

On the generalized Einstein – Yang Mills equations

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Abstract. Let $\xi = (E, p, M)$ be a vector bundle, with basis M — a real differentiable manifold of dimension n , and fiber F of dimension m . Considering the automorphisms of ξ as gauge transformations, and the set of gauge fields $\{N_i^a(x, y), L_{jk}^i(x, y), L_{bk}^a(x, y), C_{ja}^i(x, y), C_{bc}^a(x, y), g_{ij}(x, y), h_{ab}(x, y)\}$ given by a nonlinear connection, a gauge linear d -connection [9,11], and a pair of metric gauge tensor fields in local adapted coordinates, the author obtains the form of the generalized Einstein–Yang Mills equations for the general case and for the quasi-metric h - and v -symmetrical cases. These results generalise the ones obtained by G.S. ASANOV in [2,3], in a natural manner, basically using the formalism, notations and mathematical theory of distinguished geometrical object fields introduced by R. MIRON [10, 11].

Let $\{N_i^a(x, y)\}$ be the coefficients of a nonlinear connection on the vector bundle $\xi = (E, p, M)$ in local coordinates (x^i, y^a) , $i = \overline{1, n}$, $a = \overline{1, m}$ [8,11].

Definition 1. A local adapted basis in $\mathfrak{X}(E)$ is the set of vector fields $\{\delta_i, \dot{\partial}_a\}$, $i = \overline{1, n}$, $a = \overline{1, m}$, where

$$(1) \quad \delta_i = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \quad \dot{\partial}_a = \frac{\partial}{\partial y^a}.$$

Definition 2. A linear d -connection on E is a linear connection ∇ that preserves the horizontal and the vertical distributions locally generated by $\{\delta_i, i = \overline{1, n}\}$ and $\{\dot{\partial}_a, a = \overline{1, m}\}$ respectively; in the local adapted basis (1) its coefficients are given by

$$\{L_{jk}^i(x, y), L_{bk}^a(x, y), C_{ja}^i(x, y), C_{bc}^a(x, y)\}$$

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where

$$(2) \quad \begin{cases} \nabla_{\delta_j} \delta_i = L_{ij}^k \delta_k, & \nabla_{\delta_j} \dot{\partial}_a = L_{aj}^b \dot{\partial}_b \\ \nabla_{\dot{\partial}_a} \delta_j = C_{ja}^i \delta_i, & \nabla_{\dot{\partial}_b} \dot{\partial}_a = C_{ab}^d \dot{\partial}_d \end{cases}$$

Definition 3. The h - and v -covariant derivation laws associated to the linear d -connection (2) are defined by

$$(3) \quad \begin{cases} D_i w_{nb}^{ma} = \delta_i w_{nb}^{ma} + L_{ki}^m w_{nb}^{ka} - L_{ni}^k w_{kb}^{ma} + L_{di}^a w_{nb}^{md} - L_{bi}^d w_{nd}^{ma} \\ D_c w_{nb}^{ma} = \dot{\partial}_c w_{nb}^{ma} + C_{kc}^m w_{nb}^{ka} - C_{nc}^k w_{kb}^{ma} + C_{dc}^a w_{nb}^{md} - C_{bc}^d w_{nd}^{ma}. \end{cases}$$

Proposition 1. The transformation rules for the coefficients of the linear d -connection are

$$(4) \quad \begin{aligned} \bar{\partial}_m B_j^k - B_i^k \bar{L}_{jm}^i(\bar{x}, \bar{y}) + B_j^i B_m^n L_{in}^k(x, y) &= 0 \\ \bar{\partial}_m M_b^a - M_c^a \bar{L}_{bm}^c(\bar{x}, \bar{y}) + M_b^c B_m^n L_{cn}^a(x, y) &= 0 \\ B_n^i \bar{C}_{ma}^n(\bar{x}, \bar{y}) = M_a^c B_m^j C_{jc}^i(x, y) \\ M_d^a \bar{C}_{bc}^d(\bar{x}, \bar{y}) = M_b^d M_c^f C_{df}^a(x, y) \end{aligned}$$

where the coordinate transformations on E have the form

$$(5) \quad \begin{aligned} x^i &= x^i(\bar{x}), \det(\partial x^i / \partial \bar{x}^j) \neq 0 \\ y^a &= M_b^a(\bar{x}) \bar{y}^b, \det(M_b^a(\bar{x})) \neq 0 \end{aligned}$$

and we used the notations

$$B_j^i = \bar{\partial}_j x^i, \quad \bar{\partial}_j = \frac{\partial}{\partial \bar{x}^j}$$

