

A result on distributions and the change of variable

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Abstract. Let F be a distribution in $\mathcal{D}'$ and let f be a differentiable function such that $f^{(p+1)}$ is a locally summable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b) . It is proved that if F is the p -th derivative of a continuous function $F^{(-p)}$ on the interval $(f(a), (f(b)),$ (or $(f(b), f(a))$), then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle G, \varphi \rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b) , where $F_n(x) = (F * \delta_n)(x)$. This defines the distribution $F(f) = G$ on the interval (a, b) . Some examples are given.

In the following, we let N be the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\varrho(x)$ be any infinitely differentiable function having the following properties:

- (i) $\varrho(x) = 0$ for $|x| \geq 1$,
- (ii) $\varrho(x) \geq 0$,
- (iii) $\varrho(x) = \varrho(-x)$,
- (iv) $\int_{-1}^1 \varrho(x) dx = 1$.

Putting $\delta_n(x) = n\varrho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if F is an arbitrary distribution in $\mathcal{D}'$, we define

$$F_n(x) = (F * \delta_n)(x) = \langle F(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $F(x)$.

The following definition for the change of variable in distributions was given in [2].

Definition 1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that distribution $F(f(x))$ exists and is equal to the distribution G on the interval (a, b) if

$$N - \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle G, \varphi \rangle$$

for all test functions φ in $\mathcal{D}$ with support contained in the interval (a, b) , where

$$F_n(x) = (F * \delta_n)(x).$$

We now prove an existence theorem for a distribution $F(f(x))$ which in fact does not need a neutrix limit.

Theorem 1. Let F be a distribution in $\mathcal{D}'$ and let f be a differentiable function such that $f^{(p+1)}$ is a locally summable function with $f'(x) > 0$, (or < 0), for all x in the interval (a, b) . If F is the p -th derivative of a continuous function $F^{(-p)}$ on the interval $(f(a), f(b))$, (or $(f(b), f(a))$), then the distribution $F(f(x))$ exists on the interval (a, b) and

$$\begin{aligned} (1) \quad \langle F(f(x)), \varphi(x) \rangle &= (-1)^p \text{sgn}.g' \int_{-\infty}^{\infty} F^{(-p)}(x) \frac{d^p}{dx^p} [g'(x) \varphi(g(x))] dx \\ (2) \quad &= (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x)) |f'(x)| \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^p \left[\frac{\varphi(x)}{f'(x)} \right] dx \end{aligned}$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b) , where g is the inverse of f on the interval (a, b) and

$$\text{sgn}.g' = \begin{cases} 1, & g'(x) > 0, \\ -1, & g'(x) < 0. \end{cases}$$

Alternatively, if $F^{(-p)}$ is only a locally summable function but either $f^{(p+1)}$ or $F^{(-p)}$ is a bounded, locally summable function on every bounded subset of (a, b) and $(f(a), f(b))$, (or $f(b), f(a)$), respectively, then $F(f(x))$ again exists and equations (1) and (2) are satisfied.

In particular, if f is infinitely differentiable, then $F(f(x))$ is defined for every distribution F .

PROOF. Suppose first of all that $f^{(p+1)}$ is a locally summable function and $F^{(-p)}$ is a continuous function. Letting φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b) and making the substitution $t = f(x)$, we have

$$\begin{aligned}\langle F_n(f(x)), \varphi(x) \rangle &= \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx \\ &= \text{sgn.} g' \int_{-\infty}^{\infty} F_n(t) g'(t) \varphi(g(t)) dt\end{aligned}$$

Integrating by parts p times we get

$$\begin{aligned}\langle F_n(f(x)), \varphi(x) \rangle &= (-1)^p \text{sgn.} g' \int_{-\infty}^{\infty} F_n^{(-p)}(t) \frac{d^p}{dt^p} [g'(t) \varphi(g(t))] dt \\ &= (-1)^p \int_{-\infty}^{\infty} F_n^{(-p)}(f(x)) |f'(x)| \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^p \left[\frac{\varphi(x)}{f'(x)} \right] dx.\end{aligned}$$

