_{j \to \infty} M_j(m) = +\infty. \tag{10}$$

For any $0 < s < 1$, from Lemma 2.1, since $\lim_{a \to \infty} S(a) = 1$, there exists $a_0 \in \mathbb{N}$ such that for any $a \geq a_0$, $S(a) > s$. By (10), there exists $k_0 \geq 1$ such that for any $k \geq k_0$,

$$M_k(m) \geq a_0 + 1. \tag{11}$$

Define

$$E_m = \Big\{x \in (0,1] : d_j(x) = M_j(m) \text{ for all } 1 \le j \le k_0, \\ \text{and } 1 < \frac{d_{j+1}(x)}{h_j(d_j(x))} \le m \text{ for all } j \ge k_0\Big\}.$$

It is clear that $E_m \subset C_m$. Now we estimate the Hausdorff dimension of E_m.

For any $x \in E_m$, since $h_j(d) \ge d-1$ for all $j \ge 1$ and $d \ge 2$, by (5), we have, for any $k \ge k_0$,

$$\begin{aligned} d_k(x) &\ge h_{k-1}(d_{k-1}(x)) + 1 \ge d_{k-1}(x) \ge \cdots \ge d_{k_0+1}(x) \\ &\ge h_{k_0}(d_{k_0}(x)) + 1 \ge d_{k_0}(x) = M_{k_0}(m) \ge a_0 + 1, \end{aligned} \tag{12}$$

and

$$h_k(d_k(x)) \ge d_k(x) - 1 \ge a_0. \tag{13}$$

Now we introduce a symbolic space defined as follows:

For any $k \ge k_0$, let

$$D_k = \Big\{\sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{N}^k : \sigma_j = M_j(m) \text{ for all } 1 \le j \le k_0, \\ \text{and } 1 < \frac{\sigma_{j+1}}{h_j(\sigma_j)} \le m \text{ for all } k_0 \le j \le k-1\Big\},$$

and define

$$D = \bigcup_{k=k_0}^{\infty} D_k.$$

For any $k \ge k_0$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, let J_σ and I_σ denote the following closed subintervals of $(0,1]$:

$$J_\sigma = \bigcup_{h_k(\sigma_k) < d \le m h_k(\sigma_k)} \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, d_2(x) = \sigma_2, \dots, d_k(x) = \sigma_k, d_{k+1}(x) = d\},$$

$$I_\sigma = \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1,\ d_2(x) = \sigma_2, \dots, d_k(x) = \sigma_k\},$$

and each J_σ is called an interval of kth-order. Finally, define

$$E = \bigcap_{k=k_0}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma.$$

It is obvious that

$$E = E_m.$$

From the proof of Theorem 6.1 in [8], we have, for any $k \geq k_0$ and $\sigma \in D_k$,

$$|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k}, \tag{14}$$

thus by (6), we have

$$\begin{aligned}
|J_\sigma| &= \sum_{h_k(\sigma_k) < d \leq m h_k(\sigma_k)} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{(d-1)d} \\
&= \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left(\frac{1}{h_k(\sigma_k)} - \frac{1}{m h_k(\sigma_k)} \right) \\
&= \left(1 - \frac{1}{m}\right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{h_k(\sigma_k)} \\
&= \left(1 - \frac{1}{m}\right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k}.
\end{aligned} \tag{15}$$

For any $k \geq k_0$, $\sigma \in D_k$, define

$$\mu(J_\sigma) = \prod_{i=k_0}^{k-1} \left(\frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))}, \quad \text{if } k \geq k_0 + 1, \tag{16}$$

and

$$5\mu(J_\sigma) = 1, \quad \text{if } \sigma \in D_{k_0}.$$

μ is a probability mass distribution supported on E_m, because

$$\begin{aligned}
&\sum_{\sigma_{k+1} = h_k(\sigma_k)+1}^{m h_k(\sigma_k)} \mu(J_{\sigma_1 \sigma_2 \ldots \sigma_{k+1}}) \\
&\quad = \sum_{\sigma_{k+1} = h_k(\sigma_k)+1}^{m h_k(\sigma_k)} \prod_{i=k_0}^{k} \left(\frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))} = \mu(J_{\sigma_1 \sigma_2 \ldots \sigma_k}),
\end{aligned}$$

and

$$\sum_{\sigma_{k_0+1} = h_{k_0}(\sigma_{k_0})+1}^{m h_{k_0}(\sigma_{k_0})} \mu(J_{\sigma_1 \sigma_2 \ldots \sigma_{k_0+1}})$$

$$= \sum_{\sigma_{k_0+1}=h_{k_0}(\sigma_{k_0})+1}^{mh_{k_0}(\sigma_{k_0})} \left(\frac{h_{k_0}(\sigma_{k_0})}{\sigma_{k_0+1}(\sigma_{k_0+1}-1)} \right)^{S(h_{k_0}(\sigma_{k_0}))}$$

$$= 1 = \mu(J_{\sigma_1\sigma_2\ldots\sigma_{k_0}}).$$

For any $x \in E_m$, we prove that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s. \tag{17}$$

If (17) is proved, by Lemma 2.2, we have $\dim_H E_m \geq s$. Since $0 < s < 1$ is arbitrary, this implies $\dim_H C_m = 1$.

