

On generalized pseudo-projective symmetric manifolds

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Abstract. The object of the present paper is to study a type of non-flat Riemannian manifold called generalized pseudo-projective symmetric manifold.

Introduction

The notions of weakly symmetric and weakly projective symmetric manifold were introduced by L. TAMÁSSY and T. Q. BINH [1]

A non-flat Riemannian manifold (M^n, g) ($n > 2$) (these conditions will be supposed throughout this paper) is called weakly symmetric if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and a vector field F such that

$$\begin{aligned} (\nabla_X R)(Y, Z, V) = & \alpha(X)R(Y, Z, V) + \beta(Y)R(X, Z, V) + \gamma(Z)R(Y, X, V) \\ (1) \quad & + \delta(V)R(Y, Z, X) + g[R(Y, Z, V), X]F; \quad X, Y, Z, V \in \chi(M^n) \end{aligned}$$

where R is the curvature tensor of (M^n, g) . A Riemannian manifold (M^n, g) is called weakly projective symmetric if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and a vector field F such that

$$\begin{aligned} (\nabla_X W)(Y, Z, V) = & \alpha(X)W(Y, Z, V) + \beta(Y)W(X, Z, V) \\ (2) \quad & + \gamma(Z)W(Y, X, V) + \delta(V)W(Y, Z, X) + g[W(Y, Z, V), X]F \end{aligned}$$

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where W is the projective curvature tensor given by

$$(3) \quad W(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]$$

where S is the Ricci tensor of (M^n, g) . Recently, TAMÁSSY and BINH [3] further studied $(WS)_n$ with certain structures.

The object of the present paper is to study a type of non-flat Riemannian manifold (M^n, g) whose projective curvature tensor W satisfies the condition

$$(4) \quad \begin{aligned} (\nabla_X W)(Y, Z, U) &= 2A(X)W(Y, Z, U) + B(Y)W(X, Z, U) \\ &+ C(Z)W(Y, X, U) + D(U)W(Y, Z, X) + g[W(Y, Z, U), X]\rho \end{aligned}$$

where A, B, C, D are non-zero 1-forms and ρ is a vector field given by

$$(5) \quad g(X, \rho) = A(X) \quad \forall X.$$

Such a manifold will be called a generalized pseudo-projective symmetric manifold; A, B, C, D will be called its associated 1-forms and an n -dimensional manifold of this kind will be denoted by $G(PWS)_n$.

Let

$$(6) \quad \begin{aligned} g(X, \lambda) &= B(X), \quad g(X, \mu) = C(X) \quad \text{and} \\ g(X, \nu) &= D(X) \quad \forall X \in \chi(M^n). \end{aligned}$$

Then $\rho, \lambda, \nu \in \chi(M^n)$ will be called the basic vector fields of $G(PWS)_n$ corresponding to the associated 1-forms A, B, C, D respectively. If, in particular, $A = B = C = D$, then the manifold defined by (4) reduces to a pseudo-projective symmetric manifold introduced by CHAKI and SAHA [2]. This justifies the name generalized pseudo-projective symmetric manifold. (4) and (5) together are a little stronger assumptions than (2). (2) gives (4) if α and F are related by

$$(X, F) = \alpha(X) \quad \forall X.$$

So the definition of a $G(PWS)_n$ is similar to that of a weakly projective symmetric manifold mentioned above.

A non-flat Riemannian manifold is called generalized pseudo-symmetric by M.C. CHAKI [4] if the curvature tensor R satisfies

$$(7) \quad \begin{aligned} (\nabla_X R)(Y, Z, U) &= 2A(X)R(Y, Z, U) + B(Y)R(X, Z, U) \\ &+ C(Z)R(Y, X, U) + D(U)R(Y, Z, X) + g[R(Y, Z, U)X]\rho, \end{aligned}$$

where ρ is a vector field given by

$$g(X, 2\rho) = A(X) \quad \forall X.$$

He denoted such a manifold by $G(PS)_n$. Recently, CHAKI and MONDAL [5] also studied $G(PS)_n$. TAMÁSSY and BINH in their paper [1] find necessary and sufficient conditions for a weakly symmetric manifold to be a weakly projective symmetric manifold.

