

A contribution to fixed point theory in quasi-metric spaces

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1. Introduction

In [4], [6], [7] several fixed point theorems in quasi-metric spaces have been obtained. By modifying the kind of contractive self-maps used in the cited papers we show that it is possible to develop, in some directions, a unified fixed point theory in metric and quasi-metric spaces. By a quasi-metric on a set X we mean a non-negative real function d on $X \times X$ such that, for $x, y, z \in X$, we have $d(x, y) = 0$ if and only if $x = y$, and $d(x, y) \leq d(x, z) + d(z, y)$. Thus the classical conditions on a metric are relaxed here by omitting the requirement of the symmetry of d . In consequence, a quasi-metric need not be continuous and this fact makes that the proofs of fixed point theorems for quasi-metric spaces not always can be slight modifications of the proofs of their metric counterparts. However, several authors have been able to extend some fixed point theorems for metric spaces to quasi-metric spaces. In 1982 REILLY, SUBRAHMANYAM and VAMANAMURTHY proved the Banach Contraction Principle for complete Hausdorff quasi-metric spaces (Theorem 9 in [6]) and they observed that the Hausdorff condition was then essential (see Example 6 in [6]). Recently several authors have examined some more complicated conditions of a contractive type. Let us recall some of them.

Definition 1 ([2]). A self-map T of a metric space X is a Ćirić contraction if it satisfies the condition

$$(C) \quad d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for some $0 \leq h < 1$ and all x, y in X .

ĆIRIĆ's theorem (Theorem 1 in [2]) states that any Ćirić contraction on a complete metric space has a fixed point z and, for any $x \in X$,

$T^n x \rightarrow z$. It is worth underlining that there exist non-continuous Ćirić contractions.

In 1988 HICKS [4] proved among others that any *continuous* Ćirić contraction on a left K -sequentially complete (see Def. 3 in [6]) Hausdorff quasi-metric space satisfying (C) with $0 \leq h < 1/2$ has a fixed point. In 1990 ROMAGUERA and CHECA extended Hick's result to any continuous Ćirić contraction on a complete Hausdorff quasi-metric space (Theorem 2 in [7]). They also demonstrated (see Example in [7]) that the continuity of T is then essential.

In our paper we shall consider among others the following condition (C') for T :

$$(C') \quad d(Tx, Ty) \leq h \max\{d(y, x), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for some $0 \leq h < 1$ and all x, y in X .

Obviously, if (X, d) is a metric space then (C') is equivalent to (C), however, this equivalence need not hold in case of quasi-metric spaces. Moreover, while a map satisfying (C) in a complete quasi-metric space need not have fixed points, any map which fulfills (C') does have a fixed point, even if it is non-continuous and a space is not Hausdorff (see our Corollary 1). Thus we obtain a unified result for metric and quasi-metric spaces. The condition (C') seems to be the only suitable one of this kind for quasi-metric spaces: as we observed in [5], if we replaced terms $d(x, Tx)$ and $d(y, Ty)$ in (C') by $d(Tx, x)$ and $d(Ty, y)$ respectively, then a fixed point theorem would not hold.

It seems that many other fixed point theorems for metric spaces can also be carried over to quasi-metric spaces after a suitable ordering of the arguments of d in concrete inequalities. We shall illustrate this by proving Fisher's fixed point theorem [3], which is an extension of the theorem of ĆIRIĆ [2].

Definition 2 ([3]). A self-map T of a metric space X is a Fisher contraction, if it satisfies the condition

$$(F) \quad d(T^p x, T^q y) \leq h \max\{d(T^i x, T^j y), d(T^i x, T^{i'} x), d(T^j y, T^{j'} y) : \\ 0 \leq i, i' \leq p \text{ and } 0 \leq j, j' \leq q\},$$

for some fixed p, q in N , for some $0 \leq h < 1$ and all x, y in X .

FISHER's Theorem 2 ([3]) states that any continuous Fisher contraction on a complete metric space has a unique fixed point. Moreover, his Theorem 3 states that if T satisfies (F) with p (or q) = 1, then the condition that T be continuous is unnecessary. Simultaneously, this condition is essential if T satisfies (F) with $p, q > 1$ (see the example in [3]).

In our paper we shall give only an extension of FISHER's Theorem 3 to quasi-metric spaces. We are sure that after reading the proof of our

main theorem, the reader (if he wants) will be able to carry over FISHER's Theorem 2 to Hausdorff quasi-metric spaces. This is much easier than in the case of Theorem 3.

