

The maximal operator of the Fejér means of the character system of the p -series field in the Kaczmarz rearrangement

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma^{*\chi}$ of the Fejér means of the character system of the p -series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}$ to the space weak- $L_{1/2}$ and is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to FINE [1]. Later, SCHIPP [5] showed that the maximal operator $\sigma^* f$ is of weak type $(1, 1)$, from which the a.e. convergence follows by standard argument. Schipp’s result implies by interpolation also the boundedness of $\sigma^* : L_\alpha \rightarrow L_\alpha$ ($1 < \alpha \leq \infty$). This fails to hold for $\alpha = 1$ but FUJII [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 (see also SIMON [6]). Fujii’s theorem was extended by WEISZ [10]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_\alpha(I)$ to the space $L_\alpha(I)$ for $\alpha > 1/2$. SIMON [7] gave a counterexample, which shows that this boundedness does not hold for $0 < \alpha < 1/2$. In the endpoint case $\alpha = 1/2$ WEISZ [13] proved that σ^* is bounded from the Hardy space $H_{1/2}(I)$ to the space weak- $L_{1/2}(I)$.

If the Walsh system is taken in the Kaczmarz ordering, the analogue of the statement of SCHIPP [5] is due to GÁT [3]. Moreover he proved an (H_1, L_1) -type estimation. Gát’s result was extended to the Hardy space by SIMON [8], who

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proved that σ^* is of type (H_α, L_α) for $\alpha > 1/2$. WEISZ [13] showed that in the endpoint case $\alpha = 1/2$ the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

GÁT and NAGY [4] proved the a.e. convergence $\sigma_n^\chi f \rightarrow f$ ($n \rightarrow \infty$) for an integrable function $f \in L_1(G_p)$, where $\sigma_n f$ is the Fejér means of the function f with respect to the character system in the Kaczmarz rearrangement. They also proved that the maximal operator $\sigma^{*\chi}$ is of type (α, α) for all $1 < \alpha \leq +\infty$, of weak type $(1, 1)$ and $\|\sigma^* f\|_1 \leq c \|f\|_{H_1}$.

The main aim of this paper is to generalize the results of GÁT and NAGY [4] and we prove that the maximal operator $\sigma^{*\chi}$ of the Fejér means of the character system of the p -series field in the Kaczmarz rearrangement is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak- $L_{1/2}(G_p)$ and is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.

2. Definitions and notation

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $2 \leq p \in \mathbb{N}$ and denote by $\mathbb{Z}_p$ the p th cyclic group, that is, $\mathbb{Z}_p$ can be represented by the set $\{0, 1, \dots, p-1\}$, where the group operation is mod p addition and every subset is open. The Haar measure on $\mathbb{Z}_p$ is given so that

$$\mu_k(\{j\}) := \frac{1}{j} \quad (j \in \mathbb{Z}).$$

The group operation on G_p is coordinate-wise addition, the normalized Haar measure μ is the product measure. The topology on G_p is the product topology, a base for the neighborhoods of G_p can be given thus:

$$I_0(x) := G_p, \quad I_n(x) := \{y \in G_p : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

$$(x \in G_p, n \in \mathbb{N}).$$

Let $0 = (0 : i \in \mathbb{N}) \in G_p$ denote the null element of G_p , $I_n := I_n(0)$ ($n \in \mathbb{N}$). Let

$$\Delta := \{I_n(x) : x \in G_p, n \in \mathbb{N}\}.$$

The elements of Δ are intervals of G_p . Set $e_i := (0, \dots, 0, 1, 0, \dots) \in G_p$ the i th coordinate of which is 1, the rest are zeros.

The norm (or quasinorm) of the space $L_\alpha(G_p)$ is defined by

$$\|f\|_\alpha := \left(\int_{G_p} |f(x)|^\alpha d\mu(x) \right)^{1/\alpha} \quad (0 < \alpha < +\infty).$$

Let $\Gamma(p)$ denote the character group of G_p . We arrange the elements of $\Gamma(p)$ as follows: For $k \in \mathbb{N}$ and $x \in G_p$ denote by r_k the k -th generalized Rademacher function:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{p}\right) \quad (i := \sqrt{-1}, x \in G_p, k \in \mathbb{N}).$$

