

On a class of conformally invariant horizontal endomorphisms

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Abstract. We construct a class of conformally invariant horizontal endomorphisms (or Ehresmann connections) on a Finsler manifold which contains the Wagner endomorphisms studied in detail in [12].

1. Introduction

The set of all horizontal endomorphisms over a manifold M constitutes an affine space modelled on the real vector space $\Psi^1(TM)$ of semibasic vector 1-forms on TM . Motivated by a result of Sz. SZAKÁL and J. SZILASI ([7], Proposition 2.7), in this paper we study the orbit of the canonical horizontal endomorphism (the so-called Barthel endomorphism) of a Finsler manifold under the action of a subspace of $\Psi^1(TM)$ depending on the Finsler structure. The elements of this orbit will be mentioned as *L-horizontal endomorphisms* ($L \in \Psi^1(TM)$). We shall point out that Wagner endomorphisms studied in [11], [12] and [7] can be obtained as *special L-horizontal endomorphisms*. Our main result states that *the set of all conservative L-horizontal endomorphisms on a Finsler manifold is conformally closed*.

CONVENTIONS. (i) We work on an n -dimensional connected smooth manifold M whose topology is Hausdorff and has a countable base. $C^\infty(M)$ denotes the ring of smooth real-valued functions on M , $\mathcal{X}(M)$ stands for the $C^\infty(M)$ -module of (smooth) vector fields on M . $\Omega(M) := \bigoplus_{i=0}^n \Omega^i(M)$ is the graded algebra of differential forms on M , with multiplication given by the wedge product. The

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symbols d , i_X , $\mathcal{L}_X$ ($X \in \mathcal{X}(M)$) denote the exterior derivative, the substitution operator and the Lie derivative.

(ii) TM is the $2n$ -dimensional tangent manifold of M , $\overset{\circ}{TM} \subset TM$ is the open submanifold of the non-zero tangent vectors to M . f^v and f^c stand for the vertical and the complete lift of a smooth function on M into TM .

2. Preliminaries

2.1. For any vector field X on M there exist unique vector fields X^v , X^c on TM such that

$$X^v f^c = (Xf)^v, \quad X^c f^c = (Xf)^c \quad (f \in C^\infty(M)). \quad (1)$$

X^v is the *vertical lift*, X^c is the *complete lift* of X . The $C^\infty(TM)$ -module of vertical vector fields on TM will be denoted by $\mathfrak{X}^v(TM)$. The *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ is generated by the flow of positive dilatation $\delta_t : v \in TM \mapsto \delta_t(v) := e^t v \in TM$ ($t \in \mathbb{R}$). A function $F \in C^\infty(TM)$, a vector field $\xi \in \mathfrak{X}(\overset{\circ}{TM})$, and a differential form $\alpha \in \Omega(\overset{\circ}{TM})$ are called *r-homogeneous* ($r \in \mathbb{Z}$), if the relations

$$CF = rF, \quad [C, \xi] = (r-1)\xi, \quad \mathcal{L}_C \alpha = r\alpha$$

hold, respectively. Notice that

$$[C, X^v] = -X^v, \quad [C, X^c] = 0 \quad (X \in \mathcal{X}(M)), \quad (2a-b)$$

so X^v is 0-homogeneous, X^c is 1-homogeneous vector field on TM .

2.2. By a *vector k-form* on TM we mean a skew-symmetric $C^\infty(TM)$ -multilinear map $K : (\mathfrak{X}(TM))^k \rightarrow \mathfrak{X}(TM)$ if $k \in \{1, \dots, 2n\}$, and a vector field on TM , if $k = 0$. In particular, a vector 1-form on TM is just a type $(1, 1)$ tensor field. The $C^\infty(TM)$ -module of vector k -forms on TM will be denoted by $\Psi^k(TM)$. There is a unique vector 1-form $J \in \Psi^1(TM)$ such that

$$JX^v = 0, \quad JX^c = X^v \quad (X \in \mathcal{X}(M)). \quad (3a-b)$$

J is called the *vertical endomorphism*. Clearly, J is of rank n and $J^2 = 0$. A differential form $\alpha \in \Omega^k(TM)$ is *semibasic*, if $i_{J\xi}\alpha = 0$; a vector form $K \in \Psi^k(TM)$ is *semibasic*, if $i_{J\xi}K = 0$ and $J \circ K = 0$ ($k \geq 1$, $\xi \in \mathfrak{X}(TM)$).

