

The unit group of FA_4

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Abstract. A complete characterization of the unit group $\mathcal{U}(FA_4)$ of the group algebra FA_4 of the alternating group of degree 4, A_4 , over a finite field F has been obtained.

1. Introduction and result

Let FG be the group algebra of a group G over a field F . For a normal subgroup H of G , the canonical homomorphism $g \mapsto gH : G \longrightarrow G/H$ can be extended to an algebra homomorphism from FG to $F[G/H]$ defined by

$$\sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g gH,$$

for $a_g \in F$. The kernel of this homomorphism, denoted by $\omega(H)$, is the ideal of FG generated by $\{h-1 \mid h \in H\}$. Thus $FG/\omega(H) \cong F[G/H]$. The augmentation ideal $\omega(FG)$ of the group algebra FG is defined by

$$\omega(FG) = \left\{ \sum_{g \in G} a_g g \in FG \mid a_g \in F, \sum_{g \in G} a_g = 0 \right\}.$$

Clearly $\omega(G) = \omega(FG)$. In general, $\omega(H) = \omega(FH)FG = FG\omega(FH)$. For $H = G$, $FG/\omega(G) \cong F$, showing Jacobson radical of FG , $J(FG)$, is contained in $\omega(FG)$. It is known that $J(FG) = \omega(FG)$ when G is a finite p -group and the characteristic of F , $\text{char}(F)$, is p .

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The lower central chain of G is given by

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_{m+1}(G) \supseteq \cdots$$

where $\gamma_{m+1}(G) = (\gamma_m(G), G)$, for $m \geq 1$. For $g_1, g_2 \in G$ the commutator $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$. The group G is said to be nilpotent of class n if $\gamma_{n+1}(G) = (1)$ and $\gamma_n(G) \neq (1)$. In this paper, the work is on the alternating group of degree 4, A_4 , whose presentation is given by

$$A_4 = \langle \sigma, a \mid \sigma^3 = a^2 = (\sigma a)^3 = 1 \rangle$$

where $\sigma = (1, 2, 3)$ and $a = (1, 2)(3, 4)$. Thus, with $b = (1, 3)(2, 4)$ and $c = (1, 4)(2, 3)$,

$$A_4 = \{1, a, b, c, \sigma, \sigma a, \sigma b, \sigma c, \sigma^2, \sigma^2 a, \sigma^2 b, \sigma^2 c\}.$$

The distinct conjugacy classes of A_4 are $\mathcal{C}_0 = \{1\}$, $\mathcal{C}_1 = \{a, b, c\}$, $\mathcal{C}_2 = \{\sigma, \sigma a, \sigma b, \sigma c\}$, $\mathcal{C}_3 = \{\sigma^2, \sigma^2 a, \sigma^2 b, \sigma^2 c\}$. Hence $\{\widehat{\mathcal{C}}_0, \widehat{\mathcal{C}}_1, \widehat{\mathcal{C}}_2, \widehat{\mathcal{C}}_3\}$ form a basis of the center $Z(FA_4)$ (cf. Lemma 4.1.1 of [4]), where $\widehat{\mathcal{C}}_i$ denotes the class sum. We obtain the relations between $\widehat{\mathcal{C}}_0, \widehat{\mathcal{C}}_1, \widehat{\mathcal{C}}_2, \widehat{\mathcal{C}}_3$ given by

$$\begin{aligned} \widehat{\mathcal{C}}_2^2 &= 4\widehat{\mathcal{C}}_3, & \widehat{\mathcal{C}}_2^4 &= 4^3\widehat{\mathcal{C}}_2, & \widehat{\mathcal{C}}_3^2 &= 4\widehat{\mathcal{C}}_2, & \widehat{\mathcal{C}}_3^4 &= 4^3\widehat{\mathcal{C}}_3, \\ \text{and } \widehat{\mathcal{C}}_2^3 &= 4^2(\widehat{\mathcal{C}}_0 + \widehat{\mathcal{C}}_1) = \widehat{\mathcal{C}}_3^3. \end{aligned}$$