Now $F_n^{(-p)}(t)$ and $F_n^{(-p)}(f(x))$ are continuous functions converging to the continuous functions $F^{(-p)}(t)$ and $F^{(-p)}(f(x))$ respectively as n tends to infinity and

$$\frac{d^p}{dt^p} [g'(t) \varphi(g(t))], \quad |f'(x)| \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^p \left[\frac{\varphi(x)}{f'(x)} \right]$$

are locally summable functions. It follows that

$$(3) \quad \lim_{n \rightarrow \infty} \langle F_n(f(x)), \varphi(x) \rangle = (-1)^p \text{sgn.} g' \int_{-\infty}^{\infty} F^{(-p)}(t) \frac{d^p}{dt^p} [g'(t) \varphi(g(t))] dt$$

$$(4) \quad = (-1)^p \int_{-\infty}^{\infty} F^{(-p)}(f(x)) |f'(x)| \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^p \left[\frac{\varphi(x)}{f'(x)} \right] dx$$

and equations (1) and (2) follow.

Now suppose that either $f^{(p+1)}$ or $F^{(-p)}$ are bounded functions on bounded subsets. Then

$$F^{(-p)}(t) \frac{d^p}{dt^p} [g'(t) \varphi(g(t))], \quad F^{(-p)}(f(x)) |f'(x)| \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^p \left[\frac{\varphi(x)}{f'(x)} \right]$$

will be locally summable functions, ensuring that equations (3) and (4) still hold.

In the corollary the product of $\delta^{(r)}$ and an r times continuously differentiable function f is defined by

$$(5) \quad f(x)\delta^{(r)}(x) = \sum_{i=0}^r (-1)^{(r+i)} \binom{r}{i} f^{(r-i)}(0)\delta^{(i)}(x).$$

Corollary. *Let f be a continuously differentiable function having a single simple root at the point $x = \alpha$. Suppose that $f^{(p+2)}$ is a locally summable function on a neighbourhood of the point $x = \alpha$. Then*

$$(6) \quad \delta^{(r)}(f(x)) = \frac{1}{|f'(\alpha)|} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x - \alpha)$$

on the real line for $r = 0, 1, 2, \dots, p$.

PROOF. We will suppose that $\alpha = 0$ and the general result will then follow by translation. Note that $\delta^{(p)}(f(x))$ is then equal to 0 on any interval not containing the origin. We therefore only have to prove the result on some neighbourhood of the origin. Since $x = 0$ is a simple root of f there exists a neighbourhood (a, b) containing the origin on which $f'(x) > 0$, (or < 0). Now $\delta^{(r)}$ is the $(r+1)$ -th derivative of the bounded, locally summable function $H(x)$, where H denotes Heaviside's function, and so the conditions of the theorem are satisfied for the distribution $\delta^{(r)}(f(x))$ on the interval (a, b) for $r = 0, 1, 2, \dots, p$. The existence of $\delta^{(r)}(f(x))$ follows by the theorem.

Now let φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval (a, b) . Then since $\delta^{(r)}(x)$ is the $(r+2)$ -th derivative of the continuous function x_+ and supposing that $f'(x) > 0$, we have from equation (2)

$$(7) \quad \begin{aligned} \langle \delta^{(r)}(f(x)), \varphi(x) \rangle &= (-1)^{r+1} \int_0^\infty f'(x) \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r+1} \left[\frac{\varphi(x)}{f'(x)} \right] dx \\ &= -(-1)^{r+1} \int_0^\infty d \left\{ \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \left[\frac{\varphi(x)}{f'(x)} \right] \right\} \\ &= \frac{(-1)^r}{f'(0)} \frac{d}{dx} \left\{ \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r-1} \left[\frac{\varphi(x)}{f'(x)} \right] \right\} \Big|_{x=0} \end{aligned}$$

