Publ. Math. Debrecen
43 / 3–4 (1993), 297–302

Factors for generalized absolute Cesàro summability methods

By HÜSEYİN BOR (Kayseri)

Abstract. In this paper a theorem on φ–$|\mathrm{C},\alpha|_k$ $(0<\alpha\leq 1)$ summability factors, which contains some results on $|\mathrm{C},\alpha|_k$ and $|\mathrm{C},\alpha;\delta|_k$ summability factors, has been proved.

1. Introduction. Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α $(\alpha>-1)$ of the sequences (s_n) and (na_n), respectively, i.e,

$$u_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}s_v \tag{1.1}$$

$$t_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v, \tag{1.2}$$

where

$$\begin{gathered}A_n^\alpha=\binom{n+\alpha}{n}=O(n^\alpha),\quad \alpha>-1,\\ A_o^\alpha=1 \ \text{ and } \ A_{-n}^\alpha=0 \ \text{ for } \ n>0.\end{gathered} \tag{1.3}$$

The series $\sum a_n$ is said to be summable φ–$|\mathrm{C},\alpha|_k$, $k\geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}|\varphi_n(u_n^\alpha-u_{n-1}^\alpha)|^k<\infty. \tag{1.4}$$

1980 *Mathematics Subject Classification* (1985 *Revision*): 40D15, 40F05, 40G05.
Keywords: Absolute Cesàro summability, summability factors.
This paper was supported by the Scientific and Technical Research Council of Turkey (TÜBITAK, TBAG - 42).

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [3]) the condition (1.4) can also be written as

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty. \tag{1.5}$$

In the special case when $\varphi_n = n^{1-1/k}$ (resp. $\varphi_n = n^{\delta+1-1/k}$) φ–$|\mathrm{C},\alpha|_k$ summability is the same as $|\mathrm{C},\alpha|_k$ (resp. $|\mathrm{C},\alpha;\delta|_k$) summability. Also we know that (see [5]) for $k \geq 1$ and $0 < \alpha \leq 1$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n^\alpha|^k = O\left\{ \sum_{n=1}^{m} \frac{|s_n|^k}{n^{(\alpha-1)k+1}} \right\}. \tag{1.6}$$

2. Mishra and Srivastava [4] have proved the following theorem for $|\mathrm{C},1|_k$ summability factors of infinite series:

Theorem A. *Let (x_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \leq \beta_n \tag{2.1}$$

$$\beta_n \longrightarrow 0 \quad as \quad n \longrightarrow \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n |\Delta\beta_n| x_n < \infty \tag{2.3}$$

$$|\lambda_n| x_n = O(1) \quad as \quad n \longrightarrow \infty. \tag{2.4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} |s_n|^k = O(x_m) \quad as \quad m \longrightarrow \infty, \tag{2.5}$$

then the series $\sum a_n \lambda_n$ is summable $|\mathrm{C},1|_k$, $k \geq 1$.

3. The aim of this paper is to prove Theorem A for φ–$|\mathrm{C},\alpha|_k$ summability provided that $0 < \alpha \leq 1$. Now we shall prove the following.

Theorem. *Let $0 < \alpha \leq 1$. Let (x_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (2.1)–(2.4) of Theorem A are satisfied. If there exists an $\varepsilon > 0$ such that the sequence*

$(n^{\varepsilon-k}|\varphi_n|^k)$ *is nonincreasing and if the sequence* (w_n^α), *defined by*

$$w_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max\limits_{1\le u\le n} |t_u^\alpha| & (0 < \alpha < 1) \end{cases} \tag{3.1}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(|\varphi_n| w_n^\alpha)^k = O(x_m) \quad as \quad m \longrightarrow \infty, \tag{3.2}$$

then the series $\sum a_n\lambda_n$ *is summable* φ–$|\mathrm{C},\alpha|_k$, $k \ge 1$.

If we take $\alpha = 1$, $\varepsilon = 1$ and $\varphi_n = n^{1-1/k}$ in this theorem, then we get Theorem A. In fact, in this case, by (1.6), we have

$$\sum_{n=1}^{m} \frac{1}{n}(w_n)^k = \sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(1)\sum_{n=1}^{m} \frac{1}{n}|s_n|^k.$$

Also, is we take $\varepsilon = 1$ and $\varphi_n = n^{1-1/k}$ (resp. $\varepsilon = 1$ and $\varphi_n = n^{\delta+1-1/k}$), then we obtain a result for $|\mathrm{C},\alpha|_k$ (resp. $|\mathrm{C},\alpha;\delta|_k$) summability factors.