Definition 4. A gauge transformation is a automorphism of E [7,2,3], locally given by

$$(6) \quad \begin{aligned} x^i &= X^i(\tilde{x}), \det(\tilde{\partial}_j X^i) \neq 0 \\ y^a &= Y^a(\tilde{x}, \tilde{y}), \det(Y_b^a) \neq 0, \quad \tilde{\partial}_c Y_b^a = 0 \end{aligned}$$

where we denoted

$$Y_b^a = \dot{\partial}_b Y^a, \quad \dot{\partial}_b = \frac{\partial}{\partial \tilde{y}^b}, \quad \tilde{\partial}_j = \frac{\partial}{\partial \tilde{x}^j}$$

Definition 5. A (generalized [2,3]) gauge tensor field is a field on E , which obeys tensorial rules of transformation relative to (5) and (6); e.g. $\{w_{jb}^{ia}\}$ obeys

$$(7) \quad \begin{aligned} B_k^i M_c^a \bar{w}_{jb}^{kc} &= B_j^\ell M_b^d w_{\ell d}^{ia} \\ X_k^i Y_c^a \tilde{w}_{jb}^{kc} &= X_j^\ell Y_b^d w_{\ell d}^{ia}, \quad \text{where } X_k^i = \tilde{\partial}_k X^i. \end{aligned}$$

Definition 6. A gauge covariant derivation (h -resp. v -) is given by the h - and v -derivation laws in definition 3, which preserves the gauge tensorial character relative to (5), (6).

Proposition 2. *The coefficients of the h - and v -gauge covariant derivations have with respect to (6) the transformation laws*

$$(8) \quad \begin{aligned} \tilde{\partial}_m X_j^k - X_i^k \tilde{L}_{jm}^i(\tilde{x}, \tilde{y}) + X_j^i X_m^n L_{in}^k(x, y) &= 0 \\ \tilde{\partial}_m Y_b^a - Y_c^a \tilde{L}_{bm}^c(\tilde{x}, \tilde{y}) + Y_b^c X_m^n L_{cn}^a(x, y) &= 0 \\ X_n^i \tilde{C}_{ka}^n(\tilde{x}, \tilde{y}) &= Y_a^c X_k^j C_{jc}^i(x, y) \\ Y_d^a \tilde{C}_{bc}^d(\tilde{x}, \tilde{y}) &= Y_b^d Y_c^f C_{df}^a(x, y) \end{aligned}$$

Remarks.

1. $\{C_{ja}^i\}$ and $\{C_{bc}^a\}$ are gauge tensor fields.
2. The coefficients $\{L_{jk}^i, L_{bk}^a, C_{ja}^i, C_{bc}^a\}$ of the h - and v -gauge covariant derivations (3) are in fact the coefficients of a linear d -connection which satisfies the supplementary rules (8) (gauge linear d -connection).

Proposition 3. *The torsion and the curvature gauge tensor fields of a gauge linear d -connection are given by [11]*

$$(9) \quad \begin{aligned} T_{jk}^i &= L_{[jk]}^i, \quad R_{jk}^a = -\delta_{[j} N_{k]}^a, \quad P_{jc}^i = C_{jc}^i \\ P_{jb}^a &= \dot{\partial}_b N_j^a - L_{bj}^a, \quad S_{bc}^a = C_{[bc]}^a \end{aligned}$$

