

Fiberings on almost r -contact manifolds

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Abstract. The differentiable manifolds with almost contact structures were studied by many authors, for example: D.E. BLAIR [5], S. SASAKI [6], S.I. GOLDBERG and K. YANO [7], S. ISHIHARA [8], and others.

The purpose of this paper is to study strictly regular invariant r -contact manifolds related to an almost r -contact structure. This structure induces a principal bundle whose base manifold bears an almost complex structure. This paper is devoted to study the relations between the integrability conditions of almost r -contact structure and the induced complex structure. The motivation of this paper is to extend some results obtained by K. OGUIE [4].

1. Introduction

Let V_n be a C^∞ real differentiable manifold of dimension $n(= 2m+r)$. Let $\mathcal{J}(V_n)$ denote the ring of real valued differentiable functions on V_n and let $\mathcal{X}(V_n)$ be the module of the derivations of $\mathcal{J}(V_n)$. Let V_n be equipped with tensor field Φ which is a linear map $\Phi : \mathcal{X}(V_n) \rightarrow \mathcal{X}(V_n)$.

Let there be $r(C^\infty)$ 1-forms $A_1, A_2, \dots, A_r$ and $r(C^\infty)$ contravariant vector fields $T^1, T^2, \dots, T^r$ satisfying the following conditions [1]:

$$(1.1)a \quad \Phi^2 X = -X + \sum_{p=1}^r A_p(X) T^p \quad \text{for all } X \in \mathcal{X}(V_n).$$

Let us put

$$(1.1)b \quad \bar{X} = \Phi(X),$$

$$(1.2) \quad \Phi(T^p) = 0 \quad \text{for } p = 1, 2, \dots, r,$$

$$(1.3) \quad A_p(\bar{X}) = 0 \quad \text{for all } X \in \mathcal{X}(V_n),$$

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$$(1.4) \quad A_p(T^p) = 1.$$

Let a Riemannian metric g associated with Riemannian manifold V_n satisfy [1].

$$(1.5)$$

$$g(\bar{X}, \bar{Y}) = g(X, Y) - \sum_{p=1}^r A_p(X)A_p(Y), \quad \text{for all } X, Y \in \mathcal{X}(V_n).$$

$$(1.6) \quad A_q(X) = g(T^q, X), \quad \text{for } q = 1, 2, \dots, r.$$

Such a manifold satisfying all the conditions from (1.1)a to (1.6) is called an almost r -contact metric manifold. The structure endowed in V_n is called (Φ, A_p, T^p, g) -structure. Let ∇ be the Riemannian connection given by [9]

$$(1.7) \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

and the Lie derivative L_X defined by

$$(L_X \Phi)Y = [X, \Phi Y] - \Phi[X, Y].$$

2. In this section we give some definitions and obtain some results:

Definition 2.1. An almost r -contact structure (Φ, A_p, T^p, g) is said to be invariant strictly regular if contravariant vector fields $T^1, T^2, \dots, T^r$ are strictly regular vector fields, Φ and $r(C^\infty)$ 1-forms $A_1, A_2, \dots, A_r$ are invariant under the action of G , generated by $T^1, T^2, \dots, T^r$.

Definition 2.2. (V_n, M, G, Λ) is a principal G -bundle over M if M is an orbit space of $r(C^\infty)$ 1-forms and Λ is the natural projection of V_n onto M . If we define a (1,1) tensor field J and a (0,2) type tensor field g on M by

$$J_{\Lambda(x)} \bar{X} = d\Lambda \Phi_x \left(\bar{X}_x^H \right)$$

where $\bar{X}^H$ is the horizontal lift of $\bar{X} \in \mathcal{X}(M)$ at $x \in M$ with respect to $A_1, A_2, \dots, A_r$.

Let us put

$$\bar{g}(\bar{X}, \bar{Y}) \stackrel{\text{def}}{=} g(\bar{X}^H, \bar{Y}^H)$$

then $(J, \bar{g})$ is an almost Hermitian structure on M .