Let $n \in \mathbb{N}$. Then

$$n = \sum_{i=0}^{\infty} n_i p^i, \quad \text{where } 0 \leq n_i < p \quad (n_i, i \in \mathbb{N}),$$

where n is expressed in the number system with base p . Put

$$|n| := \max(j \in \mathbb{N} : n_j \neq 0) \quad \text{i.e., } p^{|n|} \leq n < p^{|n|+1}.$$

Now we define the sequence of functions $\psi := (\psi_n : n \in \mathbb{N})$ by

$$\psi_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} \quad (x \in G_p, n \in \mathbb{N}).$$

We remark that $\Gamma(p) = \{\psi_n : n \in \mathbb{N}\}$ is a complete orthogonal system with respect to the normalized Haar measure on G_p .

The character group $\Gamma(p)$ can be given in the Kaczmarz rearrangement as follows: $\Gamma(p) = \{\chi_n : n \in \mathbb{N}\}$, where

$$\chi_n(x) := r_{|n|}^{n_{|n|}}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} \quad (x \in G_p, n \in \mathbb{N}),$$

$$\chi_0(x) = 1 \quad (x \in G_p).$$

Let the transformation $\tau_A : G_p \rightarrow G_p$ be defined as follows:

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_0, x_A, x_{A+1}, \dots).$$

The transformation is measure-preserving and $\tau_A(\tau_A(x)) = x$. By the definition of τ_A , we have

$$\chi_n(x) = r_{|n|}^{n_{|n|}}(x) \psi_{n-n_{|n|}p^n}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G_p).$$

For a function f in $L_1(G_p)$ the Fourier coefficients, the partial sums of Fourier series, the Dirichlet kernels, the Fejér means and the Fejér kernels are defined as follows:

$$\begin{aligned} \hat{f}^\gamma(n) &:= \int_{G_p} f \gamma_n, \quad S_n^\gamma(f, x) := \sum_{k=0}^{n-1} \hat{f}^\gamma(k) \gamma_k(x), \quad D_n^\gamma := \sum_{k=0}^{n-1} \gamma_k, \\ \sigma_n^\gamma(f) &:= \frac{1}{n} \sum_{k=1}^n S_k^\gamma(f), \quad K_n^\gamma := \frac{1}{n} \sum_{k=1}^n D_k^\gamma(x), \end{aligned}$$

where $\gamma_n = \psi_n$ or χ_n .

Let

$$K_{a,b} := \sum_{j=a}^{a+b-1} D_j^\gamma \quad (a, b \in \mathbb{N}),$$

and

$$n^{(s)} := \sum_{i=s}^{\infty} n_i p^i \quad (n, s \in \mathbb{N}).$$

By a simple calculation we get

$$nK_n^\gamma = \sum_{s=0}^{|n|} \sum_{l=0}^{n_s-1} K_{n^{(s+1)}+lp^s, p^s}^\gamma + D_n^\gamma. \quad (1)$$

The p^n th Dirichlet kernels have a closed form:

$$D_{p^n}^\psi(x) = D_{p^n}^\chi(x) = \begin{cases} p^n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases} \quad \text{where } x \in G_p. \quad (2)$$

We define the maximal operator

$$\sigma^{*\gamma} f := \sup_{n \in \mathbb{P}} |\sigma_n^\gamma f| \quad (f \in L_1(G_p)).$$

The space weak- $L_\alpha(G_p)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_\alpha(G_p)} := \sup_{\rho > 0} \rho \mu(|f| > \rho)^{1/\alpha} < +\infty.$$

The σ -algebra generated by the intervals I_k of length p^{-k} will be denoted by F_k ($k \in \mathbb{N}$).

Denote by $f = (f^{(n)}, n \in \mathbb{N})$ the one-parameter martingale with respect to $(F_n, n \in \mathbb{N})$ (for details see, e.g., [9]–[12]) The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In case $f \in L_1(G_p)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \geq 1} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|, \quad x \in G_p.$$

For $0 < \alpha \leq \infty$ the Hardy martingale space $H_p(G_p)$ consists all martingales for which

$$\|f\|_{H_\alpha} := \|f^*\|_\alpha < \infty.$$

If $f \in L_1(G_p)$ then it is easy to show that the sequence $(S_{p^n}(f) : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \dots)$, then the Fourier coefficients must be defined in a slightly different way:

$$\widehat{f}(j) = \lim_{k \rightarrow \infty} \int_{G_p} f^{(k)}(x) \gamma_j(x) dx.$$

The Fourier coefficients of $f \in L_1(G_p)$ are the same as those of the martingale $(S_{p^n}(f) : n \in \mathbb{N})$ obtained from f .