Using these relations one may prove, by induction on r , that for $i = 2, 3$,

$$(\widehat{\mathcal{C}}_i)^{3r+1} = 4^{3r}\widehat{\mathcal{C}}_i, \quad \text{for } r \geq 0. \quad (1)$$

We define a matrix representation of A_4 ,

$$\theta : A_4 \longrightarrow \mathcal{U}(F \oplus \mathbb{M}(3, F))$$

by the assignment

$$\sigma \mapsto \left(1, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right) \quad \text{and} \quad a \mapsto \left(1, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

and can be extended to an algebra homomorphism

$$\theta^* : FA_4 \longrightarrow F \oplus \mathbb{M}(3, F),$$

where $\mathbb{M}(n, F)$ denotes the algebra of all $n \times n$ matrices over F . We use V_1 to denote $1 + J(FA_4)$, the kernel of the epimorphism from $\mathcal{U}(FA_4)$ to $\mathcal{U}(FA_4/J(FA_4))$ which is induced by the canonical homomorphism: $FA_4 \longrightarrow FA_4/J(FA_4)$.

ALLEN and HOBBY in [2], also AL-SOHEBANY in [1] have worked on the characterization of the unit group $\mathcal{U}(\mathbb{Z}A_4)$, and obtained that the group A_4 has a torsion free normal complement in $V(\mathbb{Z}A_4)$, the subgroup of the unit group of augmentation 1. Subsequently, SHARMA and GONGOPADHAYA has given presentations of the torsion free normal complement of A_4 in $V(\mathbb{Z}A_4)$ and of $V(\mathbb{Z}A_4)$ in [5], [6]. Conjugacy classes of all elements of finite order in $V(\mathbb{Z}A_4)$ have been studied by ALLEN and HOBBY in [3]. However, so far, the structure of the unit group $\mathcal{U}(FA_4)$, for $\text{char}(F) = p > 0$ is not known.

This paper gives a complete characterization of the unit group $\mathcal{U}(FA_4)$, for $\text{char}(F) = p > 0$ by proving the following:

Theorem. *Let $\mathcal{U}(FA_4)$ be the group of units of the group algebra FA_4 of the alternating group of degree 4 over a finite field F of positive characteristic p . Let F_2 be a quadratic extension of the field F and $V_1 = 1 + J(FA_4)$, where $J(FA_4)$ denotes the Jacobson radical of the group algebra FA_4 .*

- (1) *If $p = 2$, then V_1 is a nilpotent group of class 2 and $\mathcal{U}(FA_4)$ is centrally metabelian, but not metabelian.*
- (2) *If $p = 3$, then V_1 is a central subgroup of exponent $|F|$ and*

$$\mathcal{U}(FA_4)/V_1 \cong F^* \times GL(3, F)$$

- (3) *If $p > 3$ and $|F| = p^n$, then*

$$\mathcal{U}(FA_4) \cong \begin{cases} GL(3, F) \times F^* \times F^* \times F^* & \text{if } 3 \mid (p-1) \text{ or } n \text{ is even;} \\ GL(3, F) \times F_2^* \times F^* & \text{if } n \text{ is odd.} \end{cases}$$

Here $F^* = F \setminus \{0\}$ and $GL(3, F)$ is the general linear group of degree 3 over F .

