

## On the 4-dimensional hyperbolic hypercube mosaic

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**Abstract.** We construct successively belts in the 4-dimensional hyperbolic regular hypercube mosaic beginning with a vertex, then considering all elements of the mosaic adjacent to it, then all elements of the mosaic adjacent to but different from those already considered, etc. In this article we determine the limits of the ratios of the volumes of the consecutive belts for the hyperbolic mosaic  $\{4, 3, 3, 5\}$  and its dual mosaic.

### Introduction

In the 4-dimensional hyperbolic space there is a hypercube mosaic ([1]). Its Schläfli's symbol is  $\{4, 3, 3, 5\}$ . In this article we examine this mosaic and its dual mosaic  $\{5, 3, 3, 4\}$ .

For the mosaic  $\{4, 3, 3, 5\}$ , let us fix a point  $P$ , as a vertex of the mosaic and create belts around it. The belt 1 consists of the fundamental domains (hypercubes) of the mosaic containing  $P$ . (The belt 0 is the point  $P$ .) The belt 2 consists of the fundamental domains having at least one common vertex with the belt 1. If the belt  $i$  is known, let the belt  $(i + 1)$  consist of the fundamental domains that have common vertices with the belt  $i$ , but do not have with the belt  $(i - 1)$ . Let  $\Pi_i$  denote the polyhedron determined by the outer boundary of the belt  $i$  ( $\Pi_0 = P$ ).  $\Pi_i$  is the union of all the belts  $j$ , where  $j = 0 \dots i$ .

Let  $V_i$  denote the volume of the belt  $i$ , and  $F_i^k$  ( $0 \leq k < 4$ ), denote the sum of the  $k$ -dimensional volumes of the  $k$ -dimensional faces of the mosaic on the surface of  $\Pi_i$ . (The 0-dimensional volume of a vertex is 1.) Furthermore, let  $S_i = \sum_{j=0}^i V_j$ , it is the volume of  $\Pi_i$ .

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In this article we are going to prove the following theorems.

**Theorem 1.** For  $0 \leq k \leq 3$  we have  $\lim_{i \rightarrow \infty} \frac{V_{i+1}}{V_i} = \lim_{i \rightarrow \infty} \frac{S_{i+1}}{S_i} = \lim_{i \rightarrow \infty} \frac{F_{i+1}^k}{F_i^k} \approx 2381.8$  (it can be called crystal-growing ratio) and  $\lim_{i \rightarrow \infty} \frac{V_i}{S_i} \approx 0.9996$  in case of the mosaic  $\{4, 3, 3, 5\}$ .

The mosaic  $\mathcal{M}$  is the *reciprocal* (or *dual*) mosaic of the mosaic  $\mathcal{N}$ , if the elements of  $\mathcal{M}$  are the Dirichlet–Voronoi cells of the vertices of  $\mathcal{N}$ . COXETER [1] showed that the mosaic  $\{s, r, q, p\}$  is the dual of the mosaic  $\{p, q, r, s\}$ . Thus the mosaic  $\{5, 3, 3, 4\}$  is the dual of  $\{4, 3, 3, 5\}$ , and its elements are 120-cells.

In case of the dual mosaic let the belt 0 be the D-V cell of the vertex  $P$  in the original mosaic. Its centre is  $P$ . We denote it by  $\Pi_0^*$ . Let the belt 1 be formed by the elements of the dual mosaic different from  $\Pi_0^*$ , but having common vertices with  $\Pi_0^*$ . Their centres coincide with the vertices of the original mosaic on the surface of  $\Pi_1$ , as they are their D-V cells in the original mosaic. We define the further belts similarly as in the case of the original mosaic, and  $\Pi_i^*$  as the union of the 0-th, ...,  $i$ -th belts. Also  $V_i$ ,  $F_i^k$  ( $0 \leq k \leq 3$ ) and  $S_i$  are defined analogously as for the original mosaic.

**Theorem 2.** For the mosaic  $\{4, 3, 3, 5\}$  and its dual mosaic  $\{5, 3, 3, 4\}$ , for  $0 \leq k \leq 3$ , each of the limits  $\lim_{i \rightarrow \infty} \frac{V_i}{S_i}$ ,  $\lim_{i \rightarrow \infty} \frac{V_{i+1}}{V_i}$ ,  $\lim_{i \rightarrow \infty} \frac{S_{i+1}}{S_i}$  and  $\lim_{i \rightarrow \infty} \frac{F_{i+1}^k}{F_i^k}$  are equal.

KÁRTESZI [5], HORVÁTH [3] and VERMES [9] dealt with similar problems in the hyperbolic plane, ZEITLER [11] solved the hyperbolic cube mosaic  $\{4, 3, 5\}$  case and the author [6] determined the limits of  $V_{i+1}/V_i$ ,  $S_{i+1}/S_i$ ,  $V_i/S_i$  for eight mosaics in the three-dimensional hyperbolic space with unbounded fundamental domains.

We classify the vertices of a mosaic incident to the surface of  $\Pi_i$  (they are not necessarily also vertices of  $\Pi_i$ ) depending on their edge-distance from the belt  $(i-1)$  (also from  $\Pi_{i-1}$ ). The *edge-distance* of two vertices of a mosaic is  $l$  if the two vertices are connected to each other with  $l$  edges of the mosaic, but  $(l-1)$  edges do not yet connect them. The edge-distance of the vertex  $Q$  of the mosaic from  $\Pi_{i-1}$  is  $l$  if there exists a vertex of the mosaic on the surface of  $\Pi_{i-1}$  so that its edge-distance from  $Q$  is  $l$  and the edge-distances of the other vertices on the surface of  $\Pi_{i-1}$  from  $Q$  are not smaller.