Theorem 2.1. If we define a (1,1) tensor field J on M by

$$(2.1) \quad J_{\Lambda(x)}(\bar{X}) = d\Lambda \Phi_x \left(\bar{X}_x^H \right)$$

where $\overset{*}{X}_x^H$ denotes the horizontal lift of $\overset{*}{X} \in \mathcal{X}(M)$ at the point $x \in M$ with respect to $A_1, A_2, \dots, A_r$, then J is almost complex structure on M .

PROOF. For an $\overset{*}{X} \in \mathcal{X}(M)$ we have

$$\begin{aligned} J_{\Lambda(x)}^2(\overset{*}{X}) &= J_{\Lambda(x)} \left(J_{\Lambda(x)}(\overset{*}{X}) \right) = J_{\Lambda(x)} \left(d\Lambda\Phi_x \left(\overset{*}{X}_x^H \right) \right) \\ &= d\Lambda\Phi_x \left(d\Lambda\Phi_x \left(\overset{*}{X}_x^H \right) \right)^{*H} = d\Lambda\Phi\Phi \left(\overset{*}{X}_x^H \right) \end{aligned}$$

On making use of (1.1)a we obtain

$$= d\Lambda \left[-\overset{*}{X}^H + \sum_{p=1}^r A_p \left(\overset{*}{X}^H \right) T^p \right] = -d\Lambda \left(\overset{*}{X}^H \right) = -\overset{*}{X}.$$

Hence J is an almost complex structure on M .

3. Integrability and normality

We define tensor fields $\overset{*}{N}$ and $\overset{*}{\Phi}$ on V_n as follows:

$$(3.1) \quad \overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) = \left[J\overset{*}{X}, J\overset{*}{Y} \right] - J \left[J\overset{*}{X}, \overset{*}{Y} \right] - J \left[\overset{*}{X}, J\overset{*}{Y} \right] + J^2 \left[\overset{*}{X}, \overset{*}{Y} \right],$$

$$\forall \overset{*}{X}, \overset{*}{Y} \in \mathcal{X}(M).$$

$$(3.2) \quad \overset{*}{J} \left(\overset{*}{X}, \overset{*}{Y} \right) = \overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) + 2 \sum_{p=1}^r dA_p \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) T^p$$

and $N(X, Y)$ the Nijenhuis tensor of Φ is given by [2].

$$(3.3) \quad N(X, Y) = [\bar{X}, \bar{Y}] + [X, Y] - [X, \bar{Y}] - [\bar{X}, Y]$$

we now prove the following:

Lemma 3.1. *We have the following*

$$\begin{aligned} \overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) &= d\Lambda N \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \\ \text{and} \quad \overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) &= d\Lambda\overset{*}{\Phi} \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \end{aligned}$$

where $\overset{*}{X}^H$ denotes the lift of X with respect to A' 's.

PROOF. We have from (3.1)

$$\overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) = [J\overset{*}{X}, J\overset{*}{Y}] - J[J\overset{*}{X}, \overset{*}{Y}] - J[\overset{*}{X}, J\overset{*}{Y}] + J^2[\overset{*}{X}, \overset{*}{Y}]$$

