

On Nemytskii operator

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Abstract. J. MATKOWSKI has proved that every function h , which generates the Nemytskii operator mapping the Banach space of Lipschitzian functions into itself must be Jensen function with respect to the second variable (see [2], [3], [4]). This paper is devoted to Nemytskii operator defined on a space of polynomials with values in a space of functions of class C^1 .

We will start with some definitions and notations. Let $f : \langle a, b \rangle \rightarrow \mathbb{R}$, and $x_1, \dots, x_p \in \langle a, b \rangle$ be distinct points. The *divided difference* $[x_1, \dots, x_p; f]$ of f at points $x_1, \dots, x_p$ is defined by recurrence

$$[x_1, f] = f(x_1),$$

$$[x_1, \dots, x_p; f] = \frac{[x_2, \dots, x_p; f] - [x_1, \dots, x_{p-1}; f]}{x_p - x_1}, \quad p \geq 2$$

(cf. [1]). Let us consider

$$\text{lip}^2(\langle a, b \rangle) := \left\{ f : \langle a, b \rangle \rightarrow \mathbb{R} : \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, \ j \neq i}} |[t_1, t_2, t_3; f]| < \infty \right\},$$

and denote as $P^2(\langle a, b \rangle)$ the set of all restrictions of polynomials of degree at most 2 to the interval $\langle a, b \rangle$.

Mathematics Subject Classification: 47H30, 47B38.

Key words and phrases: Nemytskii operator, polynomial functions, Lipschitz functions, Jensen functional equation.

Then $P^2(\langle a, b \rangle)$ is a subspace of a real Banach space $\text{lip}^2(\langle a, b \rangle)$ with the norm

$$\|f\| := |f(a)| + |f(b)| + \sup_{\substack{t_1, t_2, t_3 \in \langle a, b \rangle \\ t_i \neq t_j, j \neq i}} |[t_1, t_2, t_3; f]|.$$

Every function $h : \langle a, b \rangle \times \mathbb{R} \rightarrow \mathbb{R}$ generates the Nemytskii operator defined on the space $P^2(\langle a, b \rangle)$ by formula

$$N\varphi(x) := h(x, \varphi(x)), \quad \varphi \in P^2(\langle a, b \rangle), \quad x \in \langle a, b \rangle.$$

Lemma. *There exists a constant $M > 0$ such that $\|A\varphi\| \leq M\|A\| \cdot \|\varphi\|$ for all $A, \varphi \in \text{lip}^2(\langle a, b \rangle)$.*

PROOF. Indeed, let us fix $x, y, z \in \langle a, b \rangle$, distinct. Then

$$|[x, y, z; A\varphi]| \leq |A(y)| \cdot \|\varphi\| + \|A\| \cdot |\varphi(z)| + \left| \frac{A(x) - A(y)}{x - y} \right| \cdot \left| \frac{\varphi(x) - \varphi(z)}{x - z} \right|.$$

It is easy to check, that

$$|\varphi(z)| \leq (2 + |a - b|^2)\|\varphi\|, \quad \left| \frac{\varphi(z) - \varphi(x)}{z - x} \right| \leq \left(2|a - b| + \frac{1}{|a - b|} \right) \|\varphi\|,$$

(and analogous formulas hold for A). Thus

$$|[x, y, z; A\varphi]| \leq 2(2 + |a - b|^2)\|A\| \cdot \|\varphi\| + \left(2|a - b| + \frac{1}{|a - b|} \right)^2 \|A\| \cdot \|\varphi\|$$

for all $x, y, z \in \langle a, b \rangle$, distinct and

$$\|A\varphi\| \leq \left(10 + 6|a - b|^2 + \frac{1}{|a - b|^2} \right) \|A\| \cdot \|\varphi\|.$$

□

Theorem. *Let $h : \langle a, b \rangle \times \mathbb{R} \rightarrow \mathbb{R}$. Then the Nemytskii operator $N : P^2(\langle a, b \rangle) \rightarrow \text{lip}^2(\langle a, b \rangle)$ defined by h satisfies the Lipschitz condition*

$$(1) \quad \|N\varphi_1 - N\varphi_2\| \leq L\|\varphi_1 - \varphi_2\|, \quad \varphi_1, \varphi_2 \in P^2(\langle a, b \rangle),$$

iff there exist functions $A, B \in \text{lip}^2(\langle a, b \rangle)$ such that

$$h(x, y) = A(x)y + B(x), \quad x \in \langle a, b \rangle, \quad y \in \mathbb{R}.$$

