

On the tangency of sets of the class $\tilde{M}_{p,k}$

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Abstract. In the present paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ in generalized metric spaces is considered.

Introduction. Let E be an arbitrary non-empty set and E_0 the family of all non-empty subsets of this set.

Let ϱ be any metric of the set E . We shall denote

$$(1) \quad \varrho(x, A) = \inf_{y \in A} \varrho(x, y) \quad \text{for} \quad x \in E, A \in E_0.$$

Let k be an arbitrary but fixed positive real number. By A' we shall denote the set of all cluster points of the set $A \in E_0$.

We assume by definition (see [5])

$$(2) \quad \tilde{M}_{p,k} = \left\{ A \in E_0 : p \in A' \text{ and there exists } \mu > 0 \text{ such that} \right. \\
\text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such} \\
\text{that for every pair of points } (x, y) \in [A, p; \mu, k] \\
\left. \text{if } \varrho(x, p) < \delta \text{ and } \frac{\varrho(x, A)}{\varrho^k(x, p)} < \delta \text{ then } \frac{\varrho(x, y)}{\varrho^k(x, p)} < \varepsilon \right\}$$

where

$$(3) \quad [A, p; \mu, k] = \\
= \{(x, y) : x \in E, y \in A \text{ and } \mu\varrho(x, A) < \varrho^k(x, p) = \varrho^k(y, p)\}.$$

Let ℓ denote any non-negative real function defined on the Cartesian product $E_0 \times E_0$. The pair (E, ℓ) we shall call a generalized metric space (see [8]).

We say that the set $A \in E_0$ has the Darboux property at the point p of the space (E, ℓ) if there exists a number $\tau > 0$ such that $A \cap S_\ell(p, r) \neq \emptyset$ for

$r \in (0, \tau)$. $S_\ell(p, r)$ denotes here the sphere with the centre at the point p and the radius r in the space (E, ℓ) . The set of all sets having the Darboux property at the point p of the space (E, ℓ) will be denoted by $D_p(E, \ell)$.

Let a, b be any non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$(4) \quad a(r) \xrightarrow[r \rightarrow 0+]{\quad} 0, \quad b(r) \xrightarrow[r \rightarrow 0+]{\quad} 0.$$

The pair (A, B) of sets $A, B \in E_0$ we call (a, b) -clustered at the point p of the space (E, ℓ) if 0 is the cluster point of the set of all real numbers $r > 0$ such that the sets $A \cap S_\ell(p, r)_{a(r)}$, $B \cap S_\ell(p, r)_{b(r)}$ are non empty.

The sets $S_\ell(p, r)_{a(r)}$, $S_\ell(p, r)_{b(r)}$ are so-called $a(r)$, $b(r)$ -neighbourhoods of the sphere $S_\ell(p, r)$ in the space (E, ℓ) (see [8]). From the above definitions it follows that the pair (A, B) is (a, b) -clustered at the point p of the space (E, ℓ) if the sets $A, B \in D_p(E, \ell)$.

Let $d_\varrho A$ denote the diameter of the set $A \in E_0$ and $\varrho(A, B)$ the distance of the sets $A, B \in E_0$ in the metric space (E, ϱ) .

By F_ϱ we shall denote the class of the functions ℓ fulfilling the conditions:

- 1° $\ell : E_0 \times E_0 \rightarrow \langle 0, \infty \rangle$,
- 2° there exist a numbers m, M such that $0 < m \leq M < \infty$ and $m\varrho(A, B) \leq \ell(A, B) \leq Md_\varrho(A \cup B)$ for $A, B \in E_0$,
- 3° the function ℓ generates on the set E the metric ℓ_0 defined by the formula: $\ell_0(x, y) = \ell(\{x\}, \{y\})$ for $x, y \in E$.

From Lemma 1.1 of the paper [5] it follows that for an arbitrary set $A \in \tilde{M}_{p,k} \cap D_p(E, \ell)$

$$(5) \quad \frac{1}{r^k} d_\ell(A \cap S_\ell(p, r)_{a(r)}) \xrightarrow[r \rightarrow 0+]{\quad} 0,$$

when

$$(6) \quad \frac{a(r)}{r^{k+1}} \xrightarrow[r \rightarrow 0+]{\quad} \alpha, \quad (\alpha < \infty).$$

According to the definition given by W. WALISZEWSKI in the paper [8], the set $A \in E_0$ is (a, b) -tangent of order k to the set $B \in E_0$ at the point p of the space (E, ℓ) , which we shall write in the form $(A, B) \in T_\ell(a, b, k, p)$, iff the pair (A, B) is (a, b) -clustered at the point p of this space and

$$(7) \quad \frac{1}{r^k} \ell(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)}) \xrightarrow[r \rightarrow 0+]{\quad} 0.$$

We call $T_\ell(a, b, k, p)$ the relation of (a, b) -tangency of sets of order k at the point p of the space (E, ℓ) .