and respectively

$$(9') \quad \begin{aligned} R_{jkl}^i &= \delta_{[l} L_{jk]}^i + L_{j[k}^h L_{h]l}^i + C_{ja}^i R_{kl}^a, \\ R_{bkl}^a &= \delta_{[l} L_{bk]}^a + L_{b[k}^c L_{cl]}^a + C_{bc}^a R_{kl}^c, \\ P_{jkc}^i &= \dot{\partial}_c L_{jk}^i - D_k C_{jc}^i + C_{jb}^i P_{kc}^b, \\ P_{bkc}^a &= \dot{\partial}_c L_{bk}^a - D_k C_{bc}^a + C_{bd}^a P_{kc}^d, \\ S_{jbc}^i &= \dot{\partial}_{[c} C_{jb]}^i + C_{j[b}^h C_{hc]}^i \\ S_{bcd}^a &= \dot{\partial}_{[c} C_{bd]}^a + C_{b[c}^e C_{ed]}^a \end{aligned}$$

where we used the notation for $[i \dots j]$:

$$L_{k[i}^h L_{hj]}^s = L_{ki}^h L_{hj}^s - L_{kj}^h L_{hi}^s$$

Proposition 4. *The following mixed Lagrangian is invariant under (5) and (6) (i.e. it is a scalar gauge field)*

$$(10) \quad L = \sum_{i \in I} n_i \cdot L_i, \quad n_i \in \mathbb{R}, \quad i \in I = \{\overline{1}, \overline{5}, \overline{11}, \overline{16}, \overline{21}, \overline{22}\}$$

where

$$\begin{aligned}
 (11) \quad & L_1 = T_{jk}^i T_i^{jk}, \quad L_2 = R_{jk}^a R_a^{jk}, \quad L_3 = P_{jc}^i P_i^{jc}, \quad L_4 = P_{jb}^a P_a^{jb} \\
 & L_5 = S_{bc}^a S_a^{bc}, \quad L_{21} = R_{jkl}^i R_i^{jkl}, \quad L_{22} = S_{bcd}^a S_a^{bcd}, \\
 & L_{11} = R_{ij}^i, \quad L_{12} = R_{bkl}^a R_a^{bkl}, \quad L_{13} = P_{jkc}^i P_i^{jkc} \\
 & L_{14} = P_{bkc}^a P_a^{bkc}, \quad L_{15} = S_{jbc}^i S_i^{jbc}, \quad L_{16} = S^{ab}_{ab}
 \end{aligned}$$

The proof is computational.

Remark. The Lagrangian L contains, relative to [2], the supplementary terms $n_3 L_3$ and $n_{15} L_{15}$, and the terms $n_{11} L_{11}$ and $n_{13} L_{13}$ are altered (are more general) since the present context doesn't impose the restrictive condition $C_{ja}^i = 0$.

The raising/lowering of the corresponding indices in (11) were performed via the gauge metric tensor fields $\{g_{ij}(x, y)\}$ and $\{h_{ab}(x, y)\}$ [3,2]. Then, introducing the Lagrangian density

$$(12) \quad \mathcal{L} = LG$$

where $G = |\det(g_{ij})|^{1/2} \cdot |\det(h_{ab})|^{1/2}$, we notice that it depends on the gauge fields

$$(13) \quad \phi \in \{N_i^a, L_{jk}^i, L_{bk}^a, C_{ja}^i, C_{bc}^a, g_{ij}, h_{ab}\}$$

and their derivatives; considering the variational principle [1,3]

$$\delta \int \mathcal{L} dx^n dy^m = 0$$

one can derive the extremum condition of vanishing the Euler–Lagrange derivatives

$$\frac{\delta \mathcal{L}}{\delta \phi} \equiv \frac{\partial}{\partial x^j} \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) + \frac{\partial}{\partial y^a} \left(\frac{\partial \mathcal{L}}{\partial (\dot{\partial}_a \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Theorem 1. *The generalized Einstein–Yang Mills equations associated to the Lagrangian (10) for the set of arguments (13) are*

$$\begin{aligned}
 (15.1) \quad & \frac{\delta L}{\delta N_k^a} = -4n_2(D_\ell^* R_a^{\ell k} + P_{a\ell}^c R_c^{\ell k} + \frac{1}{2} T_{n\ell}^k R_a^{n\ell}) - \\
 & -2n_4(D_b^* P_a^{bk} + C_{ab}^c P_c^{bk} + \underline{P_{nb}^k P_a^{nb}}) - \\
 & -4n_{21}[D_\ell^* V_a^{\ell k} + P_{a\ell}^c V_c^{\ell k} + \frac{1}{2} T_{n\ell}^k V_a^{n\ell} - R_i^{j\ell k} (P_{j\ell a}^i + D_\ell P_{ja}^i - P_{jb}^i P_{\ell a}^b)] - \\
 & -n_{11}[D_\ell^* U_a^{\ell k} + P_{a\ell}^c U_c^{\ell k} + \frac{1}{2} T_{n\ell}^k U_a^{n\ell} - w_i^{j[lk]} (P_{j\ell a}^i + \underline{D_\ell P_{ja}^i - P_{jb}^i P_{\ell a}^b})] +
 \end{aligned}$$

