

Linearizable 3-webs and the Gronwall conjecture

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Abstract. In the article [11], we studied the linearizability problem for 3-webs on 2-dimensional manifolds. Presenting in [12] the elements of the theory and the results, we announced a forthcoming paper with all the computations and explicit formulas. Four years after the publication of our article V. V. GOLDBERG and V. V. Lychagin obtained similar results by a different method in [7] and [8]. However, they obtained incorrect results on the linearizability of a certain web, as it was shown in [16]. Now, as it was announced in [12], we present the complete version of [11] with computations and explicit formulas, because this theory shows effectiveness and correctness in concrete examples.

1. Introduction

In the article [12] published in 2001 in the journal “Nonlinear Analysis”, we studied the linearizability problem for 3-webs on a 2-dimensional manifold. Using the integrability theory of over-determined partial differential systems, we computed the obstructions to linearizability and we produced an effective method to test the linearizability of 3-webs in the (real or complex) plane. We showed that, in the non-parallelizable case, there exists an algebraic submanifold $\mathcal{A}$ of the space of vector valued symmetric tensors ($S^2T^* \otimes T$), which can be expressed in terms of the curvature of the Chern connection and its covariant derivatives

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up to order 6, such that the affine deformation tensor is a section of $S^2T^* \otimes T$ with values in $\mathcal{A}$. In particular, we proved that a web is linearizable if and only if $\mathcal{A} \neq \emptyset$, and there exists at most 15 projectively nonequivalent linearizations of a nonparallelizable 3-web. In order to give a coordinate free and intrinsic presentation of the results we used tensors and covariant derivatives to find the obstructions to the linearization.

Four years after the publication of our article, V. V. GOLDBERG and V. V. LYCHAGIN in [7] and [8] obtained similar results by a different method. They criticized our article by qualifying the proofs incomplete, without giving any justification or reason for their claim. They claim that “...the main and only example of a linearizable (in their approach) 3-web ... is not linearizable at all ...” (see [8], pp. 171). In order to prove their statement they apply their theory to this particular web and find that the corresponding algebraic submanifold is empty. However, in the article [16] Z. MUZSNAY proves that, in complete accordance with the claim of [12], this particular web is indeed linearizable, by producing explicit linearizations. This shows that some of the calculations in [7] and [8] must be incorrect.

As it was announced in [12], we present here the detailed version of the theory with computations and explicit formulas, because this theory shows effectiveness in concrete examples, and we deem that the opinion of V. V. GOLDBERG and V. V. LYCHAGIN in [8] concerning our work is unjustified.

2. Introduction to the linearizability problem of 3-webs

Let M be a two-dimensional real or complex differentiable manifold. A 3-web is given in an open domain D of M by three foliations of smooth curves in general position. Two webs $\mathcal{W}$ and $\tilde{\mathcal{W}}$ are locally equivalent at $p \in M$, if there exists a local diffeomorphism on a neighborhood of p which exchanges them.

A 3-web is called *linear* (resp. *parallel*) if it is given by 3 foliations of straight lines (resp. of parallel lines). A 3-web which equivalent to a linear (resp. parallel) web is called linearizable (resp. parallelizable).

A linear connection, called Chern connection and denoted by ∇ , can be associated to a 3-web. ∇ preserves the web, i.e. the leaves are auto-parallel curves. It is not difficult to see that a 3-web is parallelizable if and only if the curvature of the Chern connection, called also Blaschke curvature, vanishes.

Basic examples of planar 3-webs comes from complex projective algebraic geometry. If $C \subset \mathbb{P}^2$ is a reduced algebraic curve of degree 3, by duality in $\check{\mathbb{P}}^2$,

one can obtain a 3-web called the algebraic web associated with $C \subset \mathbb{P}^2$ (cf. for instance [15]). Graf and Sauer proved in 1924 a theorem, which in web geometry language, can be stated as follows: a linear web is parallelizable if and only if it is associated to an algebraic curve of degree 3, i.e. its leaves are tangent lines to an algebraic curve of degree 3 ([2], page 24). This theorem is a special case of Niels Henrik Abel's theorem and its converse, the general LIE–DARBOUX–GRIFFITHS theorem [9].

The problem to give linearizability criterion is a very natural one. Such criterion is important in nomography (cf. [13]): determining whether some nomogram can be reduced to an alignment chart is equivalent to the problem of determining whether a web is linearizable. The most significant works on this subject are due to BOL ([3], [4]). In [3] he suggested how to find a criterion of linearizability, although he is unable to carry out the computation, which really need the use of computer. He shows that the number of projectively different linear 3-webs in the plane to which a non-hexagonal 3-web is equivalent is finite and less than 17. Bol's proof consists in to associate to a real 3-web two complex vector fields which play an essential role, so his proof cannot be translated in the complex case. In our computation the web can be real as well as complex.

The formulation of the linearizability problem in terms of Chern connection was suggested by AKIVIS in a lecture given in Moscow in 1973. Following AKIVIS idea GOLDBERG in [6] found all affine connection Γ^* relative to which the web leaves are geodesic lines and distinguished a linearizable 3-webs by claiming that the connection Γ^* is flat. In this paper we are using this approach to solve the problem.