Making use of (2.1), we obtain

$$\begin{aligned} &= \left[d\Lambda\Phi\overset{*}{X}^H, d\Lambda\Phi\overset{*}{Y}^H \right] - d\Lambda\Phi \left[d\Lambda\Phi\overset{*}{X}^H, d\Lambda\Phi\overset{*}{Y}^H \right]^{\ast H} \\ &\quad - d\Lambda\Phi \left[d\Lambda\overset{*}{X}^H, d\Lambda\Phi\overset{*}{Y}^H \right]^{\ast H} + d\Lambda\Phi \left(d\Lambda\Phi \left[\overset{*}{X}, \overset{*}{Y} \right]^{\ast H} \right)^{\ast H} \\ &= d\Lambda \left[\Phi\overset{*}{X}^H, \Phi\overset{*}{Y}^H \right] - d\Lambda\Phi \left[\Phi\overset{*}{X}^H, \overset{*}{Y}^H \right] - d\Lambda\Phi \left[\overset{*}{X}^H, \Phi\overset{*}{Y}^H \right] \\ &\quad + d\Lambda\Phi^2 \left[\overset{*}{X}, \overset{*}{Y} \right]^{\ast H} \\ &= d\Lambda \left\{ \left[\Phi\overset{*}{X}^H, \Phi\overset{*}{Y}^H \right] - \Phi \left[\Phi\overset{*}{X}^H, \overset{*}{Y}^H \right] - \Phi \left[\overset{*}{X}^H, \Phi\overset{*}{Y}^H \right] + \Phi^2 \left[\overset{*}{X}, \overset{*}{Y} \right] \right\} \\ &= d\Lambda N \left[\overset{*}{X}^H, \overset{*}{Y}^H \right] \end{aligned}$$

The second relation follows immediately from first and from the fact that $T^1, T^2, \dots, T^r$ are vertical.

Theorem 3.1. $\overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) = 0$ if and only if $N \left(\overset{*}{X}^H, \overset{*}{Y}^H \right)$ is vertical for all $\overset{*}{X}, \overset{*}{Y} \in \mathcal{X}(M)$.

PROOF. In view of Lemma 3.1, this follows in an obvious manner.

Definition 3.1. An almost r -contact structure is said to be integrable if $N = 0$ and is said to be normal if $\overset{*}{\Phi} = 0$.

Definition 3.2. The connection forms $A_1, A_2, \dots, A_r$ are involutive if $[X, Y]$ is horizontal for all horizontal vector fields X and Y .

Theorem 3.3. The almost r -contact structure is integrable if J is integrable and the connection forms $A_1, A_2, \dots, A_r$ are involutive.

PROOF. By Theorem 3.1 $\overset{*}{N} \left(\overset{*}{X}, \overset{*}{Y} \right) = 0$ implies that

$$N \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \quad \text{is vertical for all} \quad X, Y \in \mathcal{X}(M).$$

On the other hand from (3.1) we have

$$\sum_{p=1}^r A_p \left(N \left[\overset{*}{X}^H, \overset{*}{Y}^H \right] \right) = \sum_{p=1}^r A_p \left(\left[\Phi \overset{*}{X}^H, \Phi \overset{*}{Y}^H \right] \right) = 0,$$

since $\Phi \overset{*}{X}^H$, and $\Phi \overset{*}{Y}^H$ are horizontal. This shows that $N \left(\overset{*}{X}^H, \overset{*}{Y}^H \right)$ is horizontal. Hence we have $N \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) = 0$.

Now on the other hand it is clear that $\left[T^p, \overset{*}{X}^H \right] = 0$. Since $\overset{*}{X}^H$ is invariant under the action of the parameter group G generated by $T^1, T^2, \dots, T^p$. Hence it is easily seen that $N \left(T^p, \overset{*}{X}^H \right) = 0$. We have thus proved that $N(X, Y) = 0$ is valid for the lifts of vector fields on M and the vertical vector fields. Since N is a tensor field, $N(X, Y) = 0$ holds for any vector fields X and Y . The converse is clear.

Theorem 3.4. *The almost r -contact structure on V_n is normal if and only if J is integrable and*

$$\sum_{p=1}^r \left(J \overset{*}{X}, J \overset{*}{Y} \right) = \sum_{p=1}^r \left(\overset{*}{X}, \overset{*}{Y} \right) \quad \text{for all } \overset{*}{X}, \overset{*}{Y} \in \mathcal{X}(M),$$

where

$$\sum_{p=1}^r dA_p \left(\Phi \overset{*}{X}, \Phi \overset{*}{Y} \right) = \left(\Lambda \sum_{p=1}^r \right) \left(\overset{*}{X}, \overset{*}{Y} \right).$$