2.3. We recall that if θ_r and θ_s are graded derivations of degree r and s , respectively, of a graded algebra, then their *graded commutator* is defined by

$$[\theta_r, \theta_s] := \theta_r \circ \theta_s - (-1)^{rs} \theta_s \circ \theta_r. \quad (4)$$

Then $[\theta_r, \theta_s]$ is a graded derivation of degree $r + s$. By the *Frölicher–Nijenhuis theory* of vector forms to any vector k -form $K \in \Psi^k(TM)$ two graded derivations of $\Omega(TM)$ are associated, denoted by i_K and d_K . i_K is of degree $k - 1$, d_K is of degree k , and the following rules are prescribed:

$$i_K \upharpoonright C^\infty(TM) = 0; \quad i_K \circ \alpha = \alpha \circ K, \quad \text{if } \alpha \in \Omega^1(TM); \quad (5)$$

$$d_K := [i_K, d] \stackrel{(4)}{=} i_K \circ d - (-1)^{k-1} d \circ i_K. \quad (6)$$

Then, in particular, for all $F \in C^\infty(TM)$, $K \in \Psi^k(TM)$ we have $d_K F = dF \circ K$. For vector 0-forms $\xi \in \Psi^0(TM) = \mathfrak{X}(TM)$, i.e., for vector fields on TM , i_ξ and d_ξ reduce to the usual substitution operator and Lie derivative, respectively. To any vector forms $K \in \Psi^k(TM)$, $L \in \Psi^\ell(TM)$ there is a unique vector $(k+l)$ -form $[K, L] \in \Psi^{k+l}(TM)$, the *Frölicher–Nijenhuis bracket* of K and L such that

$$d_{[K, L]} = [d_K, d_L].$$

In this paper we are going to systematically use the Frölicher–Nijenhuis calculus of vector forms. A detailed account on the theoretical background can be found e.g. in monographs [5], [6], and (of course) in the original source [2]. A well applicable list of formulae is gathered together (among others) in the reference [7] by Sz. SZAKÁL and J. SZILASI. Concerning the vertical endomorphism and the Liouville vector field we have

$$[J, C] = J, \quad [J, J] = 0. \quad (7a-b)$$

A vector form K on TM is called *homogeneous* of degree $r \in \mathbb{Z}$ if $[C, K] = (r - 1)K$. We note finally that the complete lift f^c of a function $f \in C^\infty(M)$ is 1-homogeneous and

$$d_J f^c = d(f^v) =: (df)^v; \quad (8)$$

see [8], Lemma 2.

2.4. By a *semispray* over M we mean a C^1 vector field $S : TM \rightarrow TTM$, smooth on $\overset{\circ}{TM}$, satisfying the condition $JS = C$. A 2-homogeneous semispray is

called a *spray*. If S is a semispray over M and K is a vector 1-form on TM , then for any vector field ξ on TM we have

$$K[J\xi, S] = K\xi, \quad (9)$$

in particular

$$J[J\xi, S] = J\xi. \quad (10)$$

Indeed, by Proposition 1.7 of [3] the vector field $[J\xi, S] - \xi$ is always vertical.

Two sprays S and $\bar{S}$ over M are said to be (*pointwise*) *projectively related* if there is a smooth function P on $\mathring{TM}$ such that $\bar{S} = S + PC$ (over $\mathring{TM}$). Then the *projective factor* P is necessarily 1-homogeneous.

2.5. A vector 1-form $\mathbf{h} \in \Psi^1(TM)$, smooth – in general – only over $\mathring{TM}$ is said to be a *horizontal endomorphism* (or *Ehresmann connection*) over M if it is a projector (i.e., $\mathbf{h}^2 = \mathbf{h}$) and $\text{Ker } \mathbf{h} = \mathfrak{X}^v(TM)$, or, equivalently, if $J \circ \mathbf{h} = J$ and $\mathbf{h} \circ J = 0$. $\mathbf{h}$ is called *homogeneous* if it is 1-homogeneous in the above sense, i.e. $[C, \mathbf{h}] = 0$. The (*weak*) *torsion* of $\mathbf{h}$ is the vector 2-form $\mathbf{t} := [J, \mathbf{h}]$. If S is a semispray over M , then $S_{\mathbf{h}} := \mathbf{h} \circ S$ is also a semispray, depending only on the Ehresmann connection. $S_{\mathbf{h}}$ is called the associated semispray to $\mathbf{h}$.