A bounded measurable function a is an α -atom, if there exists an interval I , such that

- a) $\int_I a d\mu = 0$;
- b) $\|a\|_\infty \leq \mu(I)^{-1/\alpha}$;
- c) $\text{supp } a \subset I$.

3. Formulation of the main results

Theorem 1. *The maximal operator $\sigma^{*\chi}$ is bounded from the Hardy space $H_{1/2}(G_p)$ to the space weak- $L_{1/2}(G_p)$.*

Theorem 2. *The maximal operator $\sigma^{*\chi}$ is not bounded from the Hardy space $H_{1/2}(G_p)$ to the space $L_{1/2}(G_p)$.*

4. Auxiliary propositions

We shall need the following lemmas (see [4], [13]).

Lemma 1 (Weisz). *Suppose that an operator V is sublinear, and for some $0 < \alpha < 1$*

$$\sup_{\rho > 0} \rho^\alpha \mu \{x \in C_p \setminus I : |Va(x)| > \rho\} \leq c_\alpha < \infty$$

for every α -atom a , where I denotes the support of the atom. If V is bounded from L_{α_1} to L_{α_1} for a fixed $1 < \alpha_1 \leq \infty$, then

$$\|Vf\|_{\text{weak-}L_\alpha(G_p)} \leq c_\alpha \|f\|_{H_\alpha}.$$

Lemma 2 (Gát, Nagy). Suppose that $s, b, n \in \mathbb{N}$ and $x \in I_b \setminus I_{b+1}$. If $s \leq b \leq |n|$, then

$$|K_{n^{(s+1)+lp^s, p^s}}^\psi(x)| \leq cp^{s+b},$$

while if $b < s \leq |n|$, then

$$K_{n^{(s+1)+lp^s, p^s}}^\psi(x) = \begin{cases} 0 & \text{if } x - x_b e_b \notin I_s, \\ \omega_{n^{(s+1)}}(x) p^{s+b-1} & \text{if } x - x_b e_b \in I_s. \end{cases}$$

Lemma 3 (Gát, Nagy). Let $A \in \mathbb{N}$ and $n := n_A p^A + n_{A-1} p^{A-1} + \dots + n_0 p^0$. Then

$$\begin{aligned} nK_n^\chi(x) &= 1 + \sum_{j=0}^{A-1} \sum_{i=1}^{p-1} r_j^i(x) p^j K_{p^j}^\psi(\tau_j(x)) + \sum_{j=0}^{A-1} p^j D_{p^j}^\psi(x) \sum_{l=1}^{p-1} \sum_{i=0}^{l-1} r_j^l(x) \\ &\quad + p^A \sum_{l=1}^{n_A-1} r_A^l(x) K_{p^A}^\psi(\tau_A(x)) + r_A^{n_A}(x) (n - n_A p^A) K_{n-n_A p^A}^\psi(\tau_A(x)) \\ &\quad + (n - n_A p^A) \sum_{i=0}^{n_A-1} r_A^i(x) D_{p^A}^\psi(x) + p^A \sum_{j=1}^{n_A-1} \sum_{i=0}^{j-1} r_A^i(x) D_{p^A}^\psi(x). \end{aligned}$$

Corollary 1. We have

$$\sup_n \int_{G_p} |K_n^\chi(x)| d\mu(x) < +\infty.$$

Lemma 4. Let $n < p^{A+1}$, $A > N$ and $x \in I_N$ ($x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0$), $m = -1, 0, \dots, l-1$, $l = 0, \dots, N-1$. Then

$$\int_{I_N} n |K_n^w(\tau_A(x-t))| d\mu(t) \leq c \frac{p^A}{p^{m+l}},$$

where

$$\begin{aligned} &I_N(x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0) \\ &:= I_N(0, \dots, 0, x_l \neq 0, 0, \dots, 0), \quad \text{for } m = -1. \end{aligned}$$

PROOF. It is evident that for $x \in I_N$ ($x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0$) we have

$$\int_{I_N} |D_n^\chi(x-t)| d\mu(t) \leq c \sum_{j=0}^A \int_{I_N} |D_{p^j}^\psi(\tau_A(x-t))| d\mu(t) \leq c \sum_{j=0}^{A-l} \frac{p^j}{p^A} \leq \frac{c}{p^l}. \quad (3)$$

From Lemma 2 we obtain that $K_{n^{(s+1)+lp^s,p^s}}^w(\tau_A(x-t)) = 0$ for $s \geq A-m$. Hence we can suppose that $s < A-m$.