PROOF. By Theorem 3.2, $N(X, Y) = 0$ implies that $\Phi \left(\overset{*}{X}^H, \overset{*}{Y}^H \right)$ is vertical for all $\overset{*}{X}, \overset{*}{Y} \in \mathcal{X}(M)$. Also from (3.2) we have

$$\begin{aligned} \sum_{p=1}^r A_p \left(\Phi \left(\overset{*}{X}^H \overset{*}{Y}^H \right) \right) &= \sum_{p=1}^r A_p \left(\left[\Phi \overset{*}{X}^H, \Phi \overset{*}{Y}^H \right] \right) + 2 \sum_{p=1}^r dA_p \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \\ &= -2 \sum_{p=1}^r A_p \left(\Phi \overset{*}{X}^H, \Phi \overset{*}{Y}^H \right) + 2 \sum_{p=1}^r dA_p \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \\ &= -2 \left(\Lambda \sum_{p=1}^r \right) \left(\Phi \overset{*}{X}^H, \Phi \overset{*}{Y}^H \right) + 2 \left(\Lambda \sum_{p=1}^r \right) \left(\overset{*}{X}^H, \overset{*}{Y}^H \right) \end{aligned}$$

$$= -2 \sum_{p=1}^r \left(JX^*, JY^* \right) + 2 \sum_{p=1}^r \left(X^*, Y^* \right) = 0.$$

This shows that Φ^* is horizontal. Hence we have $\Phi^* = 0$. Now it is clear that $\sum_{p=1}^r dA_p \left(T^p, X^H \right) = 0$, since

$$\begin{aligned} & 2 \sum_{p=1}^r dA_p \left(T^p, X^H \right) \\ &= \sum_{p=1}^r T^p \cdot A_p \left(X^H \right) - X^H \sum_{p=1}^r A_p (T^p) - \sum_{p=1}^r A_p \left(\left[T^p, X^H \right] \right) \end{aligned}$$

Hence we have

$$\Phi^* \left(T^p, X^H \right) = N \left(T^p, X^H \right) + 2 \sum_{p=1}^r dA_p \left(T^p, X^H \right) T^p = 0$$

Thus $\Phi^*(X, Y) = 0$ is valid for the lifts of vector fields on M . The converse is clear.

References

- [1] J. VANZURA, Almost r -contact structure, *Annal Della Scuola Normale Superpipe di PISA SENE* **XXVI** (1972).
- [2] K. YANO and S. ISHIHARA, Tangent and Cotangent Bundles, *Marcel Dekker Inc., New York*, 1973.
- [3] P. STAVRE, On fibering and F structure, *Bull Math Roum* **15** (63) nr. 4 (1971).
- [4] K. OGIUE, On fiberings on almost contact manifold, *Kodai Math. Sem.* **17** (1965), 53.
- [5] D.E. BLAIR, The theory of quasi-Sasakian structures, *J. Differential Geometry* **1** (1967), 331–345.
- [6] S. SASAKI, Almost contact manifolds, *Tohoku Univ.*, 1968.
- [7] S.I. GOLDBERG and K. YANO, Nonvariant hypersurfaces of almost contact manifolds, *J. Math Society Japan* **22** (1970), 25–34.
- [8] S. ISHIHARA, On a thensor field Φ_j^i satisfying $\Phi^p = \pm I$, *Tohoku Math. J.* **13** (1961), 443–454.
- [9] S. HELGASON, Differential Geometry, Lie Groups And Symmetric spaces, *Academic Press, New York*, 1978.
- [10] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry, Vol. I., *Interscience Publishers*, 1963.
- [11] LOVEJOY S. DAS, Prolongations of F -structure to the tangent bundle of order 2, *International Journal of Mathematics and Mathematical Sciences, U.S.A.* **15** (1992).
- [12] LOVEJOY S. DAS, Complete lift of F -structure manifold, *Kyung Pook Mathematical Journal* **20** (1980), 231–237.
- [13] LOVEJOY S. DAS, On almost r -contact structures with derivations, *Comptes Rendus* **44** (1991).

- [14] LOVEJOY S. DAS, F -structure manifold with some linear connections, *Commentationes Mathematicae* **XXXVI** (1992).

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