$$\begin{aligned}
& +4n_{12}[P_{b\ell a}^c R_c^{b\ell k} + C_{ba}^c \cdot \frac{1}{4}\delta L_{12}/\delta L_{bk}^c] + \\
& +2\underline{n}_{13}[S_{jac}^i P_i^{jkc} + (D_c^* P_i^{jkc} - P_{nc}^k P_i^{jnc})P_{ja}^i] + \\
& +2n_{14}[S_{bac}^d P_d^{bkc} + (D_c^* P_d^{bkc} - P_{nc}^k P_d^{bnc})C_{ba}^d] = 0, \\
(15.2) \quad & \frac{\delta L}{\delta L_{jk}^i} = -4n_1 T_i^{jk} + 4\underline{n}_{21}(D_\ell^* R_i^{jk\ell} - \frac{1}{2}T_{n\ell}^k R_i^{jn\ell}) \\
& +n_{11}(D_\ell^* w_i^{j[k\ell]} - \frac{1}{2}T_{n\ell}^k w_i^{j[n\ell]}) + 2\underline{n}_{13}(D_c^* P_i^{jkc} - P_{nc}^k P_i^{jnc}) = 0,
\end{aligned}$$

$$\begin{aligned}
(15.3) \quad & \frac{\delta L}{\delta L_{bk}^a} = -2n_4 P_a^{bk} + 4n_{12}(D_\ell^* R_a^{bk\ell} - \frac{1}{2}T_{n\ell}^k R_a^{bn\ell}) \\
& + 2n_{14}(D_c^* P_a^{bkc} - \underline{P_{nc}^k P_a^{bnc}}) = 0,
\end{aligned}$$

$$\begin{aligned}
(15.4) \quad & \frac{\delta L}{\delta C_{ja}^i} = -2\underline{n}_3 P_i^{ja} + 2\underline{n}_{21} R_{k\ell}^a R_i^{jk\ell} - n_{11} R^{ja}{}_i - \\
& - 2n_{13}(D_\ell^* P_i^{j\ell a} - P_{b\ell}^a P_i^{j\ell b}) - \\
& - 4\underline{n}_{15}(D_d^* S_i^{jda} - \frac{1}{2}S_{bd}^a S_i^{jdb}) = 0,
\end{aligned}$$

$$\begin{aligned}
(15.5) \quad & \frac{\delta L}{\delta C_{bc}^a} = -4n_5 S_a^{bc} + 2n_{12} R_{k\ell}^c R_a^{bk\ell} + \underline{4n_{22}}(D_d^* S_a^{bcd} - \frac{1}{2}S_{ed}^c S_a^{bed}) + \\
& + n_{16}(D_d^* w_a^{b[cd]} - \frac{1}{2}S_{ed}^c w_a^{b[ed]}) - 2n_{14}(D_\ell^* P_a^{b\ell c} - P_{d\ell}^c P_a^{b\ell d}) = 0,
\end{aligned}$$

$$(15.6) \quad \frac{\delta L}{\delta g_{ij}} = -\frac{\partial L}{\partial g_{ij}} - \frac{1}{2}g^{ij}L = 0,$$

$$(15.7) \quad \frac{\delta L}{\delta h_{ab}} = -\frac{\partial L}{\partial h_{ab}} - \frac{1}{2}h^{ab}L = 0,$$

where we denoted

$$\begin{aligned}
& \begin{cases} D_\ell^* = D_\ell + V_\ell, & V_\ell = \frac{D_\ell G}{G} + T_{n\ell}^n + P_{d\ell}^d \\ D_c^* = D_c + V_c, & V_c = \frac{D_c G}{G} + P_{nc}^n + S_{dc}^d \end{cases} \\
& V_a^{k\ell} = C_{ja}^i R_i^{jk\ell}, \quad U_a^{k\ell} = C_{ja}^i w_i^{j[k\ell]} \\
& w_i^{jk\ell} = g^{jk}\delta_i^\ell, \quad w_a^{bcd} = h^{bc}\delta_a^d; \quad \frac{\delta L}{\delta \phi} \equiv \frac{1}{G} \frac{\delta \mathcal{L}}{\delta \phi}.
\end{aligned}$$