A fundamental result due to M. Crampin and J. Grifone states that *any semispray S generates a horizontal endomorphism of zero weak torsion* by the formula

$$\mathbf{h} = \frac{1}{2} (1_{\mathfrak{X}(TM)} + [J, S]). \quad (11)$$

Its associated semispray is $S_{\mathbf{h}} = \frac{1}{2} (S + [C, S])$. If S is a spray then $S_{\mathbf{h}} = S$, and $\mathbf{h}$ is homogeneous. For a recent treatment of these facts we refer to [6].

3. Some calculus on Finsler manifolds

3.1. Let a function $E : TM \rightarrow \mathbb{R}$ be given. Assume:

- (i) $E(v) > 0$ for all $v \in \mathring{TM}$, $E(0) = 0$;
- (ii) E is of class C^1 on TM , smooth on $\mathring{TM}$;
- (iii) E is (positive-)homogeneous of degree 2, i.e., $CE = 2E$;
- (iv) the *fundamental 2-form* $\omega := dd_J E$ is non-degenerate.

Then (M, E) is said to be a *Finsler manifold* with energy E . Notice that ω is semibasic and we have the relations

$$i_J \omega = 0, \quad i_C \omega = d_J E, \quad \mathcal{L}_C \omega = \omega. \quad (12a-c)$$

Due to the non-degeneracy of ω , for any 1-form $\beta \in \Omega^1(TM)$ there is unique vector field $\beta^\#$ on TM (smooth, in general, only on $\overset{\circ}{TM}$) such that

$$i_{\beta^\#} \omega = \beta. \quad (13)$$

This map $\# : \beta \rightarrow \beta^\#$ is called the (Finslerian) *sharp operator*. In particular, the *gradient* of a function $f \in C^\infty(TM)$ is the vector field $\text{grad } f := (df)^\#$.

Lemma 1. *If (M, E) is a Finsler manifold and β is a semibasic 1-form on TM , then*

$$\beta^\# \in \mathfrak{X}^v(TM), \quad [C, \beta^\#] = (\mathcal{L}_C \beta)^\# - \beta^\#. \quad (14a-b)$$

PROOF. Using (1.4g) of [7], (12a) and (13) we get

$$i_{J\beta^\#} \omega = i_{\beta^\#} \circ i_J \omega - i_J \circ i_{\beta^\#} \omega = -i_J \beta = 0,$$

since β is semibasic. Thus $J\beta^\# = 0$, therefore $\beta^\#$ is vertical. As for (14b),

$$i_{[C, \beta^\#]} \omega = \mathcal{L}_C i_{\beta^\#} \omega - i_{\beta^\#} \mathcal{L}_C \omega \stackrel{(12c), (13)}{=} \mathcal{L}_C \beta - \beta,$$

hence $[C, \beta^\#] = (\mathcal{L}_C \beta)^\# - \beta^\#$. $\square$

3.2. Following GRIFONE [3], by the *potential* of a semibasic form $K \in \Psi^k(TM)$ we mean the $(k-1)$ -form $K^\circ := i_S K$, where S is any semispray over M ($k \geq 1$). Clearly, K° is independent of the choice of S .

Lemma 2. *Let (M, E) be a Finsler manifold and L be an r -homogeneous semibasic vector 1-form on TM . Then $d_L E$, $(d_L E)^\#$ and L° are $(r+1)$ -homogeneous, while $[J, (d_L E)^\#]$ is r -homogeneous.*

PROOF. We have $[\mathcal{L}_C, d_L] = d_{[C, L]} = (r-1)d_L$, hence

$$\mathcal{L}_C d_L E = d_L \mathcal{L}_C E + (r-1)d_L E = (r+1)d_L E.$$

This proves the r -homogeneity of $d_L E$. Using this fact and (14b), we get

$$[C, (d_L E)^\#] = (\mathcal{L}_C d_L E)^\# - (d_L E)^\# = (r+1)(d_L E)^\# - (d_L E)^\# = r(d_L E)^\#,$$

as was to be shown. Since

$$[C, L]^\circ = [C, L]S = [C, L^\circ] - L[JS, S] \stackrel{(9)}{=} [C, L^\circ] - L^\circ$$

and, on the other side, $[C, L]^\circ = (r-1)L^\circ$, it follows that $[C, L^\circ] = rL^\circ$, i.e., L° is $(r+1)$ -homogeneous.