Now we prove (17).

For any $x \in E_m$, there exists $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)$ such that for any $k \geq k_0$, $(\sigma|k) := (\sigma_1, \sigma_2, \ldots, \sigma_k) \in D_k$ and $d_j(x) = \sigma_j$ for each $j \geq 1$. Thus

$$x \in J_{\sigma_1\sigma_2\ldots\sigma_k} \quad \text{for all } k \geq k_0.$$

From the proof of Theorem 6.1 in [8], we have, for any $k \geq k_0$, the right endpoint of the interval $J_{\sigma_1\sigma_2\ldots\sigma_k}$, i.e., $\max\{y \in (0,1] : y \in J_{\sigma_1\sigma_2\ldots\sigma_k}\}$, is

$$\begin{aligned}
&\frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{h_k(\sigma_k)} \\
&\quad = \frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} \\
&\qquad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k(\sigma_k - 1)} \\
&\quad = \frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{1}{\sigma_2} + \cdots + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k - 1}.
\end{aligned} \tag{18}$$

The left endpoint of the interval $J_{\sigma_1\sigma_2\ldots\sigma_k}$, i.e., $\min\{y \in (0,1] : y \in J_{\sigma_1\sigma_2\ldots\sigma_k}\}$, is

$$\begin{aligned}
&\frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{mh_k(\sigma_k)} \\
&\quad = \frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} \\
&\qquad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m\sigma_k(\sigma_k - 1)}
\end{aligned}$$

$$= \frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{1}{\sigma_2} + \dots$$
$$+ \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \left(\frac{1}{\sigma_k} + \frac{1}{m\sigma_k(\sigma_k - 1)} \right). \tag{19}$$

If $\sigma_k - 1 > h_{k-1}(\sigma_{k-1})$, from (18), (19), we know the gap between $J_{\sigma_1\sigma_2\dots\sigma_k}$ and $J_{\sigma_1\dots\sigma_{k-1}\sigma_k - 1}$ is

$$\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m(\sigma_k - 1)(\sigma_k - 2)}. \tag{20}$$

In the same way, if $\sigma_k + 1 \le mh_{k-1}(\sigma_{k-1})$, from (18), (19), we know the gap between $J_{\sigma_1\sigma_2\dots\sigma_k}$ and $J_{\sigma_1\dots\sigma_{k-1}\sigma_k+1}$ is

$$\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m\sigma_k(\sigma_k - 1)}. \tag{21}$$

For any $0 < r < \frac{1}{m}|I_{M_1(m)M_2(m)\dots M_{k_0}(m)}|$, since

$$(\sigma|k_0) = (M_1(m), M_2(m), \dots, M_{k_0}(m))$$

and $|I_{(\sigma|k)}| \to 0$ as $k \to \infty$, there exists k (depends on x) such that

$$\frac{1}{m}|I_{(\sigma|k+1)}| < r \le \frac{1}{m}|I_{(\sigma|k)}|,$$

that is,

$$\frac{1}{m}\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{\sigma_{k+1}(\sigma_{k+1} - 1)}$$
$$< r \le \frac{1}{m}\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k(\sigma_k - 1)}. \tag{22}$$

By (14),(20) and (21), $B(x, r)$ can intersect only one kth-order interval $J_{\sigma_1\sigma_2\dots\sigma_k}$.

On the other hand, for every $h_k(\sigma_k) < j \le mh_k(\sigma_k)$, from (14), we have

$$|I_{\sigma_1\sigma_2\dots\sigma_k j}| \ge \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{mh_k(\sigma_k)(mh_k(\sigma_k) - 1)}.$$

Thus $B(x, r)$ can intersect at most

$$\frac{4r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} := l$$

$(k+1)$-th-order intervals. Therefore

$$\mu(B(x,r)) \le \min\Big\{\mu(J_{\sigma_1\sigma_2\ldots\sigma_k}),\ \sum_i \mu(J_{\sigma_1\sigma_2\ldots\sigma_k i})\Big\},$$

where the sum is over all i such that $\max\{\sigma_{k+1}-l,\ h_k(\sigma_k)+1\} \le i \le \sigma_{k+1}+l$.