Hint. The Euler–Lagrange derivatives for the elementary Lagrangians (11) computed using the relations

$$\begin{aligned} D_k G &= \delta_k G - G(L_{nk}^n + L_{ak}^a) \\ D_a G &= \dot{\partial}_a G - G(C_{na}^n + C_{da}^d). \end{aligned}$$

give by addition the equations above.

Remark. The underscored terms are new with respect to [2], and the equations in [1,2] can be viewed as a particular case of (15.1)–(15.7). The notations of tensor fields and vertical indices are changed from those used by G.S. ASANOV to the corresponding ones used in the papers [8,9,11] in the theory of Finsler spaces. Also, the fact that in [1,2] the coefficients of the nonlinear connection considered in (1) are taken with opposite sign, induce related differences of sign in (15.1)–(15.7).

In the following we consider the quasi-metric case, i.e. the situation in which the gauge metric tensor fields obey

$$(16) \quad D_k g_{ij} = 0, \quad D_k h_{ab} = 0, \quad D_c h_{ab} = 0,$$

and impose for the gauge linear d -connection (2) to be h - and v -symmetrical, i.e. $T_{jk}^i = 0$ and $s_{bc}^a = 0$. Then we can state the following

Theorem 2. *The generalized Einstein–Yang Mills equations in the quasi-metric h - and v -symmetrical case, for the generalized gauge Lagrangian [2]*

$$(17) \quad \begin{aligned} L &= n_1 R_{jk}^a R_a^{jk} + n_3 P_{jb}^a P_a^{jb} + \ell_1 R^{mn}{}_{mn} + \ell_2 R^{abk\ell} R_{abk\ell} + \\ &+ \ell_{10} S^{ab}{}_{ab} + \Lambda, \quad \Lambda \in \mathcal{F}(E) \end{aligned}$$

with respect to the set of arguments

$$(18) \quad \phi \in \{N_i^a, A_{abi} = \frac{1}{2} L_{[ab]i}, C_{ja}^i, g_{ij}, h_{ab}\}$$

are the following

$$(19.1) \quad \begin{aligned} \frac{\delta L}{\delta N_i^a} &= 4n_1 (D_m^* R_a^{im} - P_{ka}^b R_b^{ik}) + 2n_3 (D_b^* P_a^{ib} - \underline{P_a^{nb} P_{nb}^i}) + \\ &+ \ell_1 [\underline{D_j^* U_a^{ij} + U_b^{ij} P_{aj}^b} - w_k^{nji} (-\dot{\partial}_a L_{nj}^k + V_n g^{km} \dot{\partial}_a g_{mj})] + \\ &+ 4\ell_2 R^{bcki} P_{bcka} + \frac{1}{2} C_{cba} \cdot \frac{\delta L}{\delta A_{cbi}} = 0, \end{aligned}$$

$$(19.2) \quad \begin{aligned} \frac{\delta L}{\delta g_{ij}} &= -\frac{1}{2} g^{ij} L + 2n_1 R^{iak} R_{ak}^j + n_3 P^{iab} P_{ab}^j + 2\ell_2 R_{abk}{}^i R^{abkj} + \\ &+ \ell_1 (R^{inj}{}_n + g^{ij} D_k^* V^k - \frac{1}{2} D^{*\{i} V^{j\}} + \frac{1}{2} \underline{P^{in}{}_a R^{ja}{}_n}) = 0, \end{aligned}$$