Finally, using the graded Jacobi identity, the $(r+1)$ -homogeneity of $(d_LE)^\#$ and (7a), we get

$$\begin{aligned} [C, [J, (d_LE)^\#]] &= -[J, [(d_LE)^\#, C]] - [(d_LE)^\#, [C, J]] \\ &= (r+1)[J, (d_LE)^\#] + [(d_LE)^\#, J] = r[J, (d_LE)^\#], \end{aligned}$$

which proves the last claim. $\square$

3.3. To conclude this section, we recall the *fundamental lemma of Finsler geometry* due to J. GRIFONE [3], see also [6]. Let (M, E) be a Finsler manifold. If

$$S_0 := -(dE)^\# \quad \text{over} \quad \overset{\circ}{TM}, \quad S_0(0) := 0$$

then S_0 is a spray over M , called the *canonical spray* of (M, E) . S_0 generates a homogeneous horizontal endomorphism $\mathbf{h}_0$ according to (11), called the *canonical horizontal endomorphism* or the *Barthel endomorphism* of (M, E) . $\mathbf{h}_0$ is conservative in the sense that $d_{\mathbf{h}_0}E = 0$.

4. L -horizontal endomorphisms on a Finsler manifold

Keeping the notation introduced in Section 3, throughout in the following we work on a Finsler manifold (M, E) .

4.1. SZ. SZAKÁL and J. SZILASI have shown in [7] that any homogeneous, conservative horizontal endomorphism $\mathbf{h}$ over M can be expressed as follows:

$$\mathbf{h} := \mathbf{h}_0 + \frac{1}{2} \mathbf{t}^\circ + \frac{1}{2} [J, (d_{\mathbf{t}^\circ}E)^\#]. \quad (15)$$

Next we consider a quite natural generalization.

Lemma 3 and definition. If L is a semibasic vector 1-form on TM and

$$\mathbf{h}_L := \mathbf{h}_0 + L + [J, (d_LE)^\#], \quad (16)$$

then $\mathbf{h}_L$ is also a horizontal endomorphism, called an *L -horizontal endomorphism* on (M, E) . In particular, for any homogeneous, conservative horizontal endomorphism $\mathbf{h}$ we have $\mathbf{h} = \mathbf{h}_{\frac{1}{2}\mathbf{t}^\circ}$.

PROOF. Indeed, the term $L + [J, (d_LE)^\#]$ in (16) is semibasic, therefore $J \circ \mathbf{h}_L = J \circ \mathbf{h}_0 = J$, $\mathbf{h}_L \circ J = \mathbf{h}_0 \circ J = 0$. The relation $\mathbf{h} = \mathbf{h}_{\frac{1}{2}\mathbf{t}^\circ}$ is just a reformulation of (15). $\square$

4.2. We gather together and prove some basic properties of L -horizontal endomorphisms.

Proposition 1. *Let L be a semibasic vector 1-form on TM .*

- (i) *If L is 1-homogeneous, then $\mathbf{h}_L$ is homogeneous.*
- (ii) *The associated semispray S_L to $\mathbf{h}_L$ is related to the canonical spray of (M, E) by*

$$S_L = S_0 + L^\circ + (\mathcal{L}_C d_LE)^\#.$$

If, in particular, L is 1-homogeneous, then S_L is a spray.

- (iii) *The weak torsion of S_L is $\mathbf{t}_L = [J, L]$.*
- (iv) *If L is 1-homogeneous and S_L is projectively related to S_0 , then the projective factor is $\frac{3}{2} \frac{L^\circ E}{E} \uparrow \overset{\circ}{TM}$.*

PROOF. The first claim is immediate: the 1-homogeneity of L implies the 1-homogeneity of $[J, (d_\mathbf{t} E)^\#]$ by Lemma 2.