We denote the vertices of the mosaic on the surface of  $\Pi_i$  whose edge-distances from  $\Pi_{i-1}$  are 1 by  $A_i$ , and by  $B_i$ ,  $C_i$  or  $D_i$  if their edge-distances are 2, 3 or 4 edges, respectively. If we can not determine the edge-distance of a vertex from  $\Pi_{i-1}$  during the actual examination (or it is not important), then we denote it

by  $W_i$ . Let  $a_i, b_i, c_i, d_i$  denote the numbers of the vertices of type  $A_i, B_i, C_i, D_i$  on the surface of  $\Pi_i$ , respectively. Let  $r_i$  be the number of the hypercubes in the belt  $i$ . Further, let  $f_i^k$  ( $0 \leq k \leq 3$ ) denote the number of the  $k$ -dimensional faces (cubes, squares, edges, vertices) of the mosaic on the surface of  $\Pi_i$ .

We define the sequence  $r_i^*$  as the number of the elements of the dual mosaic in the  $i$ -th belt, and  $f_i^{*k}$ ,  $0 \leq k < 4$ , as the number of  $k$ -dimensional faces on the surface of  $\Pi_i^*$  of the elements of the dual mosaic, and by  $f_i^{*4}$  we mean  $r_i^*$ , similarly as in the case of the original mosaic.

As the mosaic  $\{4, 3, 3, 5\}$  is regular, its elements are regular and congruent. The nearest vertices to a common vertex (i.e., at edge distance 1) determine a regular polyhedron, and these are congruent, for all the vertices. We denote this *vertex figure* belonging to the vertex  $P$  by  $\Omega_P$ , and those belonging to  $A_i, B_i, C_i$  or  $D_i$  by  $\Omega_A, \Omega_B, \Omega_C$  or  $\Omega_D$ , respectively.

### 1. The mosaic $\{4, 3, 3, 5\}$

The 4-dimensional hyperbolic hypercube mosaic consists of hypercubes  $\{4, 3, 3\}$ . The vertex figures of the mosaic are 600-cells  $\{3, 3, 5\}$ . The cells of the 600-cell are tetrahedra ( $\{3, 3\}$ ) and the neighbouring vertices of a vertex of the 600-cell form an icosahedron ( $\{3, 5\}$ ). (In Figure 1 we can see the 2-dimensional graph of the 600-cell.) The numbers of the vertices, edges, (2-dimensional) faces and cells of the 600-cell are 120, 720, 1200 and 600, respectively.

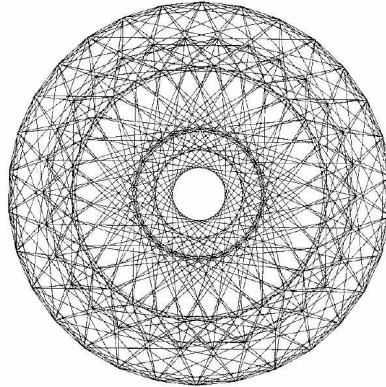


Figure 1

Examining the belt 1 in detail, we can see that on the surface of  $\Pi_1$  there are four types of the vertices. The vertices  $A_1$  are 1 edge away from  $P$ , and  $B_1$ ,  $C_1$  and  $D_1$  are 2, 3 and 4 edges away from  $P$ , respectively. Thus the edge-distances of the vertices of the mosaic on the surface of  $\Pi_1$  from  $P$  are at most 4 edges. (In Figure 2 we can see only a part — a vertex figure of one vertex — of the 600-cell, and the method how do we have to form the belt 1.) The vertices  $A_1$  are the vertices of  $\Omega_P$ . The vertices  $B_1$  correspond to the edges of  $\Omega_P$ . They are the vertices, diagonally opposite to  $P$ , of the squares with two edges leading from  $P$  to the endpoints of the edges of  $\Omega_P$ . The vertices  $C_1$  correspond to the (2-dimensional) faces of  $\Omega_P$ , while the vertices  $D_1$  correspond to the cells (3-dimensional faces) of  $\Omega_P$ , in a similar sense. They are the vertices, diagonally opposite to  $P$ , of the cubes/hypercubes with three/four edges leading from  $P$  to the vertices of 2-dimensional/3-dimensional faces of  $\Omega_P$ . We will say that the above  $A_1$ ,  $B_1$ ,  $C_1$ ,  $D_1$  *belong to* the vertices, edges, faces, cells of  $\Omega_P$ , respectively (and, analogously, for  $i > 1$ ,  $A_1, \dots, D_1, \dots, A_i, \dots, D_i$  *belong to* vertices,  $\dots$ , cells of  $\Omega_{A_{i-1}}, \dots, \Omega_{D_{i-1}}$ 's; moreover, the hypercubes containing the cells also are called *belonging to* the cell). Now, we can state  $a_1 = 120$ ,  $b_1 = 720$ ,  $c_1 = 1200$ ,  $d_1 = 600$  and  $r_1 = f_1^4 = 600$ ,  $f_1^3 = 2400$ ,  $f_1^2 = 7200$ ,  $f_1^1 = 7440$ ,  $f_1^0 = 2640$ .

It is also true, that there are 20 adjacent hypercubes of the belt 1 with a common vertex  $A_1$  (as there are 20 cells of  $\Omega_P$  containing the same vertex  $A_1$ ; another reason is that 20 hypercubes close around an edge  $PA_1$ ). Similarly, there are 5, 2 or 1 adjacent hypercubes at a vertex  $B_1$ ,  $C_1$  or  $D_1$ , respectively, because 5, 2 and 1 cells of a 600-cell have a common edge, face and cell of the 600-cell, respectively.

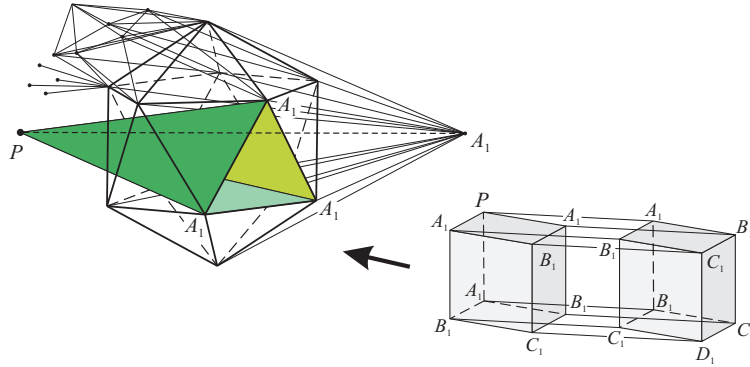


Figure 2

In the following we construct the belts.