$$\begin{aligned}
(19.3) \quad \frac{\delta L}{\delta h_{ab}} = & -\frac{1}{2}h^{ab}L - n_1 R^{kal} R_{k\ell}^b + \ell_2 D_c^* R_{k\ell}^{\{b} R^{ca\}k\ell} - \frac{1}{2}L_{dj}^{\{a} \frac{\delta L}{\delta A_{db\}j} + \\
& + n_3(P_{jc}^a P^{jcb} - P_{jc}^a P^{jbc} - \frac{1}{2}D_j^* P^{j\{ab\}}) + \\
& + \ell_{10}(S^{acb}{}_c + h^{ab}D_c^* V^c - \frac{1}{2}D^{*\{a}V^{b\}}) = 0,
\end{aligned}$$

$$(19.4) \quad \frac{\delta L}{\delta A_{abi}} = n_3 P^{i[ba]} - 4\ell_2 D_n^* R^{abni} = 0,$$

$$(19.5) \quad \frac{\delta L}{\delta C_{ja}^i} = \ell_1 R_i^{aj} = 0,$$

where we denoted

$$\begin{cases} D_\ell^* = D_\ell + V_\ell, & V_\ell = P_{d\ell}^d \\ D_a^* = D_a + V_a, & V_a = \dot{\partial}_a \ln \sqrt{|\det(g_{ij})|} \end{cases}$$

$$\begin{aligned}
U_a^{jk} &= C_{nma} w^{nmjk}, \quad w^{nmjk} = g^{n[j} g^{mk]} \\
\delta L / \delta \phi &= \frac{1}{G} \frac{\delta \mathcal{L}}{\delta \phi}; \quad \tau_{\{ij\}} = \tau_{ij} + \tau_{ji}; \quad \tau_{[ij]} = \tau_{ij} - \tau_{ji}.
\end{aligned}$$

Hint. The same procedure as in Theorem 1 can be applied, using the relations [2,11]:

$$\begin{aligned}
L_{abi} &= h_{bs} L_{ai}^s = A_{abi} + \frac{1}{2} \delta_i h_{ab} \\
C_{bc}^a &= \frac{1}{2} h^{ad} (\dot{\partial}_{\{b} h_{dc\}} - \dot{\partial}_d h_{bc}).
\end{aligned}$$

Remark. As in theorem 1, the vanishing of the underscored terms in (19.1)–(19.5) give, as a particular case, the corresponding equations in [2].

The attempt of solving the equations in theorem 2 leads to the following results

Theorem 3. *The generalized Einstein–Yang Mills equations (19.1)–(19.5) admit the solution*

$$(20) \quad \{N_i^a, g_{ij}, h_{ab}, A_{abk}, C_{ja}^i\}$$

given by

$$\begin{aligned}
 (21) \quad & N_i^a = -\frac{C_i y^a}{2C}, C \in \mathcal{F}(M), \quad D_k^* \left(\frac{C_k}{C} \right) \neq 0 \\
 & g_{ij} = e^{\lambda(v)} \cdot \tau_{ij}(x), \quad v \in \mathcal{F}(E) \\
 & h_{ab} = \gamma_{ab}(x) + b(x, y) y_a y_b, \quad \text{with } y_a \equiv \gamma_{ab} y^b, b(x, y) = \frac{1}{v^2} \\
 & A_{abk} = 0 \\
 & C_{ja}^i \text{ satisfying: } D_j U_a^{ij} = \eta_j U_a^{ij}, \quad U_a^{ij} \equiv C_{nma} g^{n[i} g^{mj]}
 \end{aligned}$$

where $\tau_{ij}(x)$ satisfies the Einstein equations of Riemannian type

$$(22) \quad E_{ij} = m[\tau_{ij}(\alpha p + \nabla_k \alpha^k) - q^{\alpha_i \alpha_j} - \frac{1}{2} \nabla_{\{i} \alpha_{j\}}]$$

and (21) are subject to the following conditions

$$\begin{aligned}
 (23) \quad & \delta_k \gamma_{ab} = 0, \quad v = [C(x) - y^2]^{-1/2} \\
 & \lambda(x, y) = \frac{2}{n} \ln \left(v \sqrt{\frac{k}{C(x)}} \right), \quad \text{with } k \in \mathbb{R}_+^*, C(x) > y^2, (y^2 \equiv y_a y^a) \\
 & \eta \equiv d \left[\ln(|C(x)|^{(1-m)/2}) \right], \quad \Lambda = -m(m-1)\ell_{10}/C(x).
 \end{aligned}$$