To verify (ii), let S be a semispray over M . Then

$$\begin{aligned} S_L &:= \mathbf{h}_L \circ S = h_0 \circ S + L^\circ + [J, (d_LE)^\#]S \\ &= S_0 + L^\circ + [C, (d_LE)^\#] - J[S, (d_LE)^\#] \\ &\stackrel{(14b), (10)}{=} S_0 + L^\circ + (\mathcal{L}_C d_LE)^\# - (d_LE)^\# + (d_LE)^\# \\ &= S_0 + L^\circ + (\mathcal{L}_C d_LE)^\#, \end{aligned}$$

as desired.

The weak torsion of $\mathbf{h}_L$ is

$$\mathbf{t}_L := [J, \mathbf{h}_L] = [J, \mathbf{h}_0 + L + [J, (d_LE)^\#]] = [J, L] + [J, [J, (d_LE)^\#]].$$

Applying the graded Jacobi identity we easily get that the last term of the right-hand side vanishes; this proves (iii).

To prove (iv), let $S_L = S_0 + PC, P \in C^\infty(\overset{\circ}{TM})$. Then $PC = L^\circ + 2(d_LE)^\#$ by the 2-homogeneity of $(d_LE)^\#$. Now we act on the fundamental 2-form ω by the substitution operators induced by PC and $L^\circ + 2(d_LE)^\#$, respectively. We find:

$$i_{PC}\omega = Pi_C\omega \stackrel{(12b)}{=} Pd_JE; \tag{17}$$

$$i_{L^\circ}\omega + 2i_{(d_LE)^\#}\omega \stackrel{(13)}{=} i_{L^\circ}\omega + 2d_LE. \tag{18}$$

Evaluating the right-hand sides of (17) and (18) on a semispray S we have

$$Pd_J E(S) = P(CE) = 2PE$$

and

$$\begin{aligned} (i_{L^\circ} \omega + 2d_L E)(S) &= \omega(L^\circ, S) + 2dE(L^\circ) = d(d_J E)(L^\circ, S) + 2dE(L^\circ) \\ &= L^\circ d_J E(S) - S d_J E(L^\circ) - d_J E([L^\circ, S]) + 2L^\circ E \\ &= 4L^\circ E - dE(J[L^\circ, S]) \stackrel{(10)}{=} 4L^\circ E - L^\circ E = 3L^\circ E. \end{aligned}$$

Thus $2PE = 3L^\circ E$, which concludes the proof. $\square$

4.3. The next proposition will imply that the Wagner endomorphisms can be considered as special L -horizontal endomorphism.

Proposition 2. *Let f be a smooth function on M . If K is a semibasic vector 1-form and*

$$L := f^c K - df^v \otimes K^\circ, \quad (19)$$

then L is also a semibasic 1-form, and

$$\begin{aligned} \mathbf{h}_L &= \mathbf{h}_0 + f^c K - df^v \otimes K^\circ + f^c [J, (d_K E)^\#] + df^v \otimes (d_K E)^\# \\ &\quad - K^\circ E[J, \text{grad } f^v] - d_J(K^\circ E) \otimes \text{grad } f^v. \end{aligned} \quad (20)$$

PROOF. It is obvious that L is indeed semibasic. To verify (20), it is enough to check that under the choice (19), $(d_L E)^\# = f^c (d_K E)^\# - (K^\circ E) \text{grad } f^v$. To see this, let X be any vector field on M . Then we get

$$\begin{aligned} i_{(d_L E)^\#} \omega(X^c) &= LX^C(E) = (f^c K(X^c) - df^v(X^c) K^\circ)E = f^c(KX^c)E \\ &\quad - (K^\circ E)df^v(X^c) = f^c d_K E(X^c) - (K^\circ E)i_{\text{grad } f^v} \omega(X^c) \\ &= i_{f^c (d_K E)^\# - (K^\circ E) \text{grad } f^v} \omega(X^c), \end{aligned}$$

which yields the desired relation. We may now apply some standard rules for calculation of the Frölicher–Nijenhuis theory and relation (8) to obtain (20). $\square$

Corollary. *The class of the L -horizontal endomorphisms of a Finsler manifold contains the Wagner endomorphisms.*