In Section 2 of this paper it is shown that in a $G(PWS)_n$ the scalar curvature r of (M^n, g) is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q defined by $g(Q, X) = A(X) + B(X)$, and also a necessary and sufficient condition for zero scalar curvature in a $G(PWS)_n$ is obtained. In Section 3 some properties of $G(PWS)_n$ have been proved. In Section 4 it is shown that an Einstein $G(PWS)_n$ reduces to a $G(PS)_n$ if $A(X) + nD(X) \neq 0$. Further, it is shown that if the vector field ρ defined by (5) is a paralld vector field, then an Einstein $G(PWS)_n$ reduces to a $G(PS)_n$ provided the vectors ρ and λ are not co-directional.

1. Preliminaries

In this section we derive some formulas which will be required in the study of a $G(PWS)_n$

Let

$$(1.1) \quad 'W(X, Y, Z, U) = g[W(X, Y, Z, \cdot), U].$$

Then form (3) we get

$$(1.2) \quad \begin{aligned} 'W(X, Y, Z, U) &= 'R(X, Y, Z, U) \\ &- \frac{1}{n-1} [g(X, U)S(Y, Z) - g(Y, U)S(X, Z)], \end{aligned}$$

where

$$(1.3) \quad {}'R(X, Y, Z, U) = g[R(X, Y, Z, \cdot), U].$$

Let

$$(1.4) \quad P(X, U) = {}'W(X, e_i, e_i, U)$$

where $\{e_i\}$, $i = 1, 2, \dots, n$ is an orthonormal basis of the tangent space at a point and i is summed for $1 \leq i \leq n$. Then using (1.2) we get

$$(1.5) \quad P(X, U) = \frac{n}{n-1} S(X, U) - \frac{r}{n-1} g(X, U)$$

where S is the Ricci tensor and r is the scalar curvature of (M^n, g) . Let ℓ and L be the symmetric endomorphisms of the tangent space at two points corresponding to the tensors P and S respectively, i.e.

$$(1.6) \quad g(\ell X, Y) = P(X, Y)$$

and

$$(1.7) \quad g(LX, Y) = S(X, Y).$$

Contracting (4) over Y , we get

$$B(W(X, Z, U)) + {}'W(\rho, Z, U, X) = 0$$

or

$${}'W(X, Z, U, \lambda) + {}'W(\rho, Z, U, X) = 0.$$

Putting $Z = U = e_i$ in the above relation and taking summation over i , $1 \leq i \leq n$, we have

$$(1.8) \quad P(X, \lambda) + P(X, \rho) = 0.$$

2. Nature of the scalar curvature of a $G(PWS)_n$ ($n > 2$)

From (1.5) and (1.8) it follows that $T(X) \frac{r}{n} = \bar{T}(X)$, where $T(X) = A(X) + B(X)$ and $\bar{T}(X) = A(LX) + B(LX)$. Hence

$$(2.1) \quad S(X, Q) = \frac{r}{n} g(X, Q),$$

where $g(X, Q) = T(X)$. This leads to the following

Theorem 1. *In a $G(PWS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q defined by $g(X, Q) = T(X)$.*

Now we obtain a necessary and sufficient condition for zero scalar curvature in a $G(PWS)_n$. First we suppose that $r = 0$ in a $G(PWS)_n$. Then from (2.1) we get $S(X, Q) = 0$. Therefore from (3) it follows that

$$(2.2) \quad W(X, Y, Q) = R(X, Y, Q).$$

Next we suppose that in a $G(PWS)_n$ the relation (2.2) holds, then from (3) we get

$$(2.3) \quad S(Y, Q)X = S(X, Q)Y.$$

Contraction of (2.3) gives $S(Y, Q) = 0$.

Hence from (2.1) we get $r = 0$, if $T(X) \neq 0$.