In (21)–(23) ϱ_{ij} , ϱ and ∇_k are the Ricci tensor field, the scalar curvature and the covariant derivative associated to τ_{ij} , and we used the notations

$$\begin{cases} C_i = \frac{\partial C}{\partial x^i}, \quad \alpha_i = -\frac{C_i}{2C}, \quad \alpha = \alpha_i \alpha_j \gamma^{ij} \\ E_{ij} = \varrho_{ij} - \frac{1}{2} \varrho \tau_{ij} \quad (\text{the Einstein tensor field}) \\ p = \frac{m(3-n)}{2(n-2)}, \quad q = \frac{m}{n-2} \end{cases}$$

PROOF. The equations (19.1)–(19.5) have a general form; it becomes possible to search for solutions of the family (22) of the form (23)–(23') under additional simplifying assumptions, namely

$$\begin{aligned}
 (A1) \quad & \delta_k h_{ab} = 0 \\
 (A2) \quad & b = \frac{1}{v^2} = b(x, z), \quad \text{with } z = y^2 \\
 (A3) \quad & \text{the differential equation of Riccati type obtained from} \\
 & \frac{\delta L}{\delta h_{ab}} = 0, \text{ to become one of Bernoulli type} \\
 (A4) \quad & \delta_k \lambda = 0, \delta_k \gamma_{ab} = 0.
 \end{aligned}$$

The cosmological constant Λ is obtained from $\frac{\delta L}{\delta g_{ij}} = 0$, equation which provides the classical Einstein equations in (24). The conditions (A1)–(A4) yield to the form of the class of solutions stated in the theorem.

For the case when ξ is the tangent bundle of M , and $L^n = (M, g_{ij}(x, y))$ is a structure of a generalised Lagrange space [11] endowed with the non-linear connection $\{N_i^a(x, y)\}$

$$(24) \quad N_i^a = \left\{ \begin{matrix} a \\ ij \end{matrix} \right\} y^j, \quad \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} \gamma^{is} \left(\frac{\partial \gamma_{sj}}{\partial x^k} + \frac{\partial \gamma_{sk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^s} \right)$$

and the fundamental tensor field

$$(25) \quad g_{ij}(x, y) = \gamma_{ij}(x) + \frac{1}{c^2} y_i y_j; \quad y_i = \gamma_{is} y^s, \quad c > 0,$$

where $\{\gamma_{ij}(x)\}$ is a Riemannian metric on M , we consider the N -lift of g_{ij} to TM ([11])

$$(26) \quad G = g_{ij}(x, y) dx^i \otimes dx^j + g_{ab}(x, y) \delta y^a \otimes \delta y^b.$$

Then, for the case of the canonic metrical h - and v -symmetric linear d -connection, we obtain the coefficients

$$(27) \quad \begin{aligned} L_{jk}^i &= \frac{1}{2} g^{in} (\delta_{\{j} g_{nk\}} - \delta_n g_{jk}) \\ C_{bc}^a &= \frac{1}{2} h^{ad} (\dot{\partial}_{\{b} h_{dc\}} - \dot{\partial}_d h_{bc}) \\ \text{with } h_{ab} &\equiv g_{ab} = g_{ij} \delta_a^i \delta_b^j, \quad \text{and} \\ L_{bk}^a &= L_{jk}^i \delta_i^a \delta_b^j, \quad C_{jc}^i = C_{bc}^a \delta_a^i \delta_j^b \end{aligned}$$

and can formulate the following

Theorem 4. *If L^n is locally Minkowskian, then the gauge fields (13) given by (24), (25), (27) provide solutions for the generalized Einstein–Yang Mills equations (19.1)–(19.5) iff $n = 2$ and*

$$\Lambda = 6\ell_{10}/(1 + 3y^2)$$

where $y^2 = \gamma_{ij} y^i y^j$ and $\left\{ \begin{smallmatrix} a \\ ij \end{smallmatrix} \right\}$ are the Christoffel coefficients for $\gamma_{ij}(x)$, (see (24)).

Remark. The vanishing of the cosmological constant Λ would infer that (19.1)–(19.5) have no solution of the given form, unless $\ell_{10} = 0$.

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