In Figures 3, 4, 5 and 6 we can see the vertex figures. A vertex  $W_{i-1}$  ( $i \geq 1$ ) and all the edges, faces, cells of  $\Omega_A$  belonging to some  $A_{i-1}$ , which contain  $W_{i-1}$  and the hypercubes belonging to the cells also containing  $W_{i-1}$  are in the belt  $i$ . An edge (it is bold in Figure 4) and all the faces, cells of  $\Omega_B$  belonging to some  $B_{i-1}$  and the hypercubes belonging to the cells which contain the edge are in the belt  $i$ . A face (it is shaded in Figure 5) and all the cells of  $\Omega_C$  belonging to some  $C_{i-1}$  and the hypercubes belonging to the cells which contain the face are in the belt  $i$ . A cell (it is shaded in Figure 6) of  $\Omega_D$  belonging to some  $D_{i-1}$  and the hypercube belonging to this cell are in the belt  $i$ .

As the edge-distance of any two vertices of a hypercube is not bigger than four edges, the edge-distance of any vertex on the outer boundary of the belt  $j$  (for any  $j$ ) from the belt  $(j - 1)$  is one, two, three or four. Thus on the outer boundary of the belt  $j$  there are only vertices of type  $A_j$ ,  $B_j$ ,  $C_j$  and  $D_j$ .

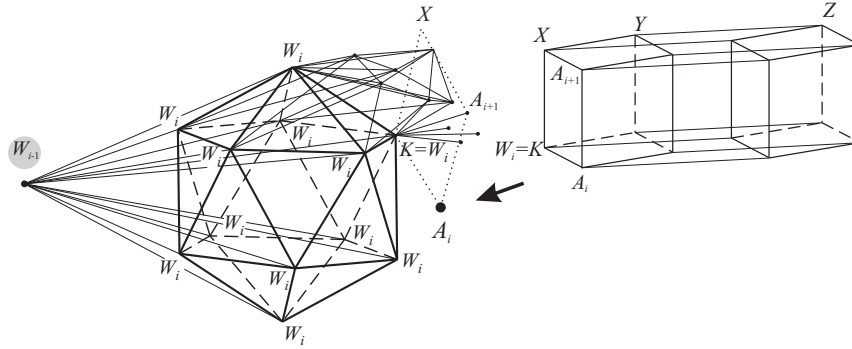


Figure 3

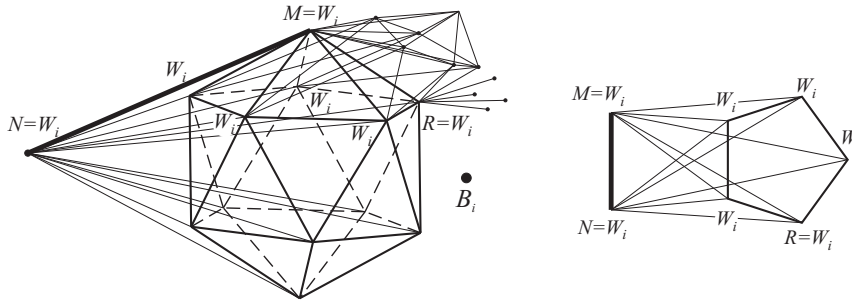


Figure 4

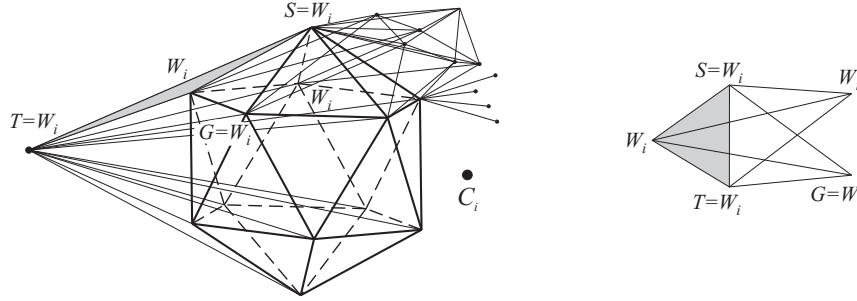


Figure 5

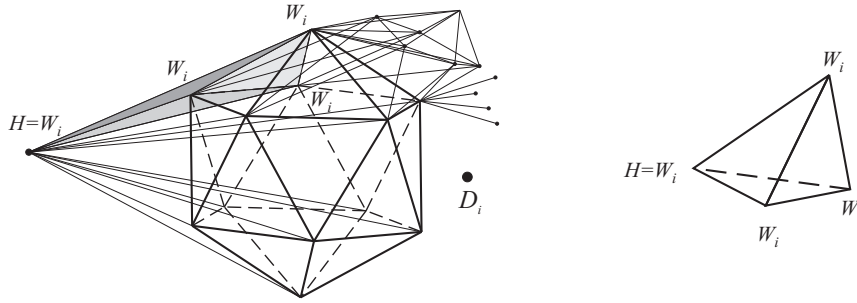


Figure 6

**Lemma 1.1.**  $a_{i+1} = 107a_i + 113b_i + 115c_i + 116d_i \quad (i \geq 1).$

PROOF. We classify the vertices of  $\Omega$ 's for determining the number of the vertices  $A_{i+1}$ . As they are one edge-distant from the current vertex, the vertices  $A_{i+1}$  come from among them. For each  $A_i$  the  $\Omega_A$  has a vertex  $W_{i-1}$  and 12 other vertices  $W_i$  (which have common edges with  $W_{i-1}$ ) (Figure 3). They are in (or on) the belt  $i$ . Thus the remaining  $120 - 13$  vertices are all  $A_{i+1}$  and they are one edge away from only the examined  $A_i$ . The number of all  $A_{i+1}$  belonging to all  $A_i$  is  $107a_i$ .

