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On super quasi Einstein manifolds

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Abstract. The notion of a super quasi Einstein manifold is introduced and some properties of such a manifold are obtained.

Introduction

The notion of a quasi Einstein manifold was introduced in a recent paper [1] by the author and R. K. Maity. According to them a non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called quasi Einstein if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1}$$

where a, b are scalars of which $b \neq 0$, A is a non-zero 1-form such that

$$g(X,U) = A(X) \forall X \text{ and } U \text{ is a unit vector field.} \tag{2}$$

In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

Subsequently, the author introduced in another recent paper [2] a generalization of a $(QE)_n$, called a generalized quasi Einstein manifold which

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was defined as follows:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called genaralized quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \quad (3)$$

where a, b, c are scalars of which $b \neq 0$, $c \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X)\,\forall X \quad (4)$$

and U, V are two unit vector fields perpendicular to each other. In such a case a, b, c are called the associated scalars, A, B are called the associated main and auxiliary 1-forms and U, V are called the main and auxiliary generators of the manifold. Such an n-dimensional manifold was denoted by the symbol $G(QE)_n$.

This paper deals with super quasi Einstein manifolds which are defined as follows:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called super quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = {} & ag(X,Y) + bA(X)A(Y) \\ & + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y) \end{aligned} \quad (5)$$

where a, b, c, d are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X)\forall X \quad (6)$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0\,\forall X. \quad (7)$$

In such a case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold. Such an n-dimenisonal manifold shall be denoted by the symbol $S(QE)n$.

In this paper it is shown that the scalars $a+b$ and $a+dD(V,V)$ are the Ricci curvatures in the directions of the vector fields U and V respectively and the scalar d is less than the ratio which the length of the Ricci tensor S bears to the length of the associated tensor D. Further, some interesting properties of the curvature tensor R of type $(1,3)$ are obtained. Moreover, it is shown that a viscous fluid space time admitting heat flux and obeying Einstein's equation without cosmological constant is a 4-dimensional semi Riemannian super quasi Einstein manifold.

1. The associated scalars of a $S(QE)_n$, $(n \geq 3)$

In this section we consider a $S(QE)n$, $(n \geq 3)$ with associated scalars a, b, c, d associated main and auxiliary 1-forms A, B, main and auxiliary generators U, V and associated symmetric $(0,2)$ tensor D.

Then (5), (6) and (7) will hold. Since U and V are mutually orthogonal unit vector fields, we have

$$g(U,U)=1, \quad g(V,V)=1 \quad \text{and} \quad g(U,V)=0. \tag{1.1}$$

Further

$$\text{trace } D = 0 \tag{1.2}$$

$$D(X,U)=0\,\forall X. \tag{1.3}$$

In virtue of (4), $g(U,V)=0$ can be expressed as

$$A(V)=B(U)=0. \tag{1.4}$$

Now contracting (5) over X and Y we get

$$r = na+b \tag{1.5}$$

where r is the scalar curvature. Again from (5) we get

$$S(U,U)=a+b, \tag{1.6}$$

$$S(V,V) = a + dD(V,V) \qquad \text{and} \tag{1.7}$$

$$S(U,V) = c. \tag{1.8}$$

If X is a unit vector field, then $S(X,X)$ is the Ricci curvature in the direction of X. Hence from (1.6) and (1.7) we can state that $a+b$ and $a + dD(V,V)$ are the Ricci curvatures in the directions of U and V respectively. Let

$$g(LX,Y) = S(X,Y) \qquad \text{and} \tag{1.9}$$

$$g(\ell X,Y) = D(X,Y). \tag{1.10}$$

Further, let ${d_1}^2$, and ${d_2}^2$ denote the squares of the lengths of the Ricci tensor S and the associated tensor D. Then

$${d_1}^2 = S\left(Le_i, e_i\right) \qquad \text{and} \tag{1.11}$$

$${d_2}^2 = D\left(\ell e_i, e_i\right) \tag{1.12}$$

where $\{e_i\}$ $i = 1, 2, \ldots, n$ is an orthonormal basis of the tangent space at a point of $S(QE)n$. Now from (5) we get

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + dS(\ell e_i, e_i). \tag{1.13}$$

Again from (5) we have

$$S(\ell e_i, e_i) = dD(\ell e_i, e_i). \tag{1.14}$$

From (1.13) and (1.14) it follows that

$$S(Le_i, e_i) = (n-1)a^2 + (a+b)^2 + d^2 D(\ell e_i, e_i). \tag{1.15}$$

Hence

$${d_1}^2 = (n-1)a^2 + (a+b)^2 + d^2(d_2)^2. \tag{1.16}$$

From (1.16) we can write ${d_1}^2 - d^2{d_2}^2 > 0$. Hence

$$d < \frac{d_1}{d_2}. \tag{1.17}$$

Summing up we can state the following theorem:

Theorem 1. *In a $S(QE)n$ $(n \geq 3)$ the scalars $a+b$ and $a+d\ D(V,V)$ are the Ricci curvatures in the directions of the generators U and V respectively and the associated scalar d is less than the ratio which the length of the Ricci tensor S bears to the length of the associated tensor D.*

2. Conformally flat $S(QE)n$ $(n > 3)$

Let R be the curvature tensor of type $(1,3)$ of a conformally flat $S(QE)n$ $(n > 3)$. Then

$$\begin{aligned} {}'R(X,Y,Z,W) = \frac{1}{n-2}&[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+ g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned} \tag{2.1}$$

where ${}'R(X,Y,Z,W) = g[R(X,Y,Z),W]$. Using (5) we can express (2.1) as follows:

$$\begin{aligned} {}'R(X,Y,Z,W) = a'&[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ b'[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &+ c'[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ &- g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ &+ g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &- g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\ &+ d'[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)] \end{aligned} \tag{2.2}$$

where

$$a' = \frac{2a(n-1) - r}{(n-1)(n-2)}, \quad b' = \frac{b}{n-2}, \quad c' = \frac{c}{n-2}, \quad d' = \frac{d}{n-2}. \tag{2.3}$$

Let $U^{\perp}$ denote the $(n-1)$-dimensional distribution orthogonal to U in a conformally flat $S(QE)n$. Then $g(X,U)=0$ if $X \in U^{\perp}$. Again if $g(X,U)=0$, then $X \in U^{\perp}$. Hence from (2.2) we get the following properties of R:

$$\begin{aligned} R(X,Y,Z) = {} & \lambda[g(Y,Z)X - g(X,Z)Y] \\ & + c'[g(Y,Z)B(X) - g(X,Z)B(Y)]U \\ & + d'[D(Y,Z)X - D(X,Z)Y + g(Y,Z)\ell X - g(X,Z)\ell Y] \end{aligned} \tag{2.4}$$

when $X, Y, Z \in U^{\perp}$ and

$$R(X,U,U) = \mu X + d'\ell X \quad \text{when} \quad X \in U^{\perp} \tag{2.5}$$

where $\lambda = \frac{(n-2)a-b}{(n-1)(n-2)}$, $\mu = \frac{a+b}{n-1}$ and c', d' have values given by (2.3). We can therefore state as follows:

Theorem 2. *In a conformally flat* $S(QE)n$ $(n > 3)$ *the curvature tensor* R *of type* $(1,3)$ *satisfies the properties given by* (2.4) *and* (2.5).

3. General relativistic viscous fluid space time admitting heat flux

Let (M^4, g) be a viscous fluid space time admitting heat flux and satisfying Einstein's equation without cosmological constant. Further, let U be the unit timelike velocity vector field of the fluid, V be the unit heat flux vector field and D be the anisotropic pressure tensor of the fluid. Then

$$g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0 \tag{3.1}$$

$$D(X,Y) = D(Y,X), \quad \text{trace } D = 0 \quad \text{and} \tag{3.2}$$

$$D(X,U) = 0 \, \forall X. \tag{3.3}$$

Let

$$g(X,U) = A(X), \quad g(X,V) = B(X) \quad \forall X. \tag{3.4}$$

Further, let T be the $(0,2)$ type of energy momentum tensor describing the matter distribution of such a fluid. Then [3]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y)$$

$$+ [A(X)B(Y) + A(Y)B(X)] + D(X,Y) \tag{3.5}$$

where σ, p denote the density and isotropic pressure and D denotes the anisotropic pressure tensor of the fluid.

It is known [4] that Einstein's equation without cosmological constant can be written as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) = kT(X,Y) \tag{3.6}$$

where k is the gravitational constant and T is the energy momentum tensor of type $(0,2)$.

In the present case (3.6) can be written as follows:

$$\begin{aligned} S(X,Y) - \frac{1}{2}rg(X,Y) = k[&(\sigma + p)A(X)A(Y) + pg(X,Y) \\ &+ A(X)B(Y) + A(Y)B(X) + D(X,Y)]. \end{aligned}$$

Hence

$$\begin{aligned} S(X,Y) = &\left(kp + \frac{1}{2}r\right) g(X,Y) + k(\sigma + p)A(X)A(Y) \\ &+ k[A(X)B(Y) + A(Y)B(X)] + kD(X,Y). \end{aligned} \tag{3.7}$$

From (3.7) it follows that the space time under consideration is a super quasi Einstein manifold with $kp+\frac{1}{2}r$, $k(\sigma+p)$, k, k as associated scalars, A and B as associated 1-forms, U, V as generators and D as the associated symmetric $(0,2)$ tensor.

Hence we can state the following theorem

Theorem 3. *A viscous fluid spacetime admitting heat flux and satisfying Einstein's equation without cosmological constant is a 4-dimensional semi Riemannian super quasi Einstein manifold.*

References

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