Further

$$(8) \quad \begin{aligned} &\left\langle \frac{1}{f'(0)} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x), \varphi(x) \right\rangle = \\ &= -\frac{1}{f'(0)} \left\langle \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r-1} \delta(x), \frac{d}{dx} \frac{\varphi(x)}{f'(x)} \right\rangle \end{aligned}$$

$$= \frac{(-1)^r}{f'(0)} \left\langle \delta(x), \frac{d}{dx} \left\{ \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^{r-1} \left[\frac{\varphi(x)}{f'(x)} \right] \right\} \right\rangle.$$

Comparing equations (7) and (8) we see that equation (6) is proved for the case $f'(x) > 0$ and $r = 0, 1, 2, \dots, p$. The case $f'(x) < 0$ follows similarly.

Note that this corollary can be extended to a function f having any number of simple roots at the points $x = \alpha_1, \alpha_2, \dots$. If then $f^{(p_i+2)}$ is a locally summable function on a neighbourhood of the point $x = \alpha_i$, we have

$$\delta^{(r)}(f(x)) = \sum_i \frac{1}{|f'(\alpha_i)|} \left[\frac{1}{f'(x)} \frac{d}{dx} \right]^r \delta(x - \alpha_i)$$

on the real line for $r = 1, 2, \dots, p$, where $p = \min\{p_i : i = 1, 2, \dots\}$. This is in agreement with Gel'fand and Shilov's definition of $\delta^{(r)}(f(x))$, see [3], but their definition was only given for infinitely differentiable f .

Example 1.

$$(9) \quad (x + x_+^p)^{-1} = x^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}}$$

on the real line for $p = 2, 3, \dots$.

PROOF. The distribution x^{-1} is the first derivative of the locally summable function $\ln|x|$ and $(x + x_+^p)'' = p(p-1)x_+^{p-2}$ is bounded on every bounded set. Using equation (2) we have

$$\begin{aligned} \langle (x + x_+^p)^{-1}, \varphi(x) \rangle &= - \int_{-\infty}^{\infty} \ln|x + x_+^p| \left[\frac{\varphi(x)}{1 + px_+^{p-1}} \right]' dx \\ &= - \int_{-\infty}^0 \ln|x| \varphi'(x) dx - \int_0^{\infty} \ln(x + x^p) \left[\frac{\varphi(x)}{1 + px^{p-1}} \right]' dx \\ &= - \langle x_-^{-1}, \varphi(x) \rangle - \int_0^1 \ln(x + x^p) \left[\frac{\varphi(x) - \varphi(0)}{1 + px^{p-1}} \right]' dx + \\ &\quad - \int_1^{\infty} \ln(x + x^{p-1}) \left[\frac{\varphi(x)}{1 + px^{p-1}} \right]' dx + \\ &\quad - \varphi(0) \int_0^1 \ln(x + x^p) [(1 + px^{p-1})^{-1}]' dx. \end{aligned}$$

Now

$$\int_0^1 \ln(x + x^p) \left[\frac{\varphi(x) - \varphi(0)}{1 + px^{p-1}} \right]' dx = \ln 2 \frac{\varphi(1) - \varphi(0)}{1 + p} - \int_0^1 \frac{\varphi(x) - \varphi(0)}{x + x^p} dx$$

$$= \ln 2 \frac{\varphi(1) - \varphi(0)}{1+p} - \int_0^1 \frac{\varphi(x) - \varphi(0)}{x} dx + \int_0^1 \frac{x^{p-2} \varphi(x)}{1+x^{p-1}} dx - \ln 2 \frac{\varphi(0)}{p-1},$$

$$\begin{aligned} \int_1^\infty \ln(x+x^p) \left[\frac{\varphi(x)}{1+px^{p-1}} \right]' dx &= -\ln 2 \frac{\varphi(1)}{1+p} - \int_1^\infty \frac{\varphi(x)}{x+x^p} dx \\ &= \ln 2 \frac{\varphi(1)}{1+p} - \int_1^\infty \frac{\varphi(x)}{x} dx + \int_1^\infty \frac{x^{p-2} \varphi(x)}{1+x^{p-1}} dx, \end{aligned}$$