Similarly for the vertices  $B_i$ , there are 7 vertices of  $\Omega_B$  in the belt  $i$  (Figure 4), so the remaining  $120 - 7$  vertices are  $A_{i+1}$ . Their edge-distances are 1 from the current  $B_i$ , and only from this vertex. Their total number is  $113b_i$ . For  $C_i$ ,  $\Omega_C$  has 5 vertices in the belt  $i$  (Figure 5), so the remaining  $120 - 5$  vertices are  $A_{i+1}$  and they belong to the current  $C_i$ . Their total number is  $115c_i$ . Similarly for  $D_i$  the number of all  $A_{i+1}$  is  $116d_i$ .

Summarizing the four statements we get the Lemma.  $\square$

**Lemma 1.2.**  $b_{i+1} = 606a_i + 652b_i + 669c_i + 678d_i \quad (i \geq 1).$

PROOF. Now we classify the edges of the  $\Omega$ 's for determining the number of the vertices  $B_{i+1}$ . There are vertices  $B_{i+1}$  only along the edges whose neither endpoints are incident to the surface of  $\Pi_i$ . If one of them is on the surface then the vertex belonging to that edge is  $A_{i+1}$  or it is in the belt  $i$ . An example is in Figure 3 the vertex  $X$ . It is 2 edges away from  $A_i$ , but its edge-distance from  $K = W_i$  (namely from  $\Pi_i$ ) is 1. Thus the vertex  $X$  is  $A_{i+1}$  which was counted for  $K$  (in Lemma 1.1).

We are going to count the edges of the  $\Omega$ 's that have at least one vertex  $W_i$  and subtract their numbers from the number of all the edges, from 720.

In case of  $\Omega_A$  for  $A_i$  the numbers of the edges joining to all the vertices  $W_i$  and  $W_{i-1}$  are 12. The number of all edges containing  $W_i$  is  $13 \cdot 12$ , but now 42 edges are counted doubly. Thus the number of the edges with one vertex  $W_i$  is  $13 \cdot 12 - 42 = 114$ . Similarly for  $\Omega_B, \Omega_C, \Omega_D$  for the vertices  $B_i, C_i$  and  $D_i$  the numbers of the edges with one vertex some  $W_i$  are  $7 \cdot 12 - 16 = 68$ ,  $5 \cdot 12 - 9 = 51$  and  $4 \cdot 12 - 6 = 42$ , respectively.

Summarizing the results, we get that  $b_{i+1} = (720 - 114)a_i + (720 - 68)b_i + (720 - 51)c_i + (720 - 42)d_i$ .  $\square$

**Lemma 1.3.**  $c_{i+1} = 970a_i + 1055b_i + 1088c_i + 1106d_i \quad (i \geq 1).$

PROOF. Now we classify the 2-dimensional faces of the  $\Omega$ 's for determining the number of the vertices  $C_{i+1}$ . There are vertices  $C_{i+1}$  only along the faces whose neither vertices are incident to the surface of  $\Pi_i$ . If a face has got at least one vertex  $W_i$ , then the vertex of the hypercube, which is 3 edges away from the examined centre of  $\Omega$ , is 2 edges away from the vertex  $W_i$ . So it is not a vertex  $C_{i+1}$ . An example is in Figure 3 the vertex  $Y$ . It is 3 edges away from  $A_i$ , but from  $K = W_i$  it is 2 edges away. The vertex  $Y$  is not an  $A_{i+1}$ .

We are going to examine the faces with at least one vertex  $W_i$  and subtract their numbers from the number of the faces.

In case of a 600-cell 30 faces join to a vertex and 5 faces to an edge. So,  $12 \cdot 30$  faces (with multiplicity) contain the vertices  $W_i$  of an  $\Omega_A$ . But we counted doubly (or triply) the faces with two (or three)  $W_i$ , so from the sum we subtract  $30 \cdot 5$ . Since the number of the faces with three  $W_i$  are subtracted doubly, now we add their number, 20. Thus, the number of the faces containing at least one  $W_i$  is  $12 \cdot 30 - 30 \cdot 5 + 20 = 230$ . (It includes the number of faces with a vertex  $W_{i-1}$  too.) Similarly, we get the numbers of the faces with  $W_i$  for the  $\Omega_B, \Omega_C, \Omega_D$ 's, they are  $7 \cdot 30 - 16 \cdot 5 + 15 = 145$ ,  $5 \cdot 30 - 9 \cdot 5 + 7 = 112$  and  $4 \cdot 30 - 6 \cdot 5 + 4 = 94$ .

Summarizing we get  $c_{i+1} = (1200 - 230)a_i + (1200 - 145)b_i + (1200 - 112)c_i + (1200 - 94)d_i$ .  $\square$

**Lemma 1.4.**  $d_{i+1} = 470a_i + 515b_i + 533c_i + 543d_i \quad (i \geq 1)$ .

PROOF. We classify the cells of the  $\Omega$ 's for determining the number of the vertices  $D_{i+1}$ . There are vertices  $D_{i+1}$  only along the cells whose neither vertices are incident to the surface of  $\Pi_i$ . An example is in Figure 3 the vertex  $Z$ . It is 4 edges away from  $A_i$ , but its edge-distance from  $K = W_i$  is 3. The vertex  $Z$  is not an  $A_{i+1}$ .

First of all we count the numbers of the cells containing one, two, three or four vertices  $W_i$ , and then we subtract their number from the number of the cells, i.e., from 600. There are 20, 5 or 2 cells containing a common vertex, edge or face, respectively. For  $\Omega_A$  the numbers of the cells containing at least one, two or three  $W_i$ 's (there is not a cell containing four  $W_i$ 's) are  $12 \cdot 20$ ,  $30 \cdot 5$  and  $20 \cdot 2$ . Thus the number of the cells is  $12 \cdot 20 - 30 \cdot 5 + 20 \cdot 2 = 130$ . (It includes the cells with a vertex  $W_{i-1}$ , too.) Similarly, the numbers of cells with  $W_i$  for the  $\Omega_B$ 's,  $\Omega_C$ 's,  $\Omega_D$ 's are  $7 \cdot 20 - 16 \cdot 5 + 15 \cdot 2 - 5 = 85$ ,  $5 \cdot 20 - 9 \cdot 5 + 7 \cdot 2 - 2 = 67$  and  $4 \cdot 20 - 6 \cdot 5 + 4 \cdot 2 - 1 = 57$ .