This leads to the following

Theorem 2. *A $G(PWS)_n$ ($n > 2$) is of zero scalar curvature if and only if the relation (2.2) holds provided $T \neq 0$.*

3. The case of $G(PWS)_n$ satisfying $A(W(X, Y, Z)) = 0$

Contracting (4) over X , we get

$$(3.1) \quad (\operatorname{div} W)(Y, Z, U) = 3A(W(Y, Z, U)),$$

where ‘div’ denotes divergence. It is known that in a Riemannian manifold (M^n, g) ($n > 2$)

$$(3.2) \quad (\operatorname{div} W)(X, Y, Z) = \frac{n-2}{n-1}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)].$$

Since $A(W(X, Y, Z)) = 0$ we get from (3.1) $(\operatorname{div} W)(X, Y, Z) = 0$.

Hence from (3.2) it follows that $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$.

Thus we can state the following

Theorem 3. *In a $G(PWS)_n$ satisfying $A(W(X, Y, Z)) = 0$ the Ricci tensor S is of Codazzi type.*

Lemma. *In order that $(\nabla_X W)(Y, Z, U) + (\nabla_Y W)(Z, X, U) + (\nabla_Z W)(X, Y, U) = 0$, it is necessary and sufficient that $(\operatorname{div} W)(X, Y, Z) = 0$.*

PROOF of the Lemma. First suppose that

$$(3.3) \quad (\nabla_X W)(Y, Z, U) + (\nabla_Y W)(Z, X, U) + (\nabla_Z W)(X, Y, U) = 0.$$

Contracting (3.3) over Z , we get $(\operatorname{div} W)(X, Y, U) = 0$.

Next suppose that $(\operatorname{div} W)(X, Y, U) = 0$. Hence from (3.2) we get

$$(3.4) \quad \frac{n-2}{n-1} [(\nabla_X S)(Y, U) - (\nabla_Y S)(X, U)] = 0.$$

Again from the Bianchi identity we get

$$(3.5) \quad (\nabla_X R)(Y, Z, U) + (\nabla_Y R)(Z, X, U) + (\nabla_Z R)(X, Y, U) = 0.$$

Hence from (3.4) and (3.5) it follows that

$$(\nabla_X W)(Y, Z, U) + (\nabla_Y W)(Z, X, U) + (\nabla_Z W)(X, Y, U) = 0. \quad \square$$

Now from (4) we get

$$(3.6) \quad \begin{aligned} & (\nabla_X W)(Y, Z, U) + (\nabla_Y W)(Z, X, U) + (\nabla_Z W)(X, Y, U) \\ &= [2A(X) - B(X) - C(X)]W(Y, Z, U) + [2A(Y) - B(Y) \\ & \quad - C(Y)]W(Z, X, U) + [2A(Z) - B(Z) - C(Z)]W(X, Y, U) \\ &= G(X)W(Y, Z, U) - G(Y)W(X, Z, U) - G(Z)W(Y, X, U), \end{aligned}$$

since

$$\begin{aligned} W(X, Y, Z) &= -W(Y, X, Z) \quad \text{and} \\ W(X, Y, Z) + W(Y, Z, X) + W(Z, X, Y) &= 0, \end{aligned}$$

where

$$G(X) = 2A(X) - B(X) - C(X).$$

Since we assume that $A(W(X, Y, Z)) = 0$, it follows from (3.1) that $(\operatorname{div} W)(X, Y, Z) = 0$. On the other hand, from the above Lemma and (3.6), it follows that

$$(3.7) \quad G(X)W(Y, Z, U) - G(Y)W(X, Z, U) - G(Z)W(Y, X, U) = 0.$$

Putting $X = \rho$ in (3.7) and applying $A(W(X, Y, Z)) = 0$ we get

$$G(\rho)W(Y, Z)U = 0.$$

Then either $G(\rho) = 0$ or the manifold is projectively flat.