Denoting by T and T^* the tangent and the cotangent bundle of M , a section L of the bundle $S^2T^* \otimes T$ on M is called *pre-linearization*, if the connection ∇^L defined by

$$\nabla_X^L Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the three families of leaves are auto-parallel curves with respect to ∇^L . A pre-linearization L is called *linearization* if the connection ∇^L is flat i.e. the curvature of the connection ∇^L given by equation (1) vanishes. This equation gives us a first order partial differential system on L . Two linearizations L and L' are projectively equivalent if the connections ∇^L and $\nabla^{L'}$ are projectively related. The equivalence classes are called classes of linearizations. They are in one-to-one correspondence with the *bases of the linearization* which is a simple projective invariant, noted by s . The linearizability condition can be reformulated with this object by a second order partial differential system. We show that the system is of finite type, and the obstruction to the linearizability

can be expressed in terms of polynomials of s , whose coefficients depends only on the curvature tensor of the Chern connection. Our main result is the following:

Theorem 2.1. *Let $\mathcal{W}$ be an analytical 3-web on a 2-dimensional real or complex manifold M , whose Chern curvature does not vanish at $p \in M$. Then, there exists an algebraic sub-manifold $\mathcal{A}$ of E over a neighborhood of p , expressed in terms of the curvature of the Chern connection and its covariant derivatives up to order 6, so that the linearizations of $\mathcal{W}$ are sections of E with values in $\mathcal{A}$. In particular:*

- (1) *The web is linearizable if and only if $\mathcal{A} \neq \emptyset$;*
- (2) *There exists at most 15 classes of linearizations.*

The explicit expression of the polynomials and its coefficients which define $\mathcal{A}$ can be found in Chapter 6 and 7.

3. Notations and definitions

Let $\mathcal{W}$ be a differential 3-web on a manifold M given by a triplet of mutually transversal foliations $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. From the definitions it follows that M is even dimensional and that the dimension of the tangent distributions of the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ is the half of the dimension of M . The foliations $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$ are called horizontal, vertical and transversal and their tangent space are denoted by T^h, T^v and T^t .

The following theorem proved by NAGY [17] gives an elegant infinitesimal characterization of 3-webs and their Chern connection.

Theorem. *A 3-web is equivalent to a pair $\{h, j\}$ of $(1, 1)$ -tensor fields on the manifold, satisfying the following conditions:*

- (1) $h^2 = h, j^2 = id$,
- (2) $jh = vj$, where $v = id - h$,
- (3) $\text{Ker } h, \text{Im } h$ and $\text{Ker}(h + id)$ are integrable distributions.

For any 3-web, there exists a unique linear connection ∇ on M which satisfies

- (1) $\nabla h = 0$,
- (2) $\nabla j = 0$,
- (3) $T(hX, vY) = 0, \forall X, Y \in TM$, T being the torsion tensor of ∇ .

∇ is called Chern connection.

In the sequel, we suppose that the dimension of M is two.

Definition 3.1. Let $\mathcal{W}$ be a 3-web and ∇ its Chern connection. A symmetrical $(1, 2)$ -tensor field L is called pre-linearization if the connection

$$\nabla_X^L Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the leaves are auto-parallel curves with respect to ∇^L . A pre-linearization is a linearization if the connection ∇^L is flat i.e. its curvature vanishes. Two pre-linearizations L and L' are projectively equivalent if the connections ∇^L and $\nabla^{L'}$ are projectively related, that is there exists $\omega \in \Lambda^1(M)$, such that

$$\nabla_X^L Y = \nabla_X^{L'} Y + \omega(X)Y + \omega(Y)X$$

Proposition 3.2. A tensor field L in $S^2 T^* \otimes T$ is a linearization if and only if

- 1) $vL(hX, hY) = 0$,
- 2) $hL(vX, vY) = 0$,
- 3) $L(hX, hY) + jL(jhX, jhY) - hL(jhX, hY) - hL(hX, jhY) - jvL(jhX, hY) - jvL(hX, jhY) = 0$,
- 4) $\nabla_X L(Y, Z) - \nabla_Y L(X, Z) + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y)Z = 0$

holds, for any $X, Y, Z \in T$, where R denotes the curvature of the Chern connection.

The proof is a straightforward verification. Properties 1), 2) and 3) means that L is a pre-linearization and follows from the fact that ∇^L preserves the web, while properties 4) expresses, that the curvature of ∇^L vanishes.

Definition 3.3. Let M be a 2-dimensional manifold, $\mathcal{W}$ a web on M and $\{e_1, e_2\}$ a frame at $p \in M$ adapted to the web, i.e. $e_1 \in T_p^h$, $e_2 = je_1 \in T_p^v$. Let L be a pre-linearization at p , whose components are L_{ij}^k , that is: $L(e_i, e_j) = L_{ij}^k e_k$, and let us set the tensor-field s represented by the components $2L_{12}^1 - L_{22}^2$. The tensor s will be called the base of L .

The following proposition is elementary, but it is the key for the proof of our main theorem.

Proposition 3.4. Two pre-linearizations L and L' are projectively equivalent if and only if they have the same base, i.e. $s = s'$.

Indeed, if L and L' are two projectively equivalent pre-linearizations, then there exists $\omega \in T^*$ such that $L' = L + \omega \odot id$, i.e. in the frame $\{e_1, e_2\}$:

$$L'_{11}^1 = L_{11}^1 + 2\omega_1, \quad L'_{22}^2 = L_{22}^2 + 2\omega_2, \quad L'_{12}^1 = L_{12}^1 + \omega_2$$

where ω_1 and ω_2 are the components of ω . This system is consistent if and only if $L'^1_{12} - L^1_{12} = \frac{1}{2}(L'^2_{22} - L^2_{22})$, i.e. $s = s'$.