Using Lemma 2 $K_{n^{(s+1)+lp^s,p^s}}^w(\tau_A(x-t)) \neq 0$ implies that

- 1) $t \in I_N(0, \dots, 0, x_N, \dots, x_{A-1})$ if $0 \leq s < A-m$;
- 2) $t \in I_A(0, \dots, 0, x_N, \dots, x_{q-1}, t_q \neq x_q, x_{q+1}, \dots, x_{A-1})$
if $A-N < s < A-l$;
- 3) $t \in I_A(0, \dots, 0, t_N, \dots, t_{A-s-1}, x_{A-s}, \dots, x_{q-1}, t_q \neq x_q, x_{q+1}, \dots, x_{A-1})$
if $1 \leq s \leq A-N$;
- 4) $t \in I_A(0, \dots, 0, t_N, \dots, t_{q-1}, t_q \neq x_q, x_{q+1}, \dots, x_{A-s}, \dots, x_{A-1})$
if $1 \leq s < A-N$;

consequently, from (1) and (3) we can write

$$\begin{aligned}
& \int_{I_N} n |K_n^w(\tau_A(x-t))| d\mu(t) \\
& \leq \sum_{s=0}^{A-m} \sum_{l=0}^{n_s-1} \int_{I_N} |K_{n^{(s+1)+lp^s,p^s}}^w(\tau_A(x-t))| d\mu(t) + \int_{I_N} |D_n^x(x-t)| d\mu(t) \\
& \leq c \left\{ \sum_{s=0}^{A-m} \frac{p^{s+A-l}}{p^A} + \sum_{s=A-N}^{A-l} \sum_{q=N}^A \frac{p^{s+A-q}}{p^A} \right. \\
& \quad \left. + \sum_{s=0}^{A-N} \sum_{q=A-s}^A \frac{p^{s+A-q} p^{A-s-N}}{p^A} + \sum_{s=0}^{A-N} \sum_{q=N}^{A-s} \frac{p^{s+A-q} p^{q-N}}{p^A} \right\} \\
& \leq c \left\{ \frac{p^A}{p^{m+l}} + \frac{p^A}{p^{N+l}} + \frac{p^A}{p^{2N}} + \sum_{s=0}^{A-N} \frac{p^s (A-s-N+1)}{p^{2N}} \right\} \leq c \frac{p^A}{p^{m+l}}.
\end{aligned}$$

Lemma 4 is proved. $\square$

Lemma 5. *Let $n \in \mathbb{N}$. Then*

$$\int_{G_p} \max_{1 \leq N \leq 2^n} \left(N |K_N^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \geq c \frac{n+1}{\log(n+2)}.$$

PROOF. It is evident that

$$\int_{G_p} D_j^\psi(\tau_n(x)) D_i^\psi(\tau_n(x)) d\mu(x) = \int_{G_p} D_j^\psi(x) D_i^\psi(x) d\mu(x) = \min\{i, j\}.$$

Then we can write

$$\begin{aligned} \int_{G_p} \left(\sum_{j=1}^N D_j^\psi(\tau_n(x)) \right)^2 d\mu(x) &= \sum_{j=1}^N \sum_{i=1}^N \int_{G_p} D_j^\psi(\tau_n(x)) D_i^\psi(\tau_n(x)) d\mu(x) \\ &= \sum_{j=1}^N \sum_{i=1}^N \min\{i, j\} \geq c_0 N^3. \end{aligned} \quad (4)$$

It is well-known that

$$\int_{G_p} \left| K_N^\psi(\tau_n(x)) \right| d\mu(x) \leq c_1 < \infty, \quad N = 1, 2, \dots, p^n, \quad n = 0, 1, \dots \quad (5)$$