Summarizing we get  $c_{i+1} = (600 - 130)a_i + (600 - 85)b_i + (600 - 67)c_i + (600 - 57)d_i$ .  $\square$

The numbers of the hypercubes of the belt  $i$  can be determined by the help of the numbers of the vertices of the belt  $i$ , of the different types  $(A_i, \dots, D_i)$ . So the sequences  $r_{i+1}$  and  $f_{i+1}^k$  ( $0 \leq k \leq 3$ ) will turn out to be homogeneous linear functions of  $a_i$   $b_i$   $c_i$   $d_i$ , cf. the following Lemmas 1.5, 1.6.

**Lemma 1.5.**  $r_{i+1} = 510a_i + \frac{2185}{4}b_i + \frac{1119}{2}c_i + \frac{1133}{2}d_i \quad (i \geq 1)$ .

PROOF. We again classify the cells of the  $\Omega$ 's for determining the number of the hypercubes in the belt  $(i + 1)$ . The hypercubes belonging to the cells whose all the four vertices are  $W_i$  ( or  $W_{i-1}$  ) are in the belt  $i$ , the others are the only new hypercubes (in the belt  $(i + 1)$ ). The new hypercubes with vertices  $W_i$  are not only connected as new hypercubes to the centre of the considered  $\Omega$  but also to other vertices.

In case of a cell with three vertices  $W_i$ , the hypercube belongs to eight vertices on the outer boundary of the belt  $i$  at the same time, to the eight vertices of the cube generated by the three vertices  $W_i$  and the centre of  $\Omega$ . So the number of the hypercubes like this is divided by eight because of the multiplicity. In case of a cell with three/two/one vertices  $W_i$ , the hypercube belongs to eight/four/two vertices on the outer boundary of the belt  $i$  at the same time. These eight/four/two



vertices of the cube/square/edge are generated by the three/two/one vertices  $W_i$  and the centre of  $\Omega$ . So the number of the hypercubes like this is divided by eight/four/two because of the multiplicity. The hypercubes connected to cells with no vertices  $W_i$  belong only to the centre of the considered  $\Omega$  as new hypercubes.

An  $\Omega_A$  has twenty cells with three vertices  $W_i$  and one vertex  $W_{i-1}$ . The vertex  $W_{i-1}$  is the common vertex of them (see Figure 3). Therefore the hypercubes along these cells are in the belt  $(i+1)$ . The  $\Omega_A$  has also twenty cells with three vertices  $W_i$ , because an icosahedron has twenty faces and two cells of  $\Omega_A$  connect to every face. The  $\Omega_A$  has thirty cells with two vertices  $W_i$ , because an icosahedron has thirty edges and five cells of  $\Omega_A$  connect to every edge, but four of them have three  $W_i$ 's. The  $\Omega_A$  has sixty cells with only one vertex  $W_i$ , because an icosahedron has twelve vertices and twenty cells of  $\Omega_A$  connect to every vertex, but fifteen of them have three or two vertices  $W_i$ . The number of the examined cells is  $20 + 20 + 30 + 60 = 130$ , so the number of the remaining cells is  $600 - 130$ . The hypercubes along these cells belong only to the considered vertices  $A_i$ . Thus the number of the hypercubes belonging to an  $A_i$  is  $20 \cdot 0 + \frac{20 \cdot (2-1)}{8} + \frac{30 \cdot (5-4)}{4} + \frac{12 \cdot (20-15)}{2} + (600 - 130) = 510$ .

An  $\Omega_B$  has five cells with only vertices  $W_i$ . They are around the edge  $MN$  (see Figure 4). There are ten cells, which have only three vertices  $W_i$ . (They connect to the edge of the pentagon with the vertices  $W_i$  and vertex  $M$  or  $N$ .) The  $\Omega_B$  has twenty cells with only two vertices  $W_i$ . There are two cells along any edge of the pentagon and one along any edge like  $NR$ . Among the cells with vertices  $W_i$  we only do not count the cells with only one vertex  $W_i$ . They are the cells with one vertex of the pentagon. The vertex  $R$  belongs to twenty cells of  $\Omega_B$ , but ten of them have three or two vertices  $W_i$ . So, there are 5·10 cells with only one vertex  $W_i$ . The remaining cells are connected to the vertex  $B_i$ . Thus the number of the hypercubes belonging to a  $B_i$  is  $5 \cdot 0 + \frac{5+5}{8} + \frac{5 \cdot 2 + 10 \cdot 1}{4} + \frac{5 \cdot 10}{2} + (600 - 85) = \frac{2185}{4}$ .

Similarly, in cases of the vertices  $C_i$ , or  $D_i$ , the number of the hypercubes are  $2 \cdot 0 + \frac{6}{8} + \frac{15}{4} + \frac{2 \cdot 10 + 3 \cdot 8}{2} + (600 - 67) = \frac{1119}{2}$ , or  $1 \cdot 0 + \frac{4}{8} + \frac{6 \cdot 2}{4} + \frac{4 \cdot 10}{2} + (600 - 57) = \frac{1133}{2}$ , respectively. Summarizing we get the lemma.  $\square$

**Lemma 1.6.**  $f_{i+1}^3 = \frac{3975}{2}a_i + \frac{8585}{4}b_i + \frac{8825}{4}c_i + \frac{4477}{2}d_i$ ,  $f_{i+1}^2 = \frac{11925}{2}a_i + \frac{25755}{4}b_i + \frac{26475}{4}c_i + \frac{13431}{2}d_i$ ,  $f_{i+1}^1 = 6128a_i + \frac{13255}{2}b_i + \frac{13635}{2}c_i + 6920d_i$  and  $f_{i+1}^0 = 2153a_i + 2335b_i + 2405c_i + 2443d_i$  ( $i \geq 1$ ).