PROOF. Suppose that h generates the Nemytskii operator N satisfying (1). For a fixed $y_0 \in \mathbb{R}$, the function $\varphi_0(x) = y_0$, $x \in \langle a, b \rangle$ belongs to $P^2(\langle a, b \rangle)$. Consequently

$$N\varphi_0 = h(\cdot, y_0) \in \text{lip}^2(\langle a, b \rangle) \quad \text{for all } y_0 \in \mathbb{R}.$$

By the proof of Theorem 1 in ([1], p. 391) the function $h(\cdot, y_0)$ is of class C^1 in $\langle a, b \rangle$, in particular h is continuous with respect to the first variable for every $y_0 \in \mathbb{R}$. Let us fix $x, y, z \in \langle a, b \rangle$, distinct, $y_1, y_2, y_3, \bar{y}_1, \bar{y}_2, \bar{y}_3 \in \mathbb{R}$ and define functions $\varphi_1, \varphi_2 : \langle a, b \rangle \rightarrow \mathbb{R}$,

$$\varphi_i(t) = A_i t^2 + B_i t + C_i, \quad t \in \langle a, b \rangle, \quad i = 1, 2,$$

where

$$\begin{aligned} A_1 &= \frac{\frac{y_3 - y_2}{z - y} - \frac{y_2 - y_1}{y - x}}{z - x}, & A_2 &= \frac{\frac{\bar{y}_3 - \bar{y}_2}{z - y} - \frac{\bar{y}_2 - \bar{y}_1}{y - x}}{z - x}, \\ B_1 &= \frac{y_3 - y_1}{z - x} - (z + x)A_1, & B_2 &= \frac{\bar{y}_3 - \bar{y}_1}{z - x} - (z + x)A_2, \\ C_1 &= A_1 z x + y_1 - \frac{y_3 - y_1}{z - x} x, & C_2 &= A_2 z x + \bar{y}_1 - \frac{\bar{y}_3 - \bar{y}_1}{z - x} x. \end{aligned}$$

Then $\varphi_1(x) = y_1$, $\varphi_1(y) = y_2$, $\varphi_1(z) = y_3$, $\varphi_2(x) = \bar{y}_1$, $\varphi_2(y) = \bar{y}_2$, $\varphi_2(z) = \bar{y}_3$, $\varphi_1, \varphi_2 \in P^2(\langle a, b \rangle)$, and

$$\begin{aligned} \|\varphi_1 - \varphi_2\| &= |(A_1 - A_2)a^2 + (B_1 - B_2)a + C_1 - C_2| + |(A_1 - A_2)b^2 \\ &\quad + (B_1 - B_2)b + C_1 - C_2| + |A_1 - A_2| \\ &\leq \left| \frac{\frac{y_3 - \bar{y}_3 - y_2 + \bar{y}_2}{z - y} - \frac{y_2 - \bar{y}_2 - y_1 + \bar{y}_1}{y - x}}{z - x} \right| (1 + 8\alpha^2) + \frac{|y_3 - y_1 - \bar{y}_3 + \bar{y}_1|}{|z - x|} (4\alpha) \\ &\quad + 2|y_1 - \bar{y}_1|, \end{aligned}$$

where $\alpha = \max\{|a|, |b|\}$.