Denote

$$A_{N_i} := \left\{ x \in G_p : \left| K_{N_i}^\psi(\tau_n(x)) \right| \leq \frac{c_0}{2c_1} N_i \right\}$$

and

$$B_{N_i} := G_p \setminus A_{N_i},$$

where

$$N_i := \frac{p^n}{n^{3i}}, \quad i = 1, 2, \dots, \left\lfloor \frac{n}{3 \log_2 n} \right\rfloor, \quad n \geq 2.$$

By (5) and from the fact that $|K_{N_i}^\psi(\tau_n(x))| = O(N_i)$ we can write

$$\begin{aligned} c_0 N^3 &\leq \int_{G_p} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^2 d\mu(x) = \int_{A_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^2 d\mu(x) \\ &\quad + \int_{B_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^2 d\mu(x) \leq \frac{c_0}{2c_1} N_i^3 \int_{A_{N_i}} |K_{N_i}^\psi(\tau_n(x))| d\mu(x) \\ &\quad + \int_{B_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{3/2} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \\ &\leq \frac{c_0}{2} N_i^3 + c_2 N_i^3 \int_{B_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x), \end{aligned}$$

consequently

$$\int_{B_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \geq c_3 > 0. \quad (6)$$

Denote

$$C_{N_i} := B_{N_i} \setminus \bigcup_{j=1}^{i-1} B_{N_j}.$$

From the definition of the set B_{N_i} we obtain

$$\frac{c_0}{2c_1} N_i \mu(B_{N_j}) < \int_{B_{N_i}} |K_{N_i}^\psi(\tau_n(x))| d\mu(x) \leq \int_{G_p} |K_{N_i}^\psi(\tau_n(x))| d\mu(x) \leq c_1,$$

hence

$$\mu(B_{N_j}) \leq \frac{c_4}{N_i}. \quad (7)$$

Combining (6) and (7) we get

$$\begin{aligned} \int_{C_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) &\geq \int_{B_{N_i}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \\ &- \sum_{j=1}^{i-1} \int_{C_{N_j}} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \geq c_3 - c_4 N_i \sum_{j=1}^{i-1} \text{mes}(C_{N_j}) \\ &\geq c_3 - c_5 N_i \sum_{j=1}^{i-1} \frac{1}{N_j} \geq c_3 - \frac{c_6}{n^3} \geq c_7, \quad \text{for } n \geq n_0. \end{aligned}$$

Consequently we can write

$$\begin{aligned} &\int_{G_p} \max_{1 \leq N \leq 2^n} \left(N |K_N^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \\ &\geq \sum_{j=1}^{\lfloor n/(3 \log n) \rfloor} \int_{C_{N_i}} \max_{1 \leq N \leq 2^n} \left(N |K_N^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \\ &\geq \sum_{j=1}^{\lfloor n/(3 \log n) \rfloor} \left(N_i |K_{N_i}^\psi(\tau_n(x))| \right)^{1/2} d\mu(x) \geq c_8 \frac{n}{\log n}, \end{aligned}$$

which completes the proof of Lemma 5. $\square$

5. Proofs of the main results

PROOF OF THEOREM 1. Let a be an arbitrary atom with support I and $\mu(I) = p^{-N}$. We may assume that $I = I_N$. It is easy to see that $\sigma_n(a) = 0$ if $n \leq p^N$. Therefore we can suppose that $n > p^N$.

From Lemma 3 and (2) we write

$$\sigma_n^\chi a(x) = \int_{G_p} a(t) K_n^\chi(x-t) d\mu(t) = \int_{I_N} a(t) K_n^\chi(x-t) d\mu(t)$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{j=N+1}^{A-1} p^j \sum_{l=1}^{p-1} \int_{I_N} a(t) r_j^l(x-t) K_{p^j}^\psi(\tau_j(x-t)) d\mu(t) \\
&\quad + \frac{p^A}{n} \sum_{l=1}^{n_A-1} \int_{I_N} a(t) r_j^l(x-t) K_{p^A}^\psi(\tau_A(x-t)) d\mu(t) \\
&\quad + \frac{1}{n} \int_{I_N} a(t) r_A^{n_A}(x-t) n^{(A-1)} K_{n^{(A-1)}}^\psi(\tau_A(x-t)) d\mu(t) \\
&= \sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x) + \sigma_n^{3,\chi} a(x). \tag{8}
\end{aligned}$$