**PROOF.** For determining the number of the 3-dimensional faces (cubes) on the outer boundary the belt  $i$  we classify the 2-dimensional faces of the  $\Omega$ 's. A 3-dimensional face determined by the centre of the considered  $\Omega$  and three vertices

$W_i$  is on the outer boundary of the belt  $i$ , and it is a cube on the outer boundary of the belt  $i$ , which is counted at each of its eight vertices. So, we divided by eight the number of the cubes like this because of the multiplicity. The other 3-dimensional faces are not on the surface of the belt  $i$ . There are twenty, ten, six or four 2-dimensional faces of  $\Omega_A$ ,  $\Omega_B$ ,  $\Omega_C$  or  $\Omega_D$ , respectively (see fig. 3-6). Naturally, the 3-dimensional faces determined by the vertex  $W_{i-1}$  of  $\Omega_A$ , by the edge of  $MN$  of  $\Omega_B$  and by the triangle  $TSW_i$  of  $\Omega_C$  are not on the outer boundary of the belt  $i$ . They are in the belt  $i$ .

Thus  $f_i^3 = \frac{20}{8}a_i + \frac{10}{8}b_i + \frac{6}{8}c_i + \frac{4}{8}d_i$ , so  $f_{i+1}^3 = \frac{20}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{10}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{6}{8}(107a_i + 113b_i + 115c_i + 116d_i) + \frac{4}{8}(107a_i + 113b_i + 115c_i + 116d_i) = \frac{3975}{2}a_i + \frac{8585}{4}b_i + \frac{8825}{4}c_i + \frac{4477}{2}d_i$ .

Similarly, for determining the number of the 2-dimensional/1-dimensional faces on the outer boundary of the belt  $i$  we classify the 1-dimensional/0-dimensional faces of the  $\Omega$ 's. A 2-dimensional/1-dimensional face determined by the centre of the considered  $\Omega$  and two/one vertices  $W_i$  are on the outer boundary of the belt  $i$ . Furthermore, they are 2-dimensional/1-dimensional faces on the outer boundary of the belt  $i$ , in case of the four/two vertices of the square/edge determined by the the centre of the considered  $\Omega$  and the vertices  $W_i$ , too. So, we divide by four/two the number of the 2-dimensional/1-dimensional faces similarly. Thus  $f_i^2 = \frac{30}{4}a_i + \frac{15}{4}b_i + \frac{9}{4}c_i + \frac{6}{4}d_i$ ,  $f_i^1 = \frac{12}{2}a_i + \frac{7}{2}b_i + \frac{5}{2}c_i + \frac{4}{2}d_i$ . (Moreover  $f_i^2 = 3f_i^1$ .) It proves the lemma.

Furthermore  $f_{i+1}^0 = a_{i+1} + b_{i+1} + c_{i+1} + d_{i+1} = 2153a_i + 2335b_i + 2405c_i + 2443d_i$ .  $\square$

*Remark 1.* Obviously,  $r_i$  and  $f_i^k$  ( $i \geq 1, 0 \leq k \leq 3$ ) are integers and  $a_1$ ,  $b_1$ ,  $c_1$  and  $d_1$  are dividable by eight.

## 2. The proof of Theorem 1

Let  $v$  denote the volume of a hypercube of the mosaic. Then  $\lim_{i \rightarrow \infty} \frac{V_{i+1}}{V_i} = \lim_{i \rightarrow \infty} \frac{v \cdot r_{i+1}}{v \cdot r_i} = \lim_{i \rightarrow \infty} \frac{r_{i+1}}{r_i}$  and  $\lim_{i \rightarrow \infty} \frac{V_i}{S_i} = \lim_{i \rightarrow \infty} \frac{v \cdot r_i}{v \cdot \sum_{j=0}^i r_j} = \lim_{i \rightarrow \infty} \frac{r_i}{\sum_{j=0}^i r_j}$  ( $i \geq 1$ ). Thus we can calculate considering the numbers of the hypercubes instead of the volumes.

From the previous lemmas we get the following linear recursion for the sequences  $a_i$ ,  $b_i$ ,  $c_i$ ,  $d_i$  ( $i \geq 1$ ):

$$a_{i+1} = 107a_i + 113b_i + 115c_i + 116d_i$$

$$b_{i+1} = 606a_i + 652b_i + 669c_i + 678d_i$$

$$\begin{aligned} c_{i+1} &= 970a_i + 1055b_i + 1088c_i + 1106d_i \\ d_{i+1} &= 470a_i + 515b_i + 533c_i + 543d_i \end{aligned} \quad (1)$$

and the sequence  $r_i$  can be given by the help of  $a_i$ ,  $b_i$ ,  $c_i$  and  $d_i$  the following way:

$$r_{i+1} = \alpha_1 a_i + \alpha_2 b_i + \alpha_3 c_i + \alpha_4 d_i, \quad (2)$$

where  $i \geq 1$ . In a shorter form

$$w_{i+1} = \mathbf{M}w_i, \quad (3)$$

$$r_{i+1} = \alpha^T w_i, \quad (4)$$

where

$$\mathbf{M} = \begin{pmatrix} 107 & 113 & 115 & 116 \\ 606 & 652 & 669 & 678 \\ 970 & 1055 & 1088 & 1106 \\ 470 & 515 & 533 & 543 \end{pmatrix}, \quad (5)$$

and  $w_j = [a_j \ b_j \ c_j \ d_j]^T$ ,  $\alpha = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]^T = [510 \ \frac{2185}{4} \ \frac{1119}{2} \ \frac{1133}{2}]^T$ .