Main theorem

Throughout this section (X, d) is a quasi-metric space and $\bar{d}$ defined by $\bar{d}(x, y) = d(y, x)$ for $x, y \in X$ is called the conjugate of d ([6]). For convenience we use the term "completeness" instead of " d -sequential completeness" ([6]). Thus, (X, d) is complete if every d -Cauchy sequence is left d -convergent. Observe that completeness of $(X, \bar{d})$ means that every d -Cauchy sequence is right d -convergent. We shall not use other notions of completeness considered in [6] — each of them implies completeness in the above sense.

Theorem. *Let T be a self-map (not necessarily continuous) of a complete quasi-metric space (X, d) (not necessarily Hausdorff). Assume that for some fixed p in $\mathbb{N}$, T satisfies the following condition:*

$$(F') \quad d(T^p x, Ty) \leq h \max\{d(y, T^i x), d(T^i x, Ty), d(T^i x, T^j x), d(y, Ty) : \\ 0 \leq i, j \leq p\},$$

for some $0 \leq h < 1$ and all x, y in X . Then T has a unique fixed point z , for any x in X the sequence $\{T^n x\}$ is left and right d -convergent to z , and z is the only limit of this sequence.

PROOF. We denote by d^* the metric defined on X by $d^*(x, y) = \max\{d(x, y), d(y, x)\}$. Then it is easy to show that T satisfies (F) in (X, d^*) , for a pair (p, p) of positive integers. From the proof of Theorem 2 in [3], for any $x \in X$, $\{T^n x\}$ is a Cauchy sequence in (X, d^*) and hence also in (X, d) . By completeness, $\{T^n x\}$ is left d -convergent to some $z \in X$, that is $d(z, T^n x) \rightarrow 0$. We shall show that $z = Tz$. By the triangular inequality we have:

$$(1) \quad d(z, Tz) \leq d(z, T^{n+p} x) + d(T^{n+p} x, Tz).$$

From (F') we get

$$(2) \quad d(T^{n+p} x, Tz) \leq \\ \leq h \max\{d(z, T^{n+i} x), d(T^{n+i} x, Tz), d(T^{n+i} x, T^{n+j} x), d(z, Tz) : \\ 0 \leq i, j \leq p\}.$$

Let $a_n := d(T^n x, Tz)$, $b_n := \max\{d(z, T^{n+i} x), d(T^{n+i} x, T^{n+j} x) : 0 \leq i, j \leq p\}$, $r := d(z, Tz)$. From (2) we have:

$$(3) \quad a_{n+p} \leq h \max\{a_n, a_{n+1}, \dots, a_{n+p}, b_n, r\}.$$

Observe, that if the above maximum is equal to a_{n+p} then $a_{n+p} = 0$, so we may omit a_{n+p} on the right side of (3).

Suppose r is positive. Since $b_n \rightarrow 0$, for sufficiently large n we have, by (3),

$$(4) \quad a_{n+p} \leq h \max\{a_n, a_{n+1}, \dots, a_{n+p-1}, r\}.$$

Without loss of generality we may assume that (4) holds for any nonnegative integer n . Now consider two cases:

a) $p = 1$. Then it is easily seen that (4) implies the inequality $a_n \leq \max\{h^n a_0, hr\}$, and hence $\limsup_{n \rightarrow \infty} a_n \leq hr$.

b) $p > 1$. We shall apply induction to show that, for any n in N , the following inequalities hold:

$$(5) \quad a_{np} \leq \max\{h^n a_0, \dots, h^n a_{p-1}, hr\},$$

$$(6) \quad a_{np+k} \leq \max\{h^{n+1} a_0, \dots, h^{n+1} a_{k-1}, h^n a_k, \dots, h^n a_{p-1}, hr\},$$

for $k = 1, \dots, p-1$.

The case $n = 0$ is trivial. Assuming that (5) and (6) hold for some n , we shall prove them for $n+1$. By (4) and the induction hypothesis, we get that

$$\begin{aligned} a_{(n+1)p} &\leq h \max\{a_{np}, a_{np+1}, \dots, a_{np+p-1}, r\} \leq \max\{h^{n+1} a_0, \dots, h^{n+1} a_{p-1}, \\ &\quad h^2 r, h^{n+2} a_0, \dots, h^{n+2} a_{k-1}, h^{n+1} a_k, \dots, h^{n+1} a_{p-1}, hr : 1 \leq k \leq p-1\} = \\ &= \max\{h^{n+1} a_0, \dots, h^{n+1} a_{p-1}, hr\}, \end{aligned}$$

and thus (5) holds for $n+1$. We leave it to the reader that (6) holds for $n+1$ (apply induction again, this time with respect to k).