4. The linearization operator

Let M be a 2-dimensional real or complex manifold and $\mathcal{W}$ a 3-web on M . $\Lambda^k T^*$ and $S^k T^*$ are the bundles of the k -skew-symmetric and symmetric forms. If $B \rightarrow M$ is a vector bundle on M , then $\text{Sec}(B)$ will denote the sheaf of the sections of B and $J_k(B)$ the vector bundle of k -jets of the sections of B . The k -th order jet of a section $s \in \text{Sec}(B)$ at a point $p \in M$ will be denoted by $j_{k,p}s$.

In the sequel E will denote the bundle of the pre-linearizations and $F := \Lambda^2 T^* \otimes T$. In order to study the linearizability of $\mathcal{W}$, we will consider the differential operator $P_1 : E \rightarrow F$ and study the integrability of the differential system $P_1(L) = 0$, where

$$\begin{aligned} (P_1(L))(X, Y, Z) &= (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) \\ &\quad + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y)Z \end{aligned} \quad (1)$$

for every $X, Y, Z \in T$.

We will use the theory of the formal integrability of Spencer ([5], [10]). The notations are those of [10], where is given also an accessible introduction to this theory. In particular, if P is a quasi-linear operator of order k and $p \in M$, then $R_{k,p}$ is the bundle of the formal solutions of order k at p , $\sigma_{k+\ell}(P)$ or simply $\sigma_{k+\ell}$ is the symbol of the ℓ -th order prolongation $p_\ell(P)$ of P . We also denote $g_{k+\ell} = \text{Ker } \sigma_{k+\ell}$ and $K = \text{Coker } \sigma_{k+1}$.

Let $L \in E$ a pre-linearization. We introduce the tensors

$$x, y, z : T^h \otimes T^h \rightarrow T^h$$

defined by

$$\begin{cases} x(hX, hY) := L(hX, hY) \\ y(hX, hY) := jL(jhX, jhY) \\ z(hX, hY) := hL(hX, jhY) \end{cases} \quad (2)$$

One denotes x^2 the (1,3) tensor defined by

$$x^2(hX, hY, hZ) := x(x(hX, hY), hZ).$$

Similarly, we define the product xy , x^3 (which is a $(1, 4)$ tensor field), etc.

The space of pre-linearizations, E is a 3-dimensional vector bundle over M , and x , y , z can be used to parameterize it. However, taking into account some symmetries of the problem and the Proposition 3.4, it is better to introduce the tensors $s, t : T^h \oplus T^h \rightarrow T^h$ defined by

$$\begin{aligned} s &:= 2z - y \\ t &:= \frac{1}{2}(x + y - 2z) \end{aligned} \quad (3)$$

and parameterize E by s, t, z where s is the base of the web (see Definition 3.3).

In order to simplify the notation, we denote by C_1 and C_2 the tensor fields $(\otimes^{p+1} T^{h*}) \otimes T^h$ defined by

$$\begin{aligned} C_1(hX, hX_1, \dots, hX_p) &= (\nabla_{hX} C)(hX_1, \dots, hX_p) \\ C_2(hX, hX_1, \dots, hX_p) &= (\nabla_{jhX} C)(hX_1, \dots, hX_p) \end{aligned} \quad (4)$$

where C is a tensor field in $(\otimes^p T^{h*}) \otimes T^h$. By recursion, we introduce the successive covariant derivatives with the convention that $C_{i_1 i_2} := (C_{i_2})_{i_1}$. Thus, $x_{i_1, \dots, i_p}$ is the $(1, p+2)$ tensor defined in an adapted frame by

$$x_{i_1, \dots, i_p}(\underbrace{e_1, \dots, e_1}_{p \text{ times}}, hX, hY) = (\underbrace{\nabla \nabla \dots \nabla}_{p \text{ times}} x)(e_{i_1}, \dots, e_{i_p}, hX, hY).$$

We denote $\mathcal{R}$ the tensor $\mathcal{R} : T^h \oplus T^h \oplus T^h \rightarrow T^h$ defined by

$$\mathcal{R}(hX, hY)hZ = R(jhX, hY)hZ \quad (5)$$

where R is the curvature of the Chern connection. With the above notation we have

$$\begin{aligned} (\nabla_i \nabla_j L_{i_1, \dots, i_m}^l) - (\nabla_j \nabla_i L_{i_1, \dots, i_m}^l) &= R_{ijk}^l L_{i_1, \dots, i_m}^k - R_{iji_1}^k L_{k, \dots, i_m}^l - \dots \\ &\quad \dots - R_{ij i_m}^k L_{i_1, \dots, k}^l. \end{aligned}$$

In particular

$$C_{12} - C_{21} = (p-1) \mathcal{R} C \quad (6)$$

for a tensor field $C \in (\otimes^p T^{h*}) \otimes T^h$.

Using these notations, and resolving two equations in z_1 and t_2 the system $P_1(L) = 0$ can be written as:

$$\begin{cases} t_1 = st + t^2, \\ t_2 = \frac{1}{3}s_1 - \frac{2}{3}s_2 + zt - \frac{1}{3}\mathcal{R}, \\ z_1 = \frac{2}{3}s_1 - \frac{1}{3}s_2 + zt + \frac{1}{3}\mathcal{R}, \\ z_2 = -zs + z^2. \end{cases} \quad (7)$$

Note that P_1 is regular because the symbol and its prolongation are regular maps. The system (7) can be seen as a Frobenius system on the variables t and z , and s being a parameter. By the formula (6), the integrability conditions are

$$\begin{cases} z_{12} - z_{21} = \mathcal{R}z, \\ t_{12} - t_{21} = \mathcal{R}t, \\ s_{12} - s_{21} = \mathcal{R}s, \end{cases}$$

and thus from (7) we can arrive at the system

$$P_2 = \begin{cases} s_{22} = 2s_{21} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2, \\ s_{11} = 2s_{21} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1, \end{cases} \quad (8)$$

(see also [14], equation “(* bis)”). The operator $P_2 : \text{Sec}(E_2) \rightarrow F_2$ is a quasi-linear second order differential operator, where $E_2 = T^{h*} \otimes T^{h*} \otimes T^h$, and $F_2 := F' \oplus F'$ with $F' := T^{h*} \otimes T^{h*} \otimes E_2$. The linearizability of the web is equivalent to the integrability of the operator P_2 . In the sequel we will consider this one and examine its integrability.