On account of (1) and the definition of the norm we have

$$\begin{aligned} & \left| \frac{h(z, y_3) - h(z, \bar{y}_3) - h(y, y_2) + h(y, \bar{y}_2)}{z - y} \right. \\ & \quad \left. - \frac{h(y, y_2) - h(y, \bar{y}_2) - h(x, y_1) + h(x, \bar{y}_1)}{y - x} \right| \\ & \leq L \left\{ (1 + 8\alpha^2) \left| \frac{y_3 - \bar{y}_3 - y_2 + \bar{y}_2}{z - y} - \frac{y_2 - \bar{y}_2 - y_1 + \bar{y}_1}{y - x} \right| \right. \\ & \quad \left. + 4\alpha|y_3 - y_1 - \bar{y}_3 + \bar{y}_1| + 2|y_1 - \bar{y}_1| |z - x| \right\} \end{aligned}$$

for all $x, y, z \in \langle a, b \rangle$, and $y_1, y_2, y_3, \bar{y}_1, \bar{y}_2, \bar{y}_3 \in \mathbb{R}$. Letting $z \rightarrow x$ it follows from the continuity of $h(\cdot, y)$ that

$$\begin{aligned} & \frac{1}{|x - y|} |h(x, y_3) - h(x, \bar{y}_3) - h(x, y_1) + h(x, \bar{y}_1)| \\ & \leq L \left\{ (1 + 8\alpha^2) \left| \frac{y_3 - \bar{y}_3 - y_1 + \bar{y}_1}{x - y} \right| + 4\alpha|y_3 - y_1 - \bar{y}_3 + \bar{y}_1| \right\} \end{aligned}$$

for all $x, y \in \langle a, b \rangle$, and $y_1, y_3, \bar{y}_1, \bar{y}_3 \in \mathbb{R}$. Multiplying the both sides of this inequality by $|x - y|$ and passing $y \rightarrow x$ we get

$$(2) \quad |h(x, y_3) - h(x, \bar{y}_3) - h(x, y_1) + h(x, \bar{y}_1)| \leq L(1 + 8\alpha^2)|y_3 - \bar{y}_3 - y_1 + \bar{y}_1|$$

for all $x \in \langle a, b \rangle$, and $y_1, y_3, \bar{y}_1, \bar{y}_3 \in \mathbb{R}$.

Putting $\bar{y}_3 = \bar{y}_1 = 0$ in (2), we have

$$|h(x, y_3) - h(x, y_1)| \leq L(1 + 8\alpha^2)|y_3 - y_1|, \quad x \in \langle a, b \rangle, \quad y_1, y_3 \in \mathbb{R}.$$

Thus for all $x \in \langle a, b \rangle$, the function $h(x, \cdot)$ is continuous.

Now, let us fix $u, v \in \mathbb{R}$. Putting $y_3 := u$, $\bar{y}_3 = y_1 := \frac{u+v}{2}$, $\bar{y}_1 := v$ in (2) we get

$$h(x, u) - 2h\left(x, \frac{u+v}{2}\right) + h(x, v) = 0 \quad \text{for } x \in \langle a, b \rangle.$$

This means that for every $x \in \langle a, b \rangle$ the function $y \rightarrow h(x, y)$ is a continuous solution of Jensen equation. According to Theorem 2, p. 316 in [1], there exist unique functions $A, B : \langle a, b \rangle \rightarrow \mathbb{R}$, such that

$$h(x, y) = A(x) + B(x)y \quad \text{for all } x \in \langle a, b \rangle, \quad y \in \mathbb{R}.$$

It is obvious that $A \in \text{lip}^2(\langle a, b \rangle)$. Since

$$B(x) = h(x, 1) - h(x, 0), \quad x \in \langle a, b \rangle$$

we also have $B \in \text{lip}^2(\langle a, b \rangle)$, which ends the first part of the proof.

Conversely, suppose that $h : \langle a, b \rangle \times \mathbb{R} \rightarrow \mathbb{R}$ is given by formula

$$h(x, y) = A(x) + B(x)y \quad \text{for all } x \in \langle a, b \rangle, y \in \mathbb{R},$$

where $A, B \in \text{lip}^2(\langle a, b \rangle)$. By the lemma, the function h generates Nemytskii operator

$$N : P^2(\langle a, b \rangle) \rightarrow \text{lip}^2(\langle a, b \rangle),$$

and

$$\|N\varphi_1 - N\varphi_2\| \leq M\|B\| \cdot \|\varphi_1 - \varphi_2\| \quad \text{for } \varphi_1, \varphi_2 \in P^2(\langle a, b \rangle)$$

for some constant $M > 0$. □

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(Received May 12, 1997; revised March 25, 1998)