As a well-known consequence of the Jordan normal form of matrices (a simple proof cf. in [6], Lemma 1.11), we have

$$r_i = g_1 z_1^i + g_2 z_2^i + g_3 z_3^i + g_4 z_4^i, \quad (6)$$

where the  $z_j$ 's are the eigenvalues of the matrix  $\mathbf{M}$ , provided that the  $z_j$ 's are different and all are different from 0. Now  $z_1 \approx 2381.8277$ ,  $z_2 \approx 8.0476$ ,  $z_3 \approx 0.1243$  and  $z_4 \approx 0.0004$ . They are all exactly real, since else there would be a conjugate complex pair among them, hence of the same absolute value. From §1 we have  $(a_1, b_1, c_1, d_1) = (120, 720, 1200, 600)$ .

By (1) we obtain in turn

|         | $a_j$         | $b_j$         | $c_j$          | $d_j$         |
|---------|---------------|---------------|----------------|---------------|
| $j = 2$ | 301800        | 1751760       | 2845200        | 1392600       |
| $j = 3$ | 718981080     | 4172659920    | 6776646000     | 3316675800    |
| $j = 4$ | 1712490229320 | 9938559168720 | 16140802146000 | 7899748243800 |

These imply by Lemma 1.5  $r_2 = 1465800$ ,  $r_3 = 3491614200$ ,  $r_4 = 8316426109800$ ,  $r_5 = 19808294143666200$ . Putting these values to (6), for  $2 \leq i \leq 5$ , we obtain a system of four linear equations for  $g_1, \dots, g_4$ . The (unique) solutions are  $g_1 \approx 615.5$ ,  $g_2 \approx -13.8$ ,  $g_3 \approx -14903.1$ ,  $g_4 \approx 4343347.8$ . We observe that  $g_1 \neq 0$ .

Since  $\lim_{i \rightarrow \infty} \left(\frac{z_j}{z_1}\right)^i = 0$  ( $2 \leq j \leq 4$ ), then

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{V_{i+1}}{V_i} &= \lim_{i \rightarrow \infty} \frac{r_{i+1}}{r_i} = \lim_{i \rightarrow \infty} \frac{g_1 z_1^{i+1} + g_2 z_2^{i+1} + g_3 z_3^{i+1} + g_4 z_4^{i+1}}{g_1 z_1^i + g_2 z_2^i + g_3 z_3^i + g_4 z_4^i} \\ &= \lim_{i \rightarrow \infty} \frac{g_1 z_1 + g_2 z_2 \left(\frac{z_2}{z_1}\right)^i + g_3 z_3 \left(\frac{z_3}{z_1}\right)^i + g_4 z_4 \left(\frac{z_4}{z_1}\right)^i}{g_1 + g_2 \left(\frac{z_2}{z_1}\right)^i + g_3 \left(\frac{z_3}{z_1}\right)^i + g_4 \left(\frac{z_4}{z_1}\right)^i} = z_1. \end{aligned}$$

Further,

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{V_i}{S_i} &= \lim_{i \rightarrow \infty} \frac{r_i}{\sum_{j=0}^i r_j} = \lim_{i \rightarrow \infty} \frac{g_1 z_1^i + g_2 z_2^i + g_3 z_3^i + g_4 z_4^i}{g_1 \sum_{j=0}^i z_1^j + g_2 \sum_{j=0}^i z_2^j + g_3 \sum_{j=0}^i z_3^j + g_4 \sum_{j=0}^i z_4^j} \\ &= \lim_{i \rightarrow \infty} \frac{g_1 + g_2 \left(\frac{z_2}{z_1}\right)^i + \cdots + g_4 \left(\frac{z_4}{z_1}\right)^i}{g_1 \frac{z_1 - \frac{1}{z_1^i}}{z_1 - 1} + g_2 \frac{z_2 \left(\frac{z_2}{z_1}\right)^i - \frac{1}{z_1^i}}{z_2 - 1} + \cdots + g_4 \frac{z_4 \left(\frac{z_4}{z_1}\right)^i - \frac{1}{z_1^i}}{z_4 - 1}} \\ &= \frac{z_1 - 1}{z_1} \approx 0.9996. \end{aligned}$$

Moreover, if  $z(k, i) = \frac{\frac{z_k^{i+1} - 1}{z_k - 1}}{\frac{z_1^{i+1} - 1}{z_1 - 1}}$  ( $1 < k \leq 4$ ), then  $\lim_{i \rightarrow \infty} z(k, i) = 0$  and

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{S_{i+1}}{S_i} &= \lim_{i \rightarrow \infty} \frac{v \cdot \sum_{j=0}^{i+1} r_j}{v \cdot \sum_{j=0}^i r_j} \\ &= \lim_{i \rightarrow \infty} \frac{g_1 z_1 \sum_{j=0}^i z_1^j + g_2 z_2 \sum_{j=0}^i z_2^j + g_3 z_3 \sum_{j=0}^i z_3^j + g_4 z_4 \sum_{j=0}^i z_4^j}{g_1 \sum_{j=0}^i z_1^j + g_2 \sum_{j=0}^i z_2^j + g_3 \sum_{j=0}^i z_3^j + g_4 \sum_{j=0}^i z_4^j} \\ &= \frac{g_1 z_1 \frac{z_1^{i+1} - 1}{z_1 - 1} + g_2 z_2 \frac{z_2^{i+1} - 1}{z_2 - 1} + g_3 z_3 \frac{z_3^{i+1} - 1}{z_3 - 1} + g_4 z_4 \frac{z_4^{i+1} - 1}{z_4 - 1}}{g_1 \frac{z_1^{i+1} - 1}{z_1 - 1} + g_2 \frac{z_2^{i+1} - 1}{z_2 - 1} + g_3 \frac{z_3^{i+1} - 1}{z_3 - 1} + g_4 \frac{z_4^{i+1} - 1}{z_4 - 1}} \\ &= \frac{g_1 z_1 + g_2 z_2 z(2, i) + g_3 z_3 z(3, i) + g_4 z_4 z(4, i)}{g_1 + g_2 z(2, i) + g_3 z(3, i) + g_4 z(4, i)} \\ &= z_1 = \lim_{i \rightarrow \infty} \frac{V_{i+1}}{V_i}. \end{aligned}$$