Proposition 4.1. *At every $p \in M$ all 2^{nd} -order solution at p of P_2 can be lifted into a 3^{rd} -order solution.*

Indeed, fixing an adapted base $\{e_1, e_2 = je_1\}$, the symbol of P_2 is the map

$$\sigma_2 : S^2T \otimes E_2 \rightarrow F_2, \quad \sigma_2(A) = (A_{22} - 2A_{21}, A_{11} - 2A_{21}),$$

where $A_{ij} = A(e_i, e_j)$. So $g_2 := \text{Ker } \sigma_2$ is defined by the equations

$$A_{22} - 2A_{21} = 0 \quad \text{and} \quad A_{11} - 2A_{21} = 0.$$

Since these equations are independent, we have $\text{rank } \sigma_2 = 2$, $\dim g_2 = 1$. On the other hand, for the symbol of the first prolongation

$$\sigma_3 : S^3T^* \otimes E_2 \rightarrow T^* \otimes F_2, \quad \sigma_3(B) = (B_{k22} - 2B_{k21}, B_{k11} - 2B_{k21}) \quad (9)$$

where $B_{ijk} = B(e_i, e_j, e_k)$, we find that $g_3 = \text{Ker } \sigma_3$ is defined by the equations

$$B_{k22} - 2B_{k21} = 0, \quad B_{k11} - 2B_{k21} = 0,$$

$k = 1, 2$. It is easy to verify that these equations are also independent. Therefore $\text{rank } \sigma_3 = 4 = \dim(T^* \otimes F_2)$ and $\dim g_3 = 0$, thus σ_3 is an isomorphism and $\text{Coker } \sigma_3 = 0$. We have the following exact diagram:

$$\begin{array}{ccccc} S^2 T^* \otimes E_2 & \xrightarrow{\sigma_3} & T^* \otimes F & \longrightarrow & \text{Coker } \sigma_3 = 0 \\ \varepsilon \downarrow & & \varepsilon \downarrow & & \\ R_3 & \longrightarrow & J_3(E_2) & \xrightarrow{p_1(P_2)} & J_1 F \\ \bar{\pi}_3 \downarrow & & \pi_3 \downarrow & & \pi_1 \downarrow \\ R_2 & \longrightarrow & J_2(E_2) & \xrightarrow{p_0(P_2)} & F \end{array}$$

Now, using homological algebraic argument it can be shown, that $\bar{\pi}_3$ is onto, i.e. every 2^{nd} -order solution of P_2 can be lifted into a 3^{rd} -order solution. $\square$

Proposition 4.2. *The operator P_2 is not 2-acyclic, i.e. there is a higher order obstruction which arises for the integrability of P_2 .*

Indeed, the sequence

$$0 \longrightarrow g_{\ell+1}(P_2) \xrightarrow{i} g_\ell(P_2) \otimes T^* \xrightarrow{\delta_\ell(P_2)} g_{\ell-1}(P_2) \otimes \Lambda^2 T^* \longrightarrow 0$$

is not exact for all $\ell \geq 2$, where δ_ℓ denotes the skew-symmetrization in the corresponding variables: for $\ell = 3$ we have

$$\text{rank } \delta_3 = 0 < \dim(g_2 \otimes \Lambda^2 T^*) = 1. \quad \square$$

5. The first obstruction obstructions to linearizability

In order to find the higher order obstruction we consider the prolongation of P_2 , i.e. the operator $P_3 := (P_2, \nabla P_2)$, where $\nabla P_2 : T^* \otimes E_2 \longrightarrow T^* \otimes F_2$ is the covariant derivative of P_2 with respect to the Chern connection. Explicitly, this

system is formed by the system (8) and by the following four equations:

$$\left\{ \begin{array}{l} s_{212} = ss_{21} - \frac{1}{3}s_1s_2 + \frac{4}{3}s_2^2 - \frac{2}{3}s_1^2 + \frac{4}{3}\mathcal{R}s_2 + 2s^2s_1 \\ \quad + \mathcal{R}s^2 + (2\mathcal{R}_2 - \mathcal{R}_1)s - \frac{2}{3}\mathcal{R}_{21} - \frac{1}{3}\mathcal{R}_{12} \\ s_{211} = -ss_{21} + \frac{1}{3}s_1s_2 + \frac{2}{3}s_2^2 - \frac{4}{3}s_1^2 + \left(\frac{5}{3}\mathcal{R} + 2s^2\right)s_2 \\ \quad - 10\mathcal{R}s_1 + (\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{1}{3}\mathcal{R}_{21} - \frac{2}{3}\mathcal{R}_{12} \\ s_{111} = -2ss_{21} - \frac{4}{3}s_1s_2 + \frac{4}{3}s_2^2 - \frac{5}{3}s_1^2 + \left(\frac{10}{3}\mathcal{R} + 2s^2\right)s_2 - \left(\frac{5}{3}\mathcal{R} - s^2\right)s_1 \\ \quad - \mathcal{R}s^2 + (2\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{2}{3}\mathcal{R}_{21} - \frac{4}{3}\mathcal{R}_{12} + \mathcal{R}_{11} \\ s_{222} = 2ss_{21} + \frac{4}{3}s_1s_2 + \frac{5}{3}s_2^2 - \frac{4}{3}s_1^2 + \left(\frac{5}{3}\mathcal{R} + s^2\right)s_2 - \left(\frac{10}{3}\mathcal{R} - 2s^2\right)s_1 \\ \quad + \mathcal{R}s^2 + (2\mathcal{R}_2 - 2\mathcal{R}_1)s - \frac{4}{3}\mathcal{R}_{21} - \frac{2}{3}\mathcal{R}_{12} + \mathcal{R}_{22} \end{array} \right. \quad (10)$$