PROOF. As VINCZE has shown in [12], the Wagner endomorphism $\bar{\mathbf{h}}$ associated to a smooth function f on M can be represented in the form

$$\bar{\mathbf{h}} = \mathbf{h}_0 + f^c J - E[J, \text{grad } f^v] - d_J E \otimes \text{grad } f^v.$$

Replacing K by $\frac{1}{2}J$ and taking into account that $J^\circ = C$, $(d_J E)^\# = C$, (20) takes the form

$$\begin{aligned} \mathbf{h}_{\frac{1}{2}f^c J - \frac{1}{2}df^v \otimes C} &= \mathbf{h}_0 + \frac{1}{2}f^c J - \frac{1}{2}df^v \otimes C + \frac{1}{2}f^c[J, C] + \frac{1}{2}df^v \otimes C \\ &\quad - \frac{1}{2}CE[J, \text{grad } f^v] - \frac{1}{2}d_J(CE) \otimes \text{grad } f^v \\ &= \mathbf{h}_0 + f^c J - E[J, \text{grad } f^v] - d_J E \otimes \text{grad } f^v = \bar{\mathbf{h}}, \end{aligned}$$

proving our claim. $\square$

5. The effect of conformal changes

We continue to assume that (M, E) is a Finsler manifold.

5.1. Let f be a smooth function on M and define a positive function on TM by

$$\varphi := \exp \circ f^v. \quad (21)$$

If $\tilde{E} := \varphi E$, then $(M, \tilde{E})$ is also a Finsler manifold (see [11] Lemma 1). We say that $(M, \tilde{E})$ has been obtained by a *conformal change* of E given by the *scale function* φ . It is known ([8]) that the Barthel endomorphism $\tilde{\mathbf{h}}_0$ and the canonical spray $\tilde{S}_0$ of $(M, \tilde{E})$ are related to the corresponding data of (M, E) by

$$\tilde{\mathbf{h}}_0 = \mathbf{h}_0 - \frac{1}{2}(f^c J + df^v \otimes C) + \frac{1}{2}E[J, \text{grad } f^v] + \frac{1}{2}d_J E \otimes \text{grad } f^v, \quad (22)$$

and

$$\tilde{S}_0 = S_0 - f^c C + E \text{grad } f^v, \quad (23)$$

respectively.

Lemma 4. Let β be a semibasic 1-form on TM . Under the conformal change with scale function given by (21) the vector field $\beta^\#$ changes by the rule $\varphi \tilde{\beta}^\# = \beta^\#$, where $\tilde{\beta}^\#$ is the sharp operator in the Finsler manifold $(M, \tilde{E})$.

PROOF. Let $\tilde{\omega}$ be the fundamental 2-form of $(M, \tilde{E})$. Then

$$\tilde{\omega} = dd_J \tilde{E} = d(d_J \varphi E) = d(\varphi d_J E) = d\varphi \wedge d_J E + \varphi \omega,$$

so for any vector field X on M we have

$$i_{\beta^\#} \tilde{\omega}(X^c) = i_{\beta^\#} (d\varphi \wedge d_J E + \varphi \omega)(X^c) = d\varphi \wedge d_J E(\beta^\#, X^c)$$

$$\begin{aligned}
& + \varphi i_{\beta^\#} \omega(X^c) = d\varphi(\beta^\#) d_J E(X^c) - d\varphi(X^c) d_J(\beta^\#) \\
& + i_{\varphi\beta^\#} \omega(X^c) \stackrel{(14a)}{=} i_{\varphi\beta^\#} \omega(X^c).
\end{aligned}$$

On the other hand, $i_{\beta^\#} \tilde{\omega} = \beta = i_{\beta^\#} \omega$; therefore $\varphi\beta^\# = \beta^\#$. $\square$

Proposition 3. *If L is a semibasic vector 1-form on TM , then the vector fields $(d_L E)^\#$ and $(\mathcal{L}_C d_L E)^\#$ are invariant under any conformal change of E .*