By (16), we have

$$\begin{aligned}
\mu(B(x,r)) &\le \mu(J_{\sigma_1\sigma_2\ldots\sigma_k})\min\left\{1,\ \sum_i \left(\frac{h_k(\sigma_k)}{i(i-1)}\right)^{S(h_k(\sigma_k))}\right\}\\
&\le \mu(J_{\sigma_1\sigma_2\ldots\sigma_k})\min\left\{1,\ 2l\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right\}\\
&= \mu(J_{\sigma_1\sigma_2\ldots\sigma_k})\min\left\{1,\ \frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right\}\\
&\le \mu(J_{\sigma_1\sigma_2\ldots\sigma_k})\cdot 1^{1-s}\cdot\left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right)^s.
\end{aligned}$$

From (13), we have, for any $n \ge k_0$,

$$h_n(\sigma_n) \ge a_0,$$

thus

$$S(h_n(\sigma_n)) \ge s \quad \text{for all } n \ge k_0. \tag{23}$$

Combining (6), (16) and (23), we have

$$\begin{aligned}
\mu(B(x,r)) &\le \left[\prod_{i=k_0}^{k-1}\frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1}-1)}\left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}\right)\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right]^s\\
&= \left(h_{k_0}(M_{k_0}(m))\frac{a_{k_0+1}(\sigma_{k_0+1})}{b_{k_0+1}(\sigma_{k_0+1})}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\frac{1}{\sigma_k(\sigma_k-1)}\right)^s\\
&\quad\cdot\left(\frac{8r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}\right)^s\cdot\left(\frac{1}{h_k(\sigma_k)}\right)^{sS(h_k(\sigma_k))}\\
&= \left(h_{k_0}(M_{k_0}(m))\cdot\frac{b_1(M_1(m))}{a_1(M_1(m))}\cdots\frac{b_{k_0}(M_{k_0}(m))}{a_{k_0}(M_{k_0}(m))}\right)^s
\end{aligned}$$

$$\cdot\left(\frac{8r(mh_k(\sigma_k))^2}{\sigma_k(\sigma_k-1)\frac{a_k(\sigma_k)}{b_k(\sigma_k)}}\cdot\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right)^s$$
$$\leq c_1^s\left(r\cdot h_k(\sigma_k)\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\right)^s,$$

where c_1 is a positive constant which does not depend on x and r.

From the definition of $S(a)$, we have

$$\begin{aligned}1&=\sum_{a<b\leq ma}\left(\frac{a}{b(b-1)}\right)^{S(a)}\\&\geq(m-1)a\left(\frac{a}{ma(ma-1)}\right)^{S(a)}\geq(m-1)a\left(\frac{a}{ma\cdot ma}\right)^{S(a)}\\&=(m-1)a\left(\frac{1}{m^2a}\right)^{S(a)},\end{aligned}$$

thus

$$\frac{a}{a^{S(a)}}\leq\frac{m^{2S(a)}}{m-1}\leq\frac{m^2}{m-1},$$

and this implies

$$h_k(\sigma_k)\left(\frac{1}{h_k(\sigma_k)}\right)^{S(h_k(\sigma_k))}\leq\frac{m^2}{m-1}.$$

Therefore

$$\mu(B(x,r))\leq c_2^s\cdot r^s, \tag{24}$$

where c_2 is a positive constant which does not depend on x and r.

From (24), we know (17) holds. This completes the proof of Theorem 2.3. □

From (8) and Theorem 2.3, we have

Corollary 2.4. *Suppose $h_j(d)\geq d-1$ for all $j\geq 1$ and $d\geq 2$, then for each $m\geq 2$, we have $\dim_H B_m=1$.*

Remark 2.5. Let $a_n(d_1,\dots,d_n)=1$, $b_n(d_1,\dots,d_n)=d_n(d_n-1)$, $(n=1,2,\dots)$. Then the algorithm (1) leads to the Lüroth expansion of x,

$$x=\frac{1}{d_1(x)}+\dots+\frac{1}{d_1(x)(d_1(x)-1)\dots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)}+\dots. \tag{25}$$

Here $h_n(j) = 1$ and $T_n(x) = d_n(x) - 1$. For the Lüroth series, with the help of the theory of self similar set, see [3], Chapter 9, the Hausdorff dimension s of the B_m is determined by the following equation

$$\sum_{2 \le b \le m+1} \left(\frac{1}{b(b-1)} \right)^s = 1.$$

To some extent, Lüroth series expansion stands as a special case to say that the assumption on h_j in the main theorem is not superfluous. Moreover, we can obtain: if $l \le h_j(d_j(x)) \le L$, for all $x \in C_m = B_{m-1}$ and j larger than some fixed integer k_0, then one can has

$$0 < \inf_{l \le a \le L} S(a) \le \dim_H C_m \le \sup_{l \le a \le L} S(a) < 1.$$

We now list some special cases which satisfy the assumption in Theorem 2.3.