Now, it follows from (5) and (6) that

$$\limsup_{n \rightarrow \infty} a_{np+k} \leq hr, \quad \text{for } k = 0, 1, \dots, p-1.$$

Thus, in both cases a) and b) we have:

$$(7) \quad \limsup_{n \rightarrow \infty} a_n \leq hr.$$

Hence and by (1) we get:

$$d(z, Tz) \leq \lim_{n \rightarrow \infty} d(z, T^{n+p}x) + \limsup_{n \rightarrow \infty} a_{n+p} \leq h d(z, Tz).$$

Since $0 \leq h < 1$, we get $z = Tz$.

To prove that $\{T^n x\}$ is right d -convergent to z , observe that since $z = Tz$, by (7), $a_n \rightarrow 0$, so $d(T^n x, z) \rightarrow 0$. Thus $\{T^n x\}$ is simultaneously

left and right d -convergent to z , so z is the only limit point of any kind of $\{T^n x\}$ (see [8]).

Finally, we shall prove the uniqueness of the fixed point. Let $z_1 = Tz_1$ and $z_2 = Tz_2$. Applying (F') we get

$$d(z_1, z_2) \leq h \max\{d(z_2, z_1), d(z_1, z_2)\} = hd^*(z_1, z_2).$$

Interchanging z_1 and z_2 we obtain that $d(z_2, z_1) \leq hd^*(z_1, z_2)$, so finally $d^*(z_1, z_2) = 0$ and thus the fixed point is unique. $\square$

The following corollary is immediate:

Corollary 1. *Let T be a self-map on a complete quasi-metric space (X, d) , and let T satisfy the condition (C'). Then T has a unique fixed point in X .*

By duality, we easily obtain the following result:

Corollary 2. *Let $(X, \bar{d})$ be complete and let T fulfil the inequality (C'') $d(Tx, Ty) \leq h \max\{d(y, x), d(Tx, x), d(Ty, y), d(Ty, x), d(Tx, y)\}$, for some $0 \leq h < 1$ and all x, y in X . Then T has a unique fixed point.*

Remark. The above corollaries extend ĆIRIĆ's theorem [2] to quasi-metric spaces. The conditions (C') and (C'') are adjusted to a type of convergence (left or right) in (X, d) , and they can be interchanged neither in Corollary 1, nor in Corollary 2 (see the example in [5]).

Corollary 3 (the Banach Contraction Principle). *Assume that (X, d) or $(X, \bar{d})$ (not necessarily Hausdorff) is complete and T satisfies the inequality*

$$(B) \quad d(Tx, Ty) \leq hd(y, x) \text{ for some } 0 \leq h < 1 \text{ and all } x, y \text{ in } X.$$

Then T has a unique fixed point.

PROOF. Apply Corollaries 1 and 2. $\square$

The following example shows that it can happen that f satisfies (B) in a quasi-metric space (X, d) which is not complete yet Corollary 3 is applicable here since $(X, \bar{d})$ is complete in this case.

Example. Let $X := [-1; 1]$ and $fx := \frac{1}{2}|x|$, for $x \in X$. Define the function d as follows:

$$d(x, y) := 2, \text{ for } 0 \leq x \leq 1 \text{ and } -1 \leq y < 0; \quad d(x, y) := |x - y|$$

in the remaining cases.

Then one can verify that the triangle inequality,

$$(8) \quad d(x, y) \leq d(x, z) + d(z, y),$$

holds for $x \in [0; 1]$, $y \in [-1; 0]$ and $z \in X$. Since $|x - y| \leq d(x, y)$ for all x, y in X and the function $(x, y) \rightarrow |x - y|$, $(x, y \in X)$ satisfies the triangle inequality we get that (8) holds in all remaining cases. Thus d is a quasi-metric. Observe that the sequence $\{-1/n\}_{n=1}^{\infty}$ is Cauchy but it is not d -convergent. We leave it to the reader to verify that, nevertheless, $(X, \bar{d})$ is complete. Finally, for all x, y in X we have, by the definition of d , that

$$d(fx, fy) = |\tfrac{1}{2}|x| - \tfrac{1}{2}|y|| \leq \tfrac{1}{2}|x - y| \leq \tfrac{1}{2}d(y, x)$$

so f satisfies (B) with $h = \frac{1}{2}$.

The following corollary can be immediately deduced from Corollary 1.

Corollary 4 (Theorem 1 in [7]). *Let (X, d) be complete and let T satisfy Bianchini's condition ([1]):*

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\},$$

for some $0 \leq h < 1$ and all x, y in X . Then T has a unique fixed point.

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