4. We need the following lemmas for the proof of our theorem:

Lemma 1 ([2]). *If* $0 < \alpha \le 1$ *and* $1 \le v \le n$, *then*

$$|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} a_p| \le \max_{1\le m\le v} \left|\sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p\right|, \tag{4.1}$$

where A_n^α *is as in (1.3).*

Lemma 2 ([4]). *Under the conditions on* (x_n), (β_n) *and* (λ_n) *as taken in the statement of Theorem A, the following conditions hold, when (2.3) is satisfied:*

$$n\beta_n x_n = o(1) \quad as \quad n \longrightarrow \infty \tag{4.2}$$

$$\sum_{n=1}^{\infty} \beta_n x_n < \infty. \tag{4.3}$$

5. Proof of the theorem. Let (T_n^α) be the n-th (C,α) means, with $0 < \alpha \le 1$, of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} \; va_v\lambda_v. \tag{5.1}$$

Applying Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned} |T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\,|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\ &\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v| + |\lambda_n| w_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n T_{n,r}^\alpha|^k < \infty \quad \text{for} \quad r = 1, 2,$$

by (1.5). Now, when $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\begin{aligned} \sum_{n=2}^{m+1} n^{-k}|\varphi_n T_{n,1}^\alpha|^k &\le \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}|\varphi_n|^k \left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \beta_v\right\}^k \\ &\le \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}|\varphi_n|^k \sum_{v=1}^{n-1} A_v^\alpha (w_v^\alpha)^k \beta_v \times \left\{\sum_{v=1}^{n-1} A_v^\alpha \beta_v\right\}^{k-1} \\ &= O(1)\sum_{n=2}^{m+1} n^{-k} n^{-\alpha k}|\varphi_n|^k (n^\alpha)^{k-1}\sum_{v=1}^{n-1} v^\alpha (w_v^\alpha)^k \beta_v \end{aligned}$$

$$= O(1)\sum_{v=1}^{m} v^\alpha (w_v^\alpha)^k \beta_v \sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k}{n^{k+\alpha}} = O(1)\sum_{v=1}^{m} v^\alpha (w_v^\alpha)^k \beta_v \sum_{n=v+1}^{m+1}\frac{n^{\varepsilon-k}|\varphi_n|^k}{n^{\alpha+\varepsilon}}$$

$$= O(1)\sum_{v=1}^{m} v^\alpha (w_v^\alpha)^k \beta_v v^{\varepsilon-k}|\varphi_v|^k \int_v^\infty \frac{dx}{x^{\alpha+\varepsilon}} = O(1)\sum_{v=1}^{m} v\beta_v v^{-k}(w_v^\alpha|\varphi_v|)^k$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v} r^{-k}(w_r^\alpha|\varphi_r|)^k + O(1)m\beta_m \sum_{v=1}^{m} v^{-k}(w_v^\alpha|\varphi_u|)^k$$

$$= O(1)\sum_{v=1}^{m-1} |\Delta(v\beta_v)|x_v + O(1)m\beta_m x_m = O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|x_v$$

$$+O(1)\sum_{v=1}^{m-1} |\beta_{v+1}|x_v + O(1)m\beta_m x_m = O(1) \quad \text{as} \quad m \longrightarrow \infty,$$

by virtue of the hypotheses and Lemma 2.
Again, since $|\lambda_n| = O(1/x_n) = O(1)$, by (2.4) we have

$$\sum_{n-1}^{m} n^{-k}|\varphi_n T_{n,2}^{\alpha}|^k = \sum_{n=1}^{m} n^{-k}|\varphi_n|^k \; |\lambda_n| \; |\lambda_n|^{k-1}(w_n^{\alpha})^k$$

$$= O(1)\sum_{n=1}^{m} |\lambda_n| n^{-k}(w_n^{\alpha}|\varphi_n|)^k = O(1)\sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} v^{-k}(w_v^{\alpha}|\varphi_v|)^k$$

$$+O(1)|\lambda_m| \sum_{v=1}^{m} v^{-k}(w_v^{\alpha}|\varphi_v|)^k = O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n|x_n + O(1)|\lambda_m|x_m$$

$$= O(1)\sum_{n=1}^{m-1} \beta_n x_n + O(1)|\lambda_m|x_m = O(1) \quad \text{as} \quad m \longrightarrow \infty,$$

by virtue of the hypotheses and Lemma 2.

Therefore, we get that

$$\sum_{n=1}^{m} n^{-k}|\varphi_n T_{n,r}^{\alpha}|^k = O(1) \quad \text{as} \quad m \longrightarrow \infty, \quad \text{for} \quad r = 1, 2.$$

This completes the proof of the theorem.

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HÜSEYİN BOR
DEPARTMENT OF MATHEMATICS
ERCİYES UNIVERSITY
KAYSERİ 38039
TURKEY

(Received March 25, 1992)