In this paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ is considered, our considerations being based on W. Waliszewski's definition.

For $k = 1$ the class $\tilde{M}_{p,k}$ contains among other things such classes of sets as H_p , A_p^* (see [1]) and the class I_p of all simple arcs with an origin at the point $p \in E$.

Moreover W. Waliszewski's definition essentially generalizes known earlier definitions of the tangency of sets in a metric space (E, ϱ) .

One of these definitions (the most general) says that the set $A \in E_0$ is tangent to the set $B \in E_0$ at the point $p \in E$ if $p \in A'$ and

$$(8) \quad \frac{\varrho(x, B)}{\varrho(p, x)} \xrightarrow{A \ni x \rightarrow p} 0.$$

The condition (8) is equivalent to the condition

$$(9) \quad \frac{1}{r} \sup\{\varrho(x, B) : x \in A \text{ and } \varrho(p, x) = r\} \xrightarrow{r \rightarrow 0+} 0.$$

If we put

$$(10) \quad \varrho_0(A, B) = \sup_{x \in A} \varrho(x, B) \text{ for } A, B \in E_0,$$

then the condition (9) can be written in the form

$$(11) \quad \frac{1}{r} \varrho_0(A \cap S_\varrho(p, r), B) \xrightarrow{r \rightarrow 0+} 0,$$

where $S_\varrho(p, r)$ denotes the sphere with centre at the point p and radius r in a metric space (E, ϱ) .

Setting $a(r) = 0$, $b(r) = r$ for $r > 0$, condition (11) can be written

$$(12) \quad \frac{1}{r} \varrho_0(A \cap S_\varrho(p, r)_{a(r)}, B \cap S_\varrho(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0,$$

which, according to the definition of the tangency relation $T_\ell(a, b, k, p)$, means that $(A, B) \in T_{\varrho_0}(a, b, 1, p)$.

This implies that if we replace the function ϱ_0 in condition (12) by an arbitrary non-negative real function ℓ defined on the Cartesian product $E \times E$, then we get the definition of the tangency of sets in the sense of W. Waliszewski.

Because the function ϱ_0 is a particular case of the function ℓ , W. Waliszewski's definition does in reality generalize the above mentioned definition of the tangency of sets in a metric space (E, ϱ) .

In this paper (by hypothesis) the function ℓ generates on the set E the metric ℓ_0 , so one can also speak about considering the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ in the ordinary metric space (E, ℓ_0) .

In Section 1 of the present paper the problem of the tangency of sets of the class $\tilde{M}_{p,k}$ for the functions ℓ belonging to the class F_ϱ is considered.

In Section 2 this problem is examined for the functions ℓ belonging to a certain class contained in the class F_ϱ .

1. Let a, b be arbitrary non-negative real functions defined in the right-hand side neighbourhood of 0 such that

$$(1.1) \quad \frac{a(r)}{r^{k+1}} \xrightarrow{r \rightarrow 0+} \alpha, \quad \frac{b(r)}{r^{k+1}} \xrightarrow{r \rightarrow 0+} \beta$$

where $\alpha, \beta \in \langle 0, \infty \rangle$.

Theorem 1.1. *If the functions a, b fulfil the condition (1.1), and the function $\ell \in F_\varrho$, then $(A, B) \in T_\ell(a, b, k, p)$ for arbitrary sets $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$ such that $A \subset B$ or $A \supset B$.*

PROOF. Let us suppose that $A \subset B$ for $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$. Then for an arbitrary function $\ell \in F_\varrho$ we have

$$(1.2) \quad \begin{aligned} 0 &\leq \frac{1}{M} \ell \left(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)} \right) \leq \\ &\leq d_\varrho \left((A \cap S_\ell(p, r)_{a(r)}) \cup (B \cap S_\ell(p, r)_{b(r)}) \right) \leq \\ &\leq d_\varrho \left((B \cap S_\ell(p, r)_{a(r)}) \cup (B \cap S_\ell(p, r)_{b(r)}) \right) \leq \\ &\leq d_\varrho \left(B \cap S_\ell(p, r)_{\max\{a(r), b(r)\}} \right). \end{aligned}$$

From the fact that the set $B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, the function $\ell \in F_\varrho$, from (1.1) and from Lemma 1.1 of the paper [5] it follows

$$(1.3) \quad \frac{1}{r^k} d_\varrho \left(B \cap S_\ell(p, r)_{\max\{a(r), b(r)\}} \right) \xrightarrow{r \rightarrow 0+} 0.$$

Hence and from (1.2) we get

$$(1.4) \quad \frac{1}{r^k} \ell \left(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)} \right) \xrightarrow{r \rightarrow 0+} 0.$$

In view of $A, B \in D_p(E, \ell)$ the pair (A, B) is (a, b) -clustered at the point p of the space (E, ℓ) .