Now $G(\rho) = 0$ implies $g(\rho, \tilde{\rho}) = 0$, where $\tilde{\rho}$ is a vector field defined by

$$(3.8) \quad g(X, \tilde{\rho}) = G(X).$$

Thus we have the following

Theorem 4. *If a $G(PWS)_n$ satisfies $A(W(X, Y, Z)) = 0$, then either the manifold is of constant curvature or the associated vector ρ is orthogonal to the vector $\tilde{\rho}$ defined by (3.8).*

4. Einstein $G(PWS)_n$ ($n > 3$)

In this section we assume that a $G(PWS)_n$ defined by (4) is an Einstein manifold. Then the Ricci tensor S satisfies

$$(4.1) \quad S(X, Y) = \frac{r}{n}g(X, Y)$$

from which it follows that

$$(4.2) \quad dr(X) = 0 \quad \text{and} \quad (\nabla_Z S)(X, Y) = 0.$$

By using (3), (4.1) and (4.2) we get from (4)

$$(4.3) \quad \begin{aligned} (\nabla_X R)(Y, Z)U &= 2A(X) \left[R(Y, Z)U - \frac{r}{n(n-1)}(g(Z, U)Y - (-g(Y, U)Z)) \right] \\ &\quad + B(Y) \left[R(X, Z)U - \frac{r}{n(n-1)}(g(Z, U)X - g(X, U)Z) \right] \\ &\quad + C(Z) \left[R(Y, X)U - \frac{r}{n(n-1)}(g(X, U)Y - g(Y, U)X) \right] \\ &\quad + D(U) \left[R(Y, Z)X - \frac{r}{n(n-1)}(g(Z, X)Y - g(Y, X)Z) \right] \\ &\quad + g \left[R(Y, Z)U - \frac{r}{n(n-1)}(g(Z, U)Y - g(Y, U)Z) \right]. \end{aligned}$$

From the Bianchi identity and (4.3) it follows that

$$\begin{aligned}
 (4.4) \quad & 3A(R(Y, Z)U) + B(R(Y, Z)U) + C(R(Y, Z)U) \\
 & + \left[2S(Z, U) - \frac{2r}{n-1}g(Z, U) \right] A(Y) \\
 & + \left[\frac{(n+1)}{n(n-1)}rg(Y, U) - 2S(Y, U) \right] A(Z) \\
 & - \frac{r}{n(n-1)}B(Y)g(Z, U) + \frac{r}{n(n-1)}B(Z)g(Y, U) \\
 & - \frac{r}{n(n-1)}C(Y)g(Z, U) + \frac{r}{n(n-1)}C(Z)g(Y, U) \\
 & - \frac{r}{n}D(U)g(Z, Y) = 0.
 \end{aligned}$$

Putting $Y = Z = e_i$ in (4.4) and taking summation over i , we get

$$r[A(U) + nD(U)] = 0.$$

Hence

$$r = 0, \quad \text{if } A(U) + nD(U) \neq 0.$$

Putting $r = 0$ in (4.3), it follows that a $G(PWS)_n$ is a $G(PS)_n$. Hence we can state the following

Theorem 5. *An Einstein $G(PWS)_n$ is a $G(PS)_n$ if $A(X) + nD(X) \neq 0$.*

Next we suppose that in an Einstein $G(PWS)_n$ the vector field ρ defined by (5) is parallel:

$$(4.5) \quad \nabla_X \rho = 0 \quad \forall X \in \chi G(PWS)_n.$$

Applying the Ricci identity we get

$$(4.6) \quad R(X, Y, \rho) = 0.$$

From (4.6) we get

$$(4.7) \quad S(Y, \rho) = 0.$$

Now by (4.5) and (4.7) it follows that

$$(4.8) \quad (\nabla_X S)(Y, \rho) = 0.$$

From (4.4) we get

$$(4.9) \quad (\nabla_X S)(Z, U) = B(R(X, Z, U)) - \frac{r}{n(n-1)} [g(Z, U)B(X) - g(X, U)B(Z)].$$

Putting $U = \rho$ in (4.9) and using (4.6), (4.7), (4.8) we get $r = 0$, if $A(X)B(Z) \neq A(Z)B(X)$

Hence we can state the following

Theorem 6. *If the vector field ρ is a paralld vector field in an Einstein $G(PWS)_n$, then $G(PWS)_n$ reduces to a $G(PS)_n$ provided the vector fields ρ and λ corresponding to the 1-forms A and B are not co-directional.*

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