Example 1. Engel expansion. Let $a_n(d_1, \dots, d_n) = 1$, $b_n(d_1, \dots, d_n) = d_n$, $(n = 1, 2, \dots)$. Then (2), together with the algorithm (1), become Engel expansion of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\dots d_n(x)} + \dots. \tag{26}$$

In this case, $h_n(j) = j - 1$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)-1} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)-1}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)-1}\right]$ otherwise. By Corollary 2.4, we have for each $m \ge 2$,

$$\dim_H\{x \in (0, 1] : 1 \le T_n(x) \le m \text{ for all } n \ge 1\} = 1.$$

Example 2. Sylvester expansion. Choose $a_n(d_1, \dots, d_n)=1$, $b_n(d_1, \dots, d_n)=1$, $(n = 1, 2, \dots)$. We get the Sylvester expansion of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_2(x)} + \cdots + \frac{1}{d_n(x)} + \dots. \tag{27}$$

Here $h_n(j) = j(j - 1)$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)}\right]$ otherwise. By Corollary 2.4, we have for each $m \ge 2$,

$$\dim_H\{x \in (0, 1] : 1 \le T_n(x) \le m \text{ for all } n \ge 1\} = 1.$$

Example 3. Cantor product. Take $a_n(d_1,\dots,d_n)=d_n+1$, $b_n(d_1,\dots,d_n)=d_n$, $(n=1,2,\dots)$, the expansion (2) yields the Cantor product,

$$1+x=\left(1+\frac{1}{d_1(x)}\right)\left(1+\frac{1}{d_2(x)}\right)\dots\left(1+\frac{1}{d_n(x)}\right)\dots. \tag{28}$$

Here $h_n(j)=j^2-1$ and $T_n(x)=\frac{d_n(x)}{d_{n-1}^2(x)-1}-1$ if $\frac{d_n(x)}{d_{n-1}^2(x)-1}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}^2(x)-1}\right]$ otherwise. By Corollary 2.4, we have for each $m\geq 2$,

$$\dim_H\{x\in(0,1]:1\leq T_n(x)\leq m \text{ for all } n\geq 1\}=1.$$

Example 4. Modified Engel expansion. Let $a_n(d_1,\dots,d_n)=1$, $b_n(d_1,\dots,d_n)=d_n-1$, $(n=1,2,\dots)$. We get the modified Engel expansion of x,

$$x=\frac{1}{d_1(x)}+\dots+\frac{1}{(d_1(x)-1)(d_2(x)-1)\dots(d_{n-1}(x)-1)d_n(x)}+\dots. \tag{29}$$

Thus $h_n(j)=j$ and $T_n(x)=\frac{d_n(x)}{d_{n-1}(x)}-1$ if $\frac{d_n(x)}{d_{n-1}(x)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)}\right]$ otherwise. By Corollary 2.4, we have for each $m\geq 2$,

$$\dim_H\{x\in(0,1]:1\leq T_n(x)\leq m \text{ for all } n\geq 1\}=1.$$

Example 5. Daróczy–Kátai-Birthday expansion. Choose $a_n(d_1,\dots,d_n)=d_n$, $b_n(d_1,\dots,d_n)=1$, $(n=1,2,\dots)$, the resulting series expansion of x takes the form,

$$x=\frac{1}{d_1(x)}+\frac{d_1(x)}{d_2(x)}+\dots+\frac{d_1(x)d_2(x)\dots d_{n-1}(x)}{d_n(x)}+\dots. \tag{30}$$

The Daróczy–Kátai-Birthday expansion was introduced for the first time in Galambos [9]. Here $h_n(j)=j^2(j-1)$ and $T_n(x)=\frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)}-1$ if $\frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}^2(x)(d_{n-1}(x)-1)}\right]$ otherwise. By Corollary 2.4, we have for each $m\geq 2$,

$$\dim_H\{x\in(0,1]:1\leq T_n(x)\leq m \text{ for all } n\geq 1\}=1.$$

Remark 2.6. A modification of (1) and (3) to the algorithm $0<x\leq 1$, $x=x_1$, and

$$\frac{1}{D_n+1}<x_n\leq\frac{1}{D_n},\quad \frac{1}{D_n}-x_n=\frac{a_n}{b_n}\cdot x_{n+1}. \tag{31}$$

generates an alternating series representation

$$x \sim \frac{1}{D_1} - \frac{a_1}{b_1}\frac{1}{D_2} + \cdots + (-1)^n \frac{a_1 a_2 \dots a_n}{b_1 b_2 \dots b_n}\frac{1}{D_{n+1}} + \dots, \tag{32}$$

called alternating Oppenheim expansion. The metric theory for the alternating Oppenheim expansion was studied recently in [10]. Using the same method, we can get the corresponding results of Theorem 2.3 and Corollary 2.4 for this expansion.

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