From here and from (1.4) it follows that $(A, B) \in T_\ell(a, b, k, p)$.

If $A \supset B$ for $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, then the proof of this theorem is analogous.

Corollary 1.1. *If the sequence of sets $A_1, A_2, \dots, A_n \in \tilde{M}_{p,k} \cap D_p(E, \ell)$ is a monotonically increasing or a monotonically decreasing sequence, and the functions a, b fulfil condition (1.1), $\ell \in F_\varrho$, then $(A_i, A_j) \in T_\ell(a, b, k, p)$ for $i, j = 1, 2, \dots, n$.*

Corollary 1.2. *If $A, B \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, the functions a, b fulfil condition (1.1), and $\ell \in F_\varrho$, then $(A, A) \in T_\ell(a, b, k, p)$, $(A, A \cup B) \in T_\ell(a, b, k, p)$ and $(A, A \cap B) \in T_\ell(a, b, k, p)$, for $A \cap B \in D_p(E, \ell)$.*

From Corollary 1.2 it follows that the tangency relation $T_\ell(a, b, k, p)$ is reflexive in the class of sets $\tilde{M}_{p,k} \cap D_p(E, \ell)$ for a function $\ell \in F_\varrho$ and functions a, b fulfilling the condition (1.1).

Theorem 1.2. *If the functions a, b fulfil the condition (1.1) and $\ell \in F_\varrho$, then for arbitrary sets $A, B, C \in \tilde{M}_{p,k} \cap D_p(E, \ell)$ such that $B \subset C$ or $B \supset C$, if $(A, B) \in T_\ell(a, b, k, p)$ then $(A, C) \in T_\ell(a, b, k, p)$.*

PROOF. Let us assume that $(A, B) \in T_\ell(a, b, k, p)$. This implies

$$(1.5) \quad \frac{1}{r^k} \ell(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

Hence and from the fact that $\ell \in F_\varrho$ we get

$$(1.6) \quad \frac{1}{r^k} \varrho(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

Let us suppose that $B \subset C$ for $B, C \in \tilde{M}_{p,k} \cap D_p(E, \ell)$. Hence and from (1.6) we have

$$(1.7) \quad \frac{1}{r^k} \varrho(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

From $A, C \in \tilde{M}_{p,k} \cap D_p(E, \ell)$, $\ell \in F_\varrho$ and from Lemma 1.1 of the paper [5] it follows

$$(1.8) \quad \frac{1}{r^k} d_\varrho(A \cap S_\ell(p, r)_{a(r)}) \xrightarrow{r \rightarrow 0+} 0, \quad \frac{1}{r^k} d_\varrho(C \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

Moreover

$$\begin{aligned} 0 &\leq \frac{1}{M} \ell(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}) \leq \\ &\leq d_\varrho((A \cap S_\ell(p, r)_{a(r)}) \cup (C \cap S_\ell(p, r)_{b(r)})) \leq \\ &\leq d_\varrho((A \cap S_\ell(p, r)_{a(r)}) + d_\varrho(C \cap S_\ell(p, r)_{b(r)})) + \\ &\quad + \varrho(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}). \end{aligned}$$

Hence from (1.7) and (1.8) we get

$$(1.9) \quad \frac{1}{r^k} \ell(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

Owing to $A, C \in D_p(E, \ell)$, the pair (A, C) is (a, b) -clustered at the point p of the space (E, ℓ) .

Hence and from (1.9) it follows that $(A, C) \in T_\ell(a, b, k, p)$.

Let us now suppose that $B \supset C$ for $B, C \in \tilde{M}_{p,k} \cap D_p(E, \ell)$. From here and from the assumption that $\ell \in F_\varrho$ we get

$$(1.10) \quad \begin{aligned} 0 &\leq \frac{1}{M} \ell(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}) \leq \\ &\leq d_\varrho((A \cap S_\ell(p, r)_{a(r)}) \cup (C \cap S_\ell(p, r)_{b(r)})) \leq \\ &\leq d_\varrho((A \cap S_\ell(p, r)_{a(r)}) \cup (B \cap S_\ell(p, r)_{b(r)})) \leq \\ &\leq d_\varrho((A \cap S_\ell(p, r)_{a(r)}) + d_\varrho(B \cap S_\ell(p, r)_{b(r)})) + \\ &\quad + \varrho(A \cap S_\ell(p, r)_{a(r)}, B \cap S_\ell(p, r)_{b(r)}). \end{aligned}$$

Now by Lemma 1.1 of [5]

$$\frac{1}{r^k} d_\varrho(A \cap S_\ell(p, r)_{a(r)}) \xrightarrow{r \rightarrow 0+} 0, \quad \frac{1}{r^k} d_\varrho(B \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0,$$

and from here, from (1.6) and from (1.10) we obtain

$$\frac{1}{r^k} \ell(A \cap S_\ell(p, r)_{a(r)}, C \cap S_\ell(p, r)_{b(r)}) \xrightarrow{r \rightarrow 0+} 0.$$

Hence and from the fact that the pair of sets (A, C) is (a, b) -clustered at the point p of the space (E, ℓ) it follows that $(A, C) \in T_\ell(a, b, k, p)$. This ends the proof.