PROOF. Consider the conformal change given by the scale function (21). Then

$$d_L \tilde{E} = d_L(\varphi E) = \varphi d_L E + E d_L \varphi = \varphi d_L E,$$

since L is semibasic and φ is a vertical lift. Hence using Lemma 4,

$$\varphi(d_L \tilde{E})^\# = (d_L \tilde{E})^\# = (\varphi d_L E)^\# = \varphi(d_L E)^\#;$$

therefore $(d_L \tilde{E})^\# = (d_L E)^\#$. Similarly, the 1-form $(\mathcal{L}_C d_L E)$ is also semibasic and

$$\mathcal{L}_C d_L \tilde{E} = \mathcal{L}_C(\varphi d_L E) = (\mathcal{L}_C \varphi) d_L E + \varphi \mathcal{L}_C d_L E = \varphi \mathcal{L}_C d_L E,$$

so, applying Lemmas 4 again, we get

$$\varphi(\mathcal{L}_C d_L \tilde{E})^\# = (\mathcal{L}_C d_L \tilde{E})^\# = (\varphi \mathcal{L}_C d_L E)^\# = \varphi(\mathcal{L}_C d_L E)^\#.$$

This yields the desired second equality. $\square$

Remark. If $L := \frac{1}{2}(f^c J - df^v \otimes C)$, our proposition leads to Proposition 3 of VINCZE's paper [11].

Proposition 4. *Let L be a semibasic vector 1-form on TM . Under the conformal change given by the scale function (21) the horizontal endomorphism $\mathbf{h}_L$ and its associated semispray S_L change as follows:*

$$\tilde{\mathbf{h}}_L = \mathbf{h}_L - \frac{1}{2}(f^c J + df^v \otimes C) + \frac{1}{2}E[J, \text{grad } f^v] + \frac{1}{2}d_J E \otimes \text{grad } f^v, \quad (24)$$

$$\tilde{S}_L = S_L - f^c C + E \text{grad } f^v. \quad (25)$$

PROOF. By the conformal invariance of $(d_L E)^\#$, (16) and (22) yield immediately (24). Similarly, applying the conformal invariance of $(\mathcal{L}_C d_L E)^\#$, Proposition 1(ii) and (23), we obtain (25). $\square$

5.2. Now we are in a position to state and prove our main observation.

Theorem. *The set of all conservative L -horizontal endomorphisms on a Finsler manifold is conformally closed.*

PROOF. Consider the conformal change $\tilde{E} := \varphi E$, $\varphi := \exp \circ f^v$ ($f \in C^\infty(M)$). Let L be a semibasic vector 1-form on TM , and define $K := \tilde{\mathbf{h}}_L - \mathbf{h}_L$. Then K is semibasic, and we have $K := \tilde{\mathbf{h}}_0 - \mathbf{h}_0$ by (16) and the conformal invariance of $(d_L E)^\#$. Since $\tilde{\mathbf{h}}_0$ and $\mathbf{h}_0$ are conservative, we get

$$\begin{aligned} 0 &= d_{\tilde{\mathbf{h}}_0} \tilde{E} = d_{\mathbf{h}_0 + K} \tilde{E} = d_{\mathbf{h}_0}(\varphi E) + d_K \tilde{E} \\ &= Ed_{\mathbf{h}_0} \varphi + \varphi d_{\mathbf{h}_0} E + d_K \tilde{E} = Ed_{\mathbf{h}_0} \varphi + d_K \tilde{E}, \end{aligned}$$

hence

$$d_K \tilde{E} = -Ed_{\mathbf{h}_0} \varphi. \quad (26)$$

Now suppose that the horizontal endomorphism $\mathbf{h}_L$ is conservative. Then

$$\begin{aligned} d_{\tilde{\mathbf{h}}_L} \tilde{E} &= d_{\mathbf{h}_L + K} \tilde{E} = d_{\mathbf{h}_L} \tilde{E} + d_K \tilde{E} \stackrel{(26)}{=} d_{\mathbf{h}_L} \tilde{E} - Ed_{\mathbf{h}_0} \varphi \\ &= d_{\mathbf{h}_L} \varphi E - Ed_{\mathbf{h}_0} \varphi = \varphi d_{\mathbf{h}_L} E + Ed_{\mathbf{h}_L} \varphi - Ed_{\mathbf{h}_0} \varphi \\ &= Ed_{\mathbf{h}_L - \mathbf{h}_0} \varphi = 0, \end{aligned}$$

since $\mathbf{h}_L - \mathbf{h}_0$ is semibasic by (16), and φ is a vertical lift. Thus $\tilde{\mathbf{h}}_L$ is also conservative, and the proof is concluded. $\square$

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