$$\begin{aligned} \varphi(0) \int_0^1 \ln(x+x^p) [(1+px^{p-1})^{-1}]' dx &= \\ = \varphi(0) \int_0^1 \ln(x+x^p) d[(1+px^{p-1})^{-1} - 1] \\ = -\ln 2 \frac{p\varphi(0)}{1+p} + \varphi(0) \int_0^1 \frac{px^{p-2}}{1+x^{p-1}} dx &= -\ln 2 \frac{p\varphi(0)}{1+p} + \ln 2 \frac{p\varphi(0)}{p-1}. \end{aligned}$$

Thus

$$\begin{aligned} \langle (x+x_+^p)^{-1}, \varphi(x) \rangle &= -\langle x_-^{-1}, \varphi(x) \rangle + \int_0^\infty x^{-1} [\varphi(x) - \varphi(0)H(1-x)] dx + \\ &\quad - \int_0^\infty \frac{x^{p-2} \varphi(x)}{1+x^{p-1}} dx = \langle x^{-1}, \varphi(x) \rangle - \left\langle \frac{x^{p-2} H(x)}{1+x^{p-1}}, \varphi(x) \right\rangle \end{aligned}$$

and equation (9) follows.

Example 2.

$$(10) \quad \delta(x+x_+^p) = \delta(x)$$

on the real line for $p = 3, 4, \dots$ and

$$(11) \quad \delta'(x+x_+^p) = \delta'(x) - \delta(x)$$

on the real line for $p = 2, 3, \dots$.

PROOF. The distribution $\delta(x)$ is the first and the distribution $\delta'(x)$ is the second derivative of the continuous function x_+ . Using equations (5) and (6), equation (10) follows immediately for $p = 2, 3, \dots$.

Using equations (5) and (6) again have

$$\delta'(x+x_+^p) = \frac{1}{1+px_+^{p-1}} \delta(x) = \delta'(x) - \delta(x),$$

giving equation (11) for $p = 3, 4, \dots$.

Example 3.

$$(12) \quad \delta(x + x^2 + x_+^p) = \delta(x + 1) + \delta(x)$$

on the real line for $p = 2, 3, \dots$ and

$$(13) \quad \delta'(x + x^2 + x_+^p) = \delta'(x + 1) + 2\delta(x + 1) + \delta'(x) + 2\delta(x)$$

on the real line for $p = 3, 4, \dots$.

PROOF. The function $x + x^2 + x_+^p$ has zeros at the points $x = -1, 0$. Using equations (5) and (6) on neighbourhoods of these points, equation (12) follows immediately for $p = 2, 3, \dots$.

Using equations (5) and (6) again neighbourhoods of these points we have

$$\begin{aligned} \delta'(x + x^2 + x_+^p) &= \frac{1}{1 + 2x + px_+^{p-1}} \delta'(x + 1) + \frac{1}{1 + 2x + px_+^{p-1}} \delta'(x) \\ &= \delta'(x + 1) + 2\delta(x + 1) + \delta'(x) + 2\delta(x), \end{aligned}$$

giving equation (13) for $p = 3, 4, \dots$.

Example 4.

$$(14) \quad (x + x_+^p + i0)^{-1} = (x + i0)^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}}$$

on the real line for $p = 2, 3, \dots$.

PROOF. The distribution $(x + i0)^{-1}$ is defined by

$$(x + i0)^{-1} = x^{-1} - i\pi\delta(x),$$

see GEL'FAND and SHILOV [3]. Using equations (9) and (10) it follows that

$$\begin{aligned} (x + x_+^p + i0)^{-1} &= (x + x_+^p)^{-1} - i\pi\delta(x + x_+^p) \\ &= x^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}} - i\pi\delta(x) = (x + i0)^{-1} - \frac{x^{p-2}H(x)}{1 + x^{p-1}}, \end{aligned}$$

giving equation (14) for $p = 2, 3, \dots$.

References

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