2. Proof of the theorem

(1) Let $\text{char}(F) = 2$ with $|F| = 2^n$. Set $K_4 = \{1, a, b, c\}$ so that K_4 is a normal subgroup of A_4 and $[A_4 : K_4] = 3$. Then, by Theorem 7.2.7 of [4],

$$J(FA_4) = J(FK_4)FA_4 = \omega(FK_4)FA_4 = \omega(K_4).$$

Hence $FA_4/J(FA_4) \cong F[A_4/K_4]$. Note that $A_4/K_4 = \langle \bar{\sigma} = \sigma K_4 \rangle$, is a cyclic group of order 3, say C_3 . Suppose $x = \alpha_0 + \alpha_1 \bar{\sigma} + \alpha_2 \bar{\sigma}^2 \in FC_3$, for $\alpha_i \in F$. If n is even, then $3 \mid (2^n - 1)$ and consequently,

$$x^{2^n} = \alpha_0^{2^n} + \alpha_1^{2^n} (\bar{\sigma})^{2^n} + \alpha_2^{2^n} (\bar{\sigma}^2)^{2^n} = x;$$

so that $o(x) \mid (2^n - 1)$, when $x \in \mathcal{U}(FC_3)$. Thus $FC_3 \cong F \oplus F \oplus F$, if n is even. When n is odd, $3 \nmid (2^n - 1)$; but $3 \mid (2^{2n} - 1)$ and as above, $x^{2^{2n}} = x, \forall x \in FC_3$. Hence if n is odd, $FC_3 \cong F_2 \oplus F$.

Now, observe that $x(a-1)y(b-1) \in Z(FA_4), \forall x, y \in A_4$, so that $\omega(K_4)^2 \subseteq Z(FA_4)$, and consequently, $\omega(K_4)^3 = 0$. For $\xi, \eta \in \omega(K_4)$, we have

$$(1 + \xi, 1 + \eta) \equiv (1 - \xi - \eta)(1 + \xi + \eta) \equiv 1 \pmod{Z(FA_4)}.$$

Thus $\gamma_2(V_1) \subseteq Z(FA_4)$ and hence $\gamma_3(V_1) = (1)$, so that V_1 is a nilpotent group of class 2.

Since $\mathcal{U}(FA_4)/V_1$ is an Abelian group, we have $\mathcal{U}(FA_4)' \subseteq V_1$, therefore $\mathcal{U}(FA_4)'' \subseteq V_1' \subseteq Z(FA_4)$. Hence $\mathcal{U}(FA_4)$ is centrally metabelian. But it is not metabelian because $((u_1, u_2), (u_1, u_3)) \neq 1$, where

$$\begin{aligned} u_1 &= 1 + \sigma + \sigma^2 a + \sigma^2 b + b \\ u_2 &= 1 + \sigma + \sigma^2 a + \sigma^2 c + b \\ u_3 &= 1 + \sigma^2 a + \sigma^2 b + \sigma^2 c + c. \end{aligned}$$

(2) Let $\text{char}(F) = 3$ and $|F| = 3^n$. Assume $x \in \text{Ker } \theta^*$ (cf. Introduction for θ^*) with

$$x = \sum_{(i,t) \in I} \alpha_i t + \alpha_{i+1} \sigma t + \alpha_{i+2} \sigma^2 t,$$

for $\alpha_i \in F$ and $I = \{(0, 1), (3, a), (6, b), (9, c)\}$. Then $\theta^*(x) = 0$ gives the following systems of equations:

$$\sum_{i=0}^{11} \alpha_i = 0 \tag{2}$$

$$\alpha_i + \alpha_j - \alpha_k - \alpha_l = 0, \tag{3}$$

where $(i, j, k, l) \in \{(0, 6, 3, 9), (1, 10, 4, 7), (2, 5, 8, 11), (2, 8, 5, 11), (0, 9, 3, 6), (1, 4, 7, 10), (1, 7, 4, 10), (2, 11, 5, 8), (0, 3, 6, 9)\}$. Solving the system over F we get

$$\alpha_i = \alpha_{i+3j}, \quad \text{for } i = 0, 1, 2 \quad \text{and } j = 1, 2, 3.$$

Further, from equation (2), $\alpha_0 + \alpha_1 + \alpha_2 = 0$. Hence

$$\text{Ker } \theta^* = \{\alpha_0(\widehat{\mathcal{C}_0} + \widehat{\mathcal{C}_1}) + \alpha_1\widehat{\mathcal{C}_2} + \alpha_2\widehat{\mathcal{C}_3} \mid \alpha_0 + \alpha_1 + \alpha_2 = 0\}.$$