Since $|a| \leq cp^{N/\alpha}$, we have

$$|\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)| \leq \frac{cp^{N/\alpha}}{n} \sum_{j=N+1}^A p^j \int_{I_N} |K_{p^j}^\psi(\tau_j(x-t))| d\mu(t). \tag{9}$$

Let

$$x \in I_N(x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0)$$

$$\text{for some } m = -1, 0, \dots, l-1, \quad l = 0, \dots, N-1.$$

Then using Lemma 2 $K_{p^j}^\psi(\tau_j(x-t)) \neq 0$ implies that

$$t \in I_j(0, \dots, 0, x_N, \dots, x_{j-1}), \quad m = l, \quad x_0 = \dots = x_{m-1} = 0.$$

Consequently we can write

$$\begin{aligned}
|\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)| &\leq \frac{cp^{N/\alpha}}{p^A} \sum_{j=N+1}^A p^j \frac{p^{j-l}}{p^j} \mathbf{1}_{I_N(0, \dots, 0, x_l \neq 0, 0, \dots, 0)}(x) \\
&\leq \frac{cp^{N/\alpha}}{p^l} \mathbf{1}_{I_N(0, \dots, 0, x_l \neq 0, 0, \dots, 0)}(x). \tag{10}
\end{aligned}$$

From Lemma 4 we have

$$\begin{aligned}
|\sigma_n^{3,\chi} a(x)| &\leq \frac{cp^{N/\alpha}}{p^A} \int_{I_N} (n - n_A p^A) |K_{n - n_A p^A}^\psi(\tau_A(x-t))| d\mu(t) \\
&\leq \frac{cp^{N/\alpha}}{p^A} \frac{p^A}{p^{m+l}} \leq \frac{cp^{N/\alpha}}{p^{m+l}}. \tag{11}
\end{aligned}$$

Combining (8)–(11) we get

$$\sigma^{*\chi} a(x) \leq \frac{cp^{N/\alpha}}{p^l} \mathbf{1}_{I_N(0, \dots, 0, x_l \neq 0, 0, \dots, 0)}(x) + \frac{cp^{N/\alpha}}{p^{m+l}} \tag{12}$$

for

$$x \in I_N (x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0),$$

$$m = -1, 0, \dots, l-1, \quad l = 0, \dots, N-1.$$

Now we apply Lemma 1. We may suppose that $a \in L_\infty(G_p)$ is a $1/2$ -atom with respect to $I_N (n \in \mathbb{N})$. Denote

$$I_N^{m,l} := I_N (x_0, \dots, x_m \neq 0, 0, \dots, 0, x_l \neq 0, 0, \dots, 0),$$

$$m = -1, 0, \dots, l-1, \quad l = 0, \dots, N-1.$$

Then it is evident that

$$G_p \setminus I_N = \bigcup_{l=0}^{N-1} \bigcup_{m=-1}^{l-1} \bigcup_{x_0=0}^{p-1} \cdots \bigcup_{x_{m-1}=0}^{p-1} \bigcup_{x_m=1}^{p-1} \bigcup_{x_l=1}^{p-1} I_N^{m,l}.$$

Suppose that $\rho = cp^\lambda$ for some $\lambda \in \mathbb{N}$. Then from (10) we have

$$p^{\lambda/2} \mu \left\{ x \in G_p \setminus I_N : \sup_n |\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)| > cp^\lambda \right\} = 0$$

for $\lambda > 2N - l$. Hence we can suppose that $\lambda \leq 2N - l$ and $x \in I_N (0, \dots, x_l \neq 0, 0, \dots, 0)$ for some $l = 0, \dots, N-1$. Now we get

$$\begin{aligned} p^{\lambda/2} \mu \left\{ x \in G_p \setminus I_N : \sup_n |\sigma_n^{1,\chi} a(x) + \sigma_n^{2,\chi} a(x)| > p^\lambda \right\} \\ \leq cp^{\lambda/2} \sum_{l=0}^{2N-\lambda} \sum_{x_l=1}^{p-1} \frac{1}{p^N} \leq c \frac{N - \lambda/2}{p^{N-\lambda/2}} \leq c < \infty. \end{aligned}$$