Since (10) can be solved with respect to the 3^{rd} -order derivatives, the existence of a 2^{nd} -order formal solution implies the existence of 3^{rd} -order solutions.

In the sequel, we will use the notion of involutivity of a differential system in the sense used in monograph [5], p. 121 (cf. the discussion of this notion on page 2)¹. We have then

Lemma 5.1. *The symbol of P_3 is involutive. Moreover, any 3^{rd} -order solution of P_3 can be lifted into a 4^{th} -order solution if and only if $\varphi \equiv 0$, where*

$$\begin{aligned} \varphi(s) = & -24\mathcal{R}s_{21} - (24\mathcal{R}s + 12\mathcal{R}_1 - 6\mathcal{R}_2)s_1 + (24\mathcal{R}s + 6\mathcal{R}_1 - 8\mathcal{R}_2)s_2 \\ & + 3\mathcal{R}s^3 + (-4\mathcal{R}_2 - 3\mathcal{R}_{22} + \mathcal{R}_{21} + 2\mathcal{R}_{12} - 13\mathcal{R}^2 - 3\mathcal{R}_{11})s \\ & + 2\mathcal{R}_{122} - \mathcal{R}_{221} - \mathcal{R}_{112} - 5\mathcal{R}\mathcal{R}_1 - 2\mathcal{R}_{121} - 11\mathcal{R}\mathcal{R}_2 \end{aligned} \quad (11)$$

PROOF. The symbol of P_3 is just $\sigma_3(P_3) : S^3T^* \otimes E_2 \longrightarrow T^* \otimes F_2$ introduced in (9). On the other hand, $\sigma_4(P_3) : S^4T^* \otimes E_2 \longrightarrow S^2T^* \otimes F_2$ and $g_4 := \text{Ker } \sigma_4$ is defined by the equations

$$D_{ij}^1 := C_{ij22} - 2C_{ij21} = 0, \quad D_{ij}^2 := C_{ij11} - 2C_{ij21} = 0, \quad i, j = 1, 2.$$

¹Sometimes there is a confusion between different terminologies. The involutivity here (and also in the mentioned [5] and [10]) means the involutivity of the symbol i.e. that the Cartan's test for involutivity holds. It does not mean the integrability, which is the case in some another terminologies.

There is one relation between these equations: $D_{11}^1 - 2D_{12}^1 - D_{22}^2 + 2D_{12}^2 = 0$. Therefore the rank of this system is 5, so if K_2 denotes the cokernel of σ_4 i.e. $K_2 = (S^2T^* \otimes F_2) / \text{Im } \sigma_4$, then $\dim K_2 = 1$. If we define a map $\tau : S^2T^* \otimes F_2 \rightarrow \mathbb{C}$ by $\tau(D) = D_{11}^1 - 2D_{12}^1 + 2D_{12}^2 - D_{22}^2$, then, the sequence

$$0 \longrightarrow S^4T^* \otimes E_2 \xrightarrow{\sigma_4} S^2T^* \otimes F_2 \xrightarrow{\tau} K_2 \longrightarrow 0$$

is exact. We can deduce that the obstruction to the integrability of P_3 is given by $\varphi_p = 0$, where $\varphi : R_3 \rightarrow K_2$ is defined by

$$\varphi(s) = [\nabla(P_3(s))]_{11}^1 - 2[\nabla(P_3(s))]_{12}^1 + 2[\nabla(P_3(s))]_{12}^2 - [\nabla(P_3(s))]_{22}^2.$$

Using the equations (8) and (10), we obtain

$$\begin{aligned} \varphi(s) = & \nabla_{11}[2s_{21} - s_{22} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2] \\ & - 2\nabla_{12}[2s_{21} - s_{22} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2] \\ & + 2\nabla_{12}[2s_{21} - s_{11} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1] \\ & - \nabla_{22}[2s_{21} - s_{11} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1] \end{aligned}$$

By the formula (6) we can eliminate the 4^{th} -order derivatives and find (11). Moreover, we can remark that $\dim g_{3,p} = 0$ and therefore $\dim g_{k,p} = 0$ for every $k > 3$. It follows that P_3 is involutive. $\square$

Remark. If $\mathcal{R} = 0$, then $\varphi = 0$, therefore, all 3^{rd} -order solution of P_3 can be lifted into a 4^{th} -solution. Since P_3 is involutive P_3 is formally integrable and consequently, it is integrable in the analytical case. We have the following result:

Corollary 5.2. *If $\mathcal{W}$ is a parallelizable 3-web on the plane, then for all $L_0 \in E_p$ there exists a germ of linearizations L which prolongs L_0 .*

In accord of the Graf-Sauer Theorem, one can deduce that for a parallelizable web, there exist non projectively equivalent linearizations. Indeed, it is sufficient to consider $L_0, L'_0 \in E_p$ with $s_p \neq s'_p$ and to prolong them in germs of linearization to obtain two non projectively equivalent germs of linearization.