This theorem implies the

Corollary 1.3. *If the functions a, b fulfil the condition (1.1) and $\ell \in F_\varrho$, then for arbitrary sets $A, B_1, B_2, \dots, B_n \in \tilde{M}_{p,k} \cap D_p(E, \ell)$ such that the sequence $B_1, B_2, \dots, B_n$ is monotonically increasing or monotonically decreasing, if $(A, B_i) \in T_\ell(a, b, k, p)$ then $(A, B_j) \in T_\ell(a, b, k, p)$ for $i, j = 1, 2, \dots, n$.*

2. Let F_ϱ^* be the class of functions ℓ fulfilling the conditions:

$$1^\circ \quad \ell : E_0 \times E_0 \rightarrow \langle 0, \infty \rangle,$$

$$2^\circ \quad \varrho(A, B) \leq \ell(A, B) \leq d_\varrho(A \cup B) \quad \text{for } A, B \in E_0.$$

From the definition of the classes of functions F_ϱ and F_ϱ^* it follows that $F_\varrho^* \subset F_\varrho$. Moreover, any function $\ell \in F_\varrho^*$ generates on the set E the metric ϱ because

$$(2.1) \quad \varrho(x, y) = \varrho(\{x\}, \{y\}) \leq \ell(\{x\}, \{y\}) \leq d_\varrho(\{x\} \cup \{y\}) = \varrho(x, y).$$

Let a_i, b_i ($i = 1, 2$) be arbitrary non-negative real functions defined in a right-hand side neighbourhood of 0 such that

$$(2.2) \quad \frac{a_i(r)}{r^{k+1}} \xrightarrow{r \rightarrow 0+} \alpha_i, \quad \frac{b_i(r)}{r^{k+1}} \xrightarrow{r \rightarrow 0+} \beta_i,$$

where $\alpha_i, \beta_i \in (0, \infty)$.

Hence and from Corollary 1.1 of the paper [5] there follows the equivalence

$$(2.3) \quad (A, B) \in T_{\ell_1}(a_1, b_1, k, p) \equiv (A, B) \in T_{\ell_2}(a_2, b_2, k, p),$$

for any sets $A, B \in \tilde{M}_{p,k} \cap D_p(E, \varrho)$ and functions $\ell_1, \ell_2 \in F_\varrho^*$.

Let C be an arbitrary set of the class $\tilde{M}_{p,k} \cap D(E, \varrho)$ such that $C \subset B$ or $C \supset B$.

From here and from Theorem 1.2 the present paper there follows the implication

$$(2.4) \quad (A, B) \in T_{\ell_2}(a_2, b_2, k, p) \implies (A, C) \in T_{\ell_2}(a_2, b_2, k, p)$$

for $\ell_2 \in F_\varrho^*$ and $A, B, C \in \tilde{M}_{p,k} \cap D_p(E, \varrho)$.

Hence and from (2.3) we get

Theorem 2.1. *If the functions a_i, b_i ($i = 1, 2$) fulfil the condition (2.2) and $\ell_1, \ell_2 \in F_\varrho^*$, then for arbitrary sets $A, B, C \in \tilde{M}_{p,k} \cap D_p(E, \varrho)$ such that $B \subset C$ or $B \supset C$, if $(A, B) \in T_{\ell_1}(a_1, b_1, k, p)$ then $(A, C) \in T_{\ell_2}(a_2, b_2, k, p)$.*

From this theorem we get

Corollary 2.1. *If the functions a_k, b_k ($k = 1, 2$) fulfil the condition (2.2), and $\ell_1, \ell_2 \in F_\varrho^*$, then for arbitrary sets $A, B_1, B_2, \dots, B_n \in \tilde{M}_{p,k} \cap D_p(E, \varrho)$ such that the sequence $B_1, B_2, \dots, B_n$ is monotonically increasing or monotonically decreasing if $(A, B_i) \in T_{\ell_1}(a_1, b_1, k, p)$ then $(A, B_j) \in T_{\ell_2}(a_2, b_2, k, p)$ for $i, j = 1, 2, \dots, n$.*

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(Received April 7, 1992; revised November 26, 1992)