Using the estimation (11) we have

$$p^{\lambda/2} \mu \left\{ x \in G_p \setminus I_N : \sup_n |\sigma_n^{3,\chi} a(x)| > cp^\lambda \right\} = 0$$

for $\lambda > 2N - m - l$. Therefore we can suppose that $\lambda \leq 2N - m - l$. Then we obtain

$$\begin{aligned} p^{\lambda/2} \mu \left\{ x \in G_p \setminus I_N : \sup_n |\sigma_n^{3,\chi} a(x)| > cp^\lambda \right\} \\ \leq cp^{\lambda/2} \sum_{l=0}^{N-1} \sum_{m=-1}^{l-1} \sum_{x_0=0}^{p-1} \cdots \sum_{x_{m-1}=0}^{p-1} \sum_{x_m=1}^{p-1} \sum_{x_l=1}^{p-1} \mu \left\{ x \in I_N^{m,l} : \sup_n |\sigma_n^{3,\chi} a(x)| > cp^\lambda \right\} \\ \leq cp^{\lambda/2} \left\{ \sum_{l=0}^{N-\lambda/2} \sum_{m=0}^l \frac{p^m}{p^N} + \sum_{l=N-\lambda/2}^{2N-\lambda} \sum_{m=0}^{2N-\lambda-l} \frac{p^m}{p^N} \right\} \leq c < \infty. \end{aligned}$$

Theorem 1 is proved. $\square$

PROOF OF THEOREM 2. Let $n \in \mathbb{P}$ and

$$f_n(x) := D_{p^{n+1}}^X(x) - D_{p^n}^X(x) = D_{p^{n+1}}^\psi(x) - D_{p^n}^\psi(x).$$

It is evident that

$$\widehat{f}_n^X(v) = \begin{cases} 1 & \text{if } v = p^n, \dots, p^{n+1} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can write that

$$S_k^X(f_n; x) = \begin{cases} 0, & \text{if } k = 0, \dots, p^n, \\ D_k^X(x) - D_{p^n}^X(x), & \text{if } k = p^n + 1, \dots, p^{n+1} - 1, \\ f_n(x), & \text{if } k \geq p^{n+1}. \end{cases} \quad (13)$$

We have

$$\begin{aligned} f_n^{*X}(x) &= \sup_k |S_{p^k}^X(f_n; x)| = |f_n(x)|, \\ \|f_n\|_{H_\alpha} &= \|f_n^*\|_\alpha = \|D_{p^n}^X(x)\|_\alpha = p^{n(1-1/\alpha)}. \end{aligned} \quad (14)$$

Since

$$D_{k+p^n}^X(x) - D_{p^n}^X(x) = w_{p^n}(x) D_k^\psi(\tau_n(x)), \quad k = 1, 2, \dots, p^n,$$

from (13) we obtain

$$\begin{aligned} \sigma^{*X} f_n(x) &\geq \max_{1 \leq N \leq p^n} |\sigma_{p^{n+N}}^X(f_n; x)| \\ &= \max_{1 \leq N \leq p^n} \frac{1}{p^n + N} \left| \sum_{k=p^n+1}^{p^n+N} S_k^X(f_n; x) \right| \\ &\geq \frac{1}{2p^n} \max_{1 \leq N \leq p^n} \left| \sum_{k=p^n+1}^{p^n+N} (D_k^X(x) - D_{p^n}^X(x)) \right| \\ &= \frac{1}{2p^n} \max_{1 \leq N \leq p^n} \left| \sum_{k=1}^N (D_{k+p^n}^X(x) - D_{p^n}^X(x)) \right| \\ &= \frac{1}{2p^n} \max_{1 \leq N \leq p^n} \left| \sum_{k=1}^N D_k^\psi(\tau_n(x)) \right|. \end{aligned}$$

From Lemma 5 we get

$$\begin{aligned} \frac{\|\sigma^{*X} f_n\|_{1/2}}{\|f_n\|_{1/2}} &\geq \frac{1}{2p^n p^{-n}} \left(\int_{I_d} \max_{1 \leq N \leq 2^n} (N |K_N(x)|)^{1/2} d\mu(x) \right)^2 \\ &\geq c \left(\frac{n+1}{\log(n+2)} \right)^2 \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned} \quad (15)$$

Combining (14) and (15) we complete the proof of Theorem 2. $\square$

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