6. Second obstruction to linearizability

In the sequel we will suppose that $\mathcal{R} \neq 0$. In this case the compatibility condition (11) is not satisfied, so we have to introduce into our differential system and consider the second order quasi-linear system $P_\varphi = 0$:

$$P_\varphi := (P_2, \varphi),$$

where P_2 is defined by (8) and φ is given by (11). The commutative diagram associated to P_φ is:

$$\begin{array}{ccccc}
S^3 T^* \otimes E_2 & \xrightarrow{\sigma_3(P_\varphi)} & (T^* \otimes F_2) \oplus (T^* \otimes K_2) & \xrightarrow{\tau_3} & K_3 \longrightarrow 0 \\
\downarrow \varepsilon & & \downarrow \varepsilon & & \\
R_3 \longrightarrow & J_3 E_2 & \xrightarrow{p_1(P_\varphi)} & J_1 F_2 \oplus J_1 K_2 & \\
\downarrow \bar{\pi}_3 & \downarrow \pi_3 & & \downarrow \pi_1 & \\
R_2 \longrightarrow & J_2 E_2 & \xrightarrow{p_0(P_\varphi)} & F_2 \oplus K_2 &
\end{array}$$

Lemma 6.1. *A 2^{nd} -order formal solution $j_{2,p}s$ of P_φ at $p \in M$, can be lifted into a 3^{rd} -order solution if and only if:*

$$\begin{cases} \psi_p^1 s := 24\mathcal{R}(s_2)^2 - 48\mathcal{R}s_1s_2 + \alpha(s)s_1 + \beta(s)s_2 + \gamma(s) = 0 \\ \psi_p^2 s := -24\mathcal{R}(s_1)^2 + 48\mathcal{R}s_1s_2 + \hat{\alpha}(s)s_1 + \hat{\beta}(s)s_2 + \hat{\gamma}(s) = 0. \end{cases}$$

where $\alpha, \beta, \hat{\alpha}, \hat{\beta}$ are polynomials in s of degree 2 with coefficients $\mathcal{R}$ and its derivatives up to order 2, γ and $\hat{\gamma}$ are polynomials in s of degree 3 with coefficients $\mathcal{R}$ and its derivatives up to order 4. Their explicit expressions are given in Appendix.

PROOF. The symbol of differential operator φ and its prolongation are given by

$$\begin{aligned} \sigma_2(\varphi) : S^2 T^* \otimes E_2 &\longrightarrow K_2, & \sigma_2(\varphi)(A) &= -24\mathcal{R}A_{21} \\ \sigma_3(\varphi) : S^3 T^* \otimes E_2 &\longrightarrow T^* \otimes K_2, & \sigma_3(\varphi)(B)(e_i) &= -24\mathcal{R}B_{i21}, \quad i = 1, 2 \end{aligned}$$

where $A_{21} := A(e_2, e_1)$ and $B_{i21} = B(e_i, e_2, e_1)$ are the components of the corresponding tensors with respect to the adapted basis $\{e_1, e_2\}$.

Note, that we have $g_2(P_\varphi) = g_2(P_2) \cap \text{Ker } \sigma_2(\varphi) = 0$, therefore for every $\ell > 2$ we obtain that $g_\ell(P_\varphi) = 0$, and so the symbol of P_φ is involutive.

The kernel of the symbol of the first prolongation of P_φ is $g_3(P_\varphi)$ defined by the system

$$\begin{cases} A_1^1 := B_{122} - 2B_{112} = 0, \\ A_2^1 := B_{222} - 2B_{122} = 0, \\ A_1^2 := B_{111} - 2B_{112} = 0, \\ A_2^2 := B_{112} - 2B_{122} = 0, \\ C_1 := -24\mathcal{R}B_{112} = 0, \\ C_2 := -24\mathcal{R}B_{122} = 0. \end{cases} \quad (12)$$

There are two relations in this system (12). Namely

$$24\mathcal{R}A_1^1 - 2C_1 + C_2 = 0, \quad 24\mathcal{R}A_2^2 + C_1 - 2C_2 = 0.$$

So $\text{rank } \sigma_3(P_\varphi) = 4$, and

$$\dim K_3 = \dim((T^* \otimes F_2) \oplus (T^* \otimes K_2) / \text{Im } \sigma_3) = 2.$$

Moreover, if we define

$$\begin{aligned} \tau_3 : \quad (T^* \otimes F_2) \oplus (T^* \otimes K_2) &\longrightarrow K_3 \simeq \mathbb{C}^2 \\ (A, C) &\longmapsto (D^1, D^2) \end{aligned}$$

by the components as

$$D^1 := 24\mathcal{R}A_1^1 - 2C_1 + C_2, \quad \text{and} \quad D^2 := 24\mathcal{R}A_2^2 + C_1 - 2C_2,$$

then the sequence

$$0 \longrightarrow S^3 T^* \otimes E_2 \xrightarrow{\sigma_\varphi} (T^* \otimes F_2) \oplus (T^* \otimes K_2) \xrightarrow{\tau_3} K_3 \longrightarrow 0$$

is exact. We can deduce that a 2^{nd} order solution $(j_2 s)_p$ of P_φ can be lifted into a 3^{rd} order solution if and only if $[\tau_3 \nabla(P_\varphi(s))]_p = 0$. Let

$$(\psi^1, \psi^2)_p := \tau_3 \nabla(P_\varphi(s))_p$$

We have:

$$\begin{aligned} \psi^1 &= 24\mathcal{R}[\nabla(P_2(s))]_1^1 + [\nabla(\varphi)]_2 - 2[\nabla(\varphi)]_1 \\ \psi^2 &= 24\mathcal{R}[\nabla(P_2(s))]_2^2 + [\nabla(\varphi)]_1 - 2[\nabla(\varphi)]_2 \end{aligned}$$