Thus $x^{3^n} = 0$, $\forall x \in \text{Ker } \theta^*$, so that $\text{Ker } \theta^* \subseteq J(FA_4)$. Also, since θ^* is onto, $\theta^*(J(FA_4)) \subseteq J(F \oplus \mathbb{M}(3, F)) = 0$ and so $J(FA_4) \subseteq \text{Ker } \theta^*$. Hence $\text{Ker } \theta^* = J(FA_4)$, therefore

$$(FA_4)/J(FA_4) \cong F \oplus \mathbb{M}(3, F).$$

Now, since $\mathcal{U}(FA_4)/V_1 \cong \mathcal{U}(FA_4/J(FA_4))$, we have

$$\mathcal{U}(FA_4)/V_1 \cong F^* \times GL(3, F).$$

Since $x^{3^n} = 0$, for all $x \in J(FA_4)$, we have V_1 is a central subgroup of exponent 3^n .

(3) Assume $p > 3$. Since $p \nmid |A_4|$, by Artin–Wedderburn theorem we have

$$FA_4 \cong \mathbb{M}(n_1, D_1) \oplus \mathbb{M}(n_2, D_2) \oplus \cdots \oplus \mathbb{M}(n_r, D_r),$$

where D_i 's are finite dimensional division algebras over F . Thus D_i 's are finite fields, as F is finite. Since FA_4 is noncommutative, there exists a k such that $n_k > 1$, so that n_k will be either 2 or 3. Further, since $\dim_F(Z(FA_4)) = 4$, we will get either of the following two possibilities only.

$$FA_4 \cong \mathbb{M}(3, F) \oplus F \oplus F \oplus F$$

$$FA_4 \cong \mathbb{M}(3, F) \oplus F_2 \oplus F$$

If $3 \mid (p-1)$, then $p^n \equiv 1 \pmod{3}$, for all n . Using the equation (1) in Section 1, we compute that

$$\widehat{\mathcal{C}_2}^{p^n} = \widehat{\mathcal{C}_2}^{3r+1} = (4^{3r})\widehat{\mathcal{C}_2} = (4^{p^n-1})\widehat{\mathcal{C}_2} = \widehat{\mathcal{C}_2}.$$

Similarly, we have $\widehat{\mathcal{C}_3}^{p^n} = \widehat{\mathcal{C}_3}$. Also, note that $(\widehat{\mathcal{C}_0} + \widehat{\mathcal{C}_1})^{p^n} = \widehat{\mathcal{C}_0} + \widehat{\mathcal{C}_1}$. Thus $x^{p^n} = x$, for all $x \in Z(FA_4)$. In particular, if $x \in \mathcal{U}(Z(FA_4))$, then $o(x) \mid (p^n - 1)$. Hence $FA_4 \cong \mathbb{M}(3, F) \oplus F \oplus F \oplus F$. Also, when $3 \nmid (p-1)$ but n is even, then $3 \mid (p^n - 1)$, so that

$$FA_4 \cong \mathbb{M}(3, F) \oplus F \oplus F \oplus F.$$

If $3 \nmid (p-1)$ and n is odd, we get $3 \nmid (p^n - 1)$ and so $3 \mid (p^n + 1)$. Then $3 \mid (p^{2n} - 1)$, which implies $x^{p^{2n}} = x$ for all $x \in Z(FA_4)$. Thus

$$FA_4 \cong \mathbb{M}(3, F) \oplus F_2 \oplus F.$$

Hence

$$\mathcal{U}(FA_4) \cong \begin{cases} GL(3, F) \times F^* \times F^* \times F^* & \text{if } 3 \mid (p-1) \text{ or } n \text{ is even;} \\ GL(3, F) \times F_2^* \times F^* & \text{if } n \text{ is odd.} \end{cases}$$

This completes the proof of the Theorem.

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