Now let us consider  $f_i^k$  for  $0 \leq k \leq 3$ . From the above values of  $(a_1, \dots, d_1), \dots, (a_4, \dots, d_4)$  we have

|         | $f_2^k$  | $f_3^k$     | $f_4^k$         | $f_5^k$            |
|---------|----------|-------------|-----------------|--------------------|
| $k = 0$ | 6291360  | 14984962800 | 35691599787840  | 85011241243420560  |
| $k = 1$ | 17840160 | 42493162800 | 101211400319040 | 241068117493415760 |
| $k = 2$ | 17323200 | 41262300000 | 98279700796800  | 234085314374992800 |
| $k = 3$ | 5774400  | 13754100000 | 32759900265600  | 78028438124997600  |

Analogously to the case  $r_i$ , this leads to the analogue of (6), with

$$\begin{aligned}
 (g_1 \neq 0, g_2, g_3, g_4) &\approx (2641.4, 13.8, -63968.8, 18640343.4), & \text{or} \\
 &\approx (7490.2, -7.8, -181396.9, 52858803.6), & \text{or} \\
 &\approx (7273.3, -32.5, -176142.1, 51327690.2), & \text{or} \\
 &\approx (2424.4, -10.8, -58714, 17109230.1),
 \end{aligned}$$

for  $f_i^0$ , or  $f_i^1$ , or  $f_i^2$ , or  $f_i^3$ , respectively. The rest of the proof is word by word the same as for  $r_i$ .  $\square$

*Remark 2.* From the proof of Theorem 1 we get that we can choose a hypercube for  $\Pi_0$  instead of  $P$ . Then for  $r_i, f_i^3, \dots$ , or  $f_i^0$ , we have  $g_1 \approx 9106.7 \neq 0$ ,  $35880.7 \neq 0$ ,  $107642.2 \neq 0$ ,  $110853.2 \neq 0$ , or  $39091.7 \neq 0$ , respectively, and the present limits are equal to those in the previous case  $\Pi_0 = P$ .

### 3. Dual mosaics

**3.1. The proof of Theorem 2.** The belt  $(i+1)$  ( $i \geq 0$ ) of the dual mosaic is formed by the D-V cells of the original mosaic, whose centres are the vertices on the surface of  $\Pi_{i+1}$ . They are the vertices  $A_{i+1}, B_{i+1}, C_{i+1}$  and  $D_{i+1}$ . (We will not need to classify the vertices of  $\Pi_{i+1}^*$ , and determine their respective numbers.) Thus  $r_{i+1}^* = f_{i+1}^{*4} = a_{i+1} + b_{i+1} + c_{i+1} + d_{i+1} = f_{i+1}^0$  ( $i \geq 0$ ).

Two elements of the dual mosaic (D-V cells for the original mosaic) join to each 3-dimensional face of the dual mosaic (common face of the D-V cells for the original mosaic). Thus on the surface of  $\Pi_{i+1}^*$  there are as many 3-dimensional faces of the dual mosaic, as the number of the D-V cells of the original mosaic joining to  $\Pi_{i+1}^*$  with 3-dimensional faces. They are the D-V cells for the original mosaic with centres  $A_{i+2}$ . That is why  $f_{i+1}^{*3} = a_{i+2}$  ( $i \geq 0$ ).

Similarly, the D-V cells with centres  $B_{i+2}$ ,  $C_{i+2}$  and  $D_{i+2}$  join to the surface of  $\Pi_{i+1}^*$  with 2, 1, 0-dimensional faces. Thus  $f_{i+1}^{*2} = b_{i+2}$ ,  $f_{i+1}^{*1} = c_{i+2}$ ,  $f_{i+1}^{*0} = d_{i+2}$  ( $i \geq 0$ ).

Thus we have determined the sequences  $r_{i+1}^*$  and  $f_{i+1}^{*k}$  ( $0 \leq k \leq 3$ ) by the sequences  $a_i$ ,  $b_i$ ,  $c_i$  and  $d_i$ , as certain linear combinations of these four sequences (the numbers  $a_{i+1}, \dots, d_{i+1}$ , or  $a_{i+2}, \dots, d_{i+2}$  can be calculated from  $a_i, \dots, d_i$  by applying (1) once or twice, respectively). These equalities are the analogues of Lemmas 1.5, 1.6 for the original mosaic. Observe that also now we have the same matrix  $\mathbf{M}$  as in (5). Thus, rather than (6), we have the same formula, with its right hand side containing the same  $z_1, \dots, z_4$ , and with its left hand side replaced, in turn, by  $r_{i+1}^*$ ,  $f_{i+1}^{*3}$ ,  $\dots$ , or  $f_{i+1}^{*0}$ , respectively. Again calculating  $r_{i+1}^*$ ,  $f_{i+1}^{*3}$ ,  $\dots$ ,  $f_{i+1}^{*0}$ , for  $0 \leq i \leq 3$ , and putting them to the analogue of (6), we can uniquely solve the respective systems of linear equations, for  $g_1, \dots, g_4$ . We obtain, in the above order,  $(g_1, g_2, g_3, g_4) \approx (615.5, -13.8, -14903.1, 4343347.8)$ ,  $(2424.4, -10.8, -58714, 17109230.1)$ ,  $(7273.3, -32.5, -176142.1, 51327690.2)$ ,  $(7490.3, -7.8, -181396.9, 52858803.6)$ , or  $(2641.4, 13.8, -63968.8, 18640343.4)$ , respectively. Observe that in each case we have  $g_1 \neq 0$ . Thus, analogously to the proof of Theorem 1, the limits for  $V_i, F_i^3, \dots, F_i^0$  for the dual mosaic, i.e., those for  $r_i^*, f_i^{*3}, \dots, f_i^{*0}$ , are equal to those for  $r_i, f_i^3, \dots, f_i^0$ , i.e., to those for  $V_i, F_i^3, \dots, F_i^0$  for the original mosaic, namely to  $z_1$ .  $\square$

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