Using the equations $P_2(s)_p = 0$ and $\varphi(s)_p = 0$ and the permutation formula (6), we find that $\psi^1(p)$ and $\psi^2(p)$ can be written as a function of s and its derivatives at $p \in M$, up to order 3. Nevertheless, using the formula (6) we can also eliminate the 3^{th} order derivatives of s at p . On the other hand, with the help of the equation $P_2 = 0$ and $\varphi = 0$ we can express the 2^{nd} order derivatives of s too with the 1^{st} order derivatives of s . The calculation carried out with MAPLE gives the formulas. $\square$

7. The linearization theorem

Since the compatibility conditions $\psi^1 = 0$ and $\psi^2 = 0$ found in the previous section are not identically satisfied, we have to introduce them into the system P_φ . We arrive at the system:

$$P_\psi = (P_2, \varphi, \psi^1, \psi^2).$$

Differentiating the equations $\psi^1 = 0$ and $\psi^2 = 0$ with respect to e_1 and e_2 we find 4 equations: $\psi_j^i = 0$, $i, j = 1, 2$, where

$$\begin{cases} \psi_1^1 = 24\mathcal{R}_1 s_2^2 + \tilde{\beta} s_{12} - 48\mathcal{R}_1 s_1 s_2 + (\alpha - 48\mathcal{R} s_2) s_{11} + \alpha_1 s_1 + \beta_1 s_2 + \gamma_1 \\ \psi_2^1 = 24\mathcal{R}_2 s_2^2 + \tilde{\beta} s_{22} - 48\mathcal{R}_2 s_1 s_2 + (\alpha - 48\mathcal{R} s_2) s_{21} + \alpha_2 s_1 + \beta_2 s_2 + \gamma_2 \\ \psi_2^2 = -24\mathcal{R}_1 s_1^2 + \tilde{\alpha} s_{11} + 48\mathcal{R}_1 s_1 s_2 + (48\mathcal{R} s_1 + \hat{\beta}) s_{12} + \hat{\alpha}_1 s_1 + \hat{\beta}_1 s_2 + \hat{\gamma}_1 \\ \psi_2^2 = -24\mathcal{R}_2 s_1^2 + \tilde{\alpha} s_{21} + 48\mathcal{R}_2 s_1 s_2 + (48\mathcal{R} s_1 + \hat{\beta}) s_{22} + \hat{\alpha}_2 s_1 + \hat{\beta}_2 s_2 + \hat{\gamma}_2 \end{cases}$$

In this expression, we can eliminate the second order derivatives using the equation $P_2 = 0$ and $\varphi = 0$, and with the help of the equation $\psi^1 = 0$ and $\psi^2 = 0$, we can express the terms s_1^2 and s_2^2 as a function of s_1 , s_2 and the product $s_1 s_2$. Therefore the system

$$P_\psi = 0, \quad \nabla P_{\psi^1} = 0, \quad \nabla P_{\psi^2} = 0$$

is equivalent to the system formed by the equation $P_\psi = 0$ and the four *linear* equations in s_1 , s_2 and $s_1 s_2$:

$$\mathcal{S} = \begin{cases} a^1 s_1 + b^1 s_2 + c^1 s_1 s_2 = d^1, \\ a^2 s_1 + b^2 s_2 + c^2 s_1 s_2 = d^2, \\ a^3 s_1 + b^3 s_2 + c^3 s_1 s_2 = d^3, \\ a^4 s_1 + b^4 s_2 + c^4 s_1 s_2 = d^4, \end{cases} \quad (13)$$

where a^i , b^i , $i = 1, \dots, 4$ are polynomials in s of degree 3, whose coefficients are $\mathcal{R}$ and its derivatives up to order 3, c^1 and c^4 are polynomials in s of degree 1 with coefficients $\mathcal{R}$, $\mathcal{R}_1$ and $\mathcal{R}_2$, c^2 and c^3 can be expressed as a function of $\mathcal{R}$, $\mathcal{R}_1$ and $\mathcal{R}_2$, and d^1 , d^4 (resp. d^2 and d^3) are polynomials in s of degree 5 (resp. 4), with coefficients $\mathcal{R}$ and its derivatives up to order 5. Its explicit expressions will be given in Appendix.

The direct computation by MAPLE shows us that the determinant

$$\begin{vmatrix} a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

is identically null, so that the system $\mathcal{S}$ is compatible. On the other hand, the 3^{rd} -order minors of the system $\mathcal{S}$ are polynomials in s of degree 7 which are not identically zero. There is a open dense $\mathcal{U} \subset \mathbb{C}^2$ on which,

$$D(s) = \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \neq 0.$$

Solving on $\mathcal{U}$ the system $\mathcal{S}$ for s_1 , s_2 and $s_1 s_2$ we obtain:

$$s_1 = F(s) = \frac{A(s)}{D(s)}, \quad s_2 = G(s) = \frac{B(s)}{D(s)} \quad (14)$$

and

$$s_1 s_2 = H(s) = \frac{C(s)}{D(s)}, \quad (15)$$

where $A = A(s)$, $B = B(s)$ and $C = C(s)$ are polynomials in s of degrees 8, 8, and 11 respectively:

$$A = \begin{vmatrix} -d^1 & b^1 & c^1 \\ -d^2 & b^2 & c^2 \\ -d^3 & b^3 & c^3 \end{vmatrix}, \quad B = \begin{vmatrix} a^1 & -d^1 & c^1 \\ a^2 & -d^2 & c^2 \\ a^3 & -d^3 & c^3 \end{vmatrix}, \quad C = \begin{vmatrix} a^1 & b^1 & -d^1 \\ a^2 & b^2 & -d^2 \\ a^3 & b^3 & -d^3 \end{vmatrix}.$$

By (15) we must find $F(s) \cdot G(s) = H(s)$. Thus, the solution of s for the linearization system must be in the algebraic manifold defined by

$$Q_1(s) := AB - CD = 0. \quad (16)$$

On the other hand, the compatibility condition of the system (14) is

$$s_{12} - s_{21} = \mathcal{R}s.$$

Computing it explicitly we find that s must be in the algebraic manifold defined by

$$Q_2(s) = 0,$$

where Q_2 is polynomial in s of degree 15. Indeed, if $A(s) = \sum_{i=1}^8 A_i s^i$, $B(s) =$

$\sum_{i=1}^8 B_i s^i$, and $D(s) = \sum_{i=1}^7 D_i s^i$ where A_i , B_i and C_i are function on M , then using (14) we obtain

$$\begin{aligned} Q_2(s) = & \left(\sum_{i=1}^8 (\nabla_2 B_i) s^i \right) \left(\sum_{i=1}^7 D_i s^i \right) - \left(\sum_{i=1}^8 B_i s^i \right) \left(\sum_{i=1}^7 (\nabla_2 D_i) s^i \right) \\ & - \left(\sum_{i=1}^8 (\nabla_1 A_i) s^i \right) \left(\sum_{i=1}^7 D_i s^i \right) - \left(\sum_{i=1}^8 A_i s^i \right) \left(\sum_{i=1}^7 (\nabla_1 D_i) s^i \right) \\ & + \left(\sum_{i=1}^8 B_i s^{i-1} \right) \left(\sum_{i=1}^8 A_i s^i \right) - \left(\sum_{i=1}^8 B_i s^i \right) \left(\sum_{i=1}^8 A_i s^{i-1} \right) - \mathcal{R} s D^2. \end{aligned}$$

Moreover, we must impose that s_1 and s_2 given by (14) verify the 5 equations of P_ψ , this implies 5 polynomial equations $Q_i = 0$, $i = 3, \dots, 7$. Finally, we arrive at the conclusion that if the web is linearizable then s must be in the algebraic manifold $\mathcal{A}$, where $\mathcal{A}$ is defined by the equations $Q_i = 0$, $i = 1, \dots, 7$:

$$\mathcal{A} := \{Q_i = 0 \mid i = 1, \dots, 7\}.$$

So the compatibility system (therefore the linearization system) has a solution in the neighborhood of a point $p \in M$ if and only if the algebraic variety $\mathcal{A}$ is not empty. If $\mathcal{A} \neq \emptyset$, then for all smooth point $s_0 \in \mathcal{A}$, there exists a neighborhood U of s_0 so that all $s \in U$ can be prolonged in a germ $\tilde{s}$ as a basis of linearization. The explicit expression of the polynomials Q_i can be computed with the help of MAPLE. The degree of these polynomials Q_i , $i = 1 \dots 7$ are 18, 15, 23, 23, 24, 17 and 17 respectively. One obtains the following results:

Theorem 7.1. *A non-parallelizable 3-web $\mathcal{W}$ is linearizable if and only if there is an open set U of M on which the polynomials $Q_1, \dots, Q_7$ have common zeros. Moreover, if this condition is satisfied, then for all $p \in U$ and all pre-linearization $L_0 \in E_p$ whose base is in $\mathcal{A} = \{Q_i = 0 \mid i = 1, \dots, 7\}$, there exists a unique linearization L so that $L_p = L_0$.*

Since the lowest degree of the polynomials defining $\mathcal{A}$ is 15 we arrive at the

Theorem 7.2. *For a non parallelizable 3-web, there exists at most 15 projectively non equivalent linearizations.*

An old problem related to the linearizability of webs is the following conjecture:

Gronwall Conjecture (1912) [13]: *If a non-parallelizable 3-web $\mathcal{W}$ in the (real or complex) plane is linearizable, then, up to a projective transformation, there is a unique diffeomorphism which maps $\mathcal{W}$ into a linear 3-web.*

Using Theorem 7.2, the Gronwall conjecture can be expressed in the following way: for any non parallelizable 3-web in the (real or complex) plane

$$\deg\{\gcd(Q_1, \dots, Q_7)\} \leq 1,$$

where $\gcd$ denotes the greatest common divisor of the corresponding polynomials and $\deg$ is its degree.

Examples

- (1) Consider the web $\mathcal{W}$ defined by $x = cte$, $y = cte$, $f(x, y) = cte$, where $f(x, y) := (x + y)e^{-x}$. This web is not parallelizable in a neighborhood of $(0, 0)$ because the Chern connection is not flat. Indeed, the component of the curvature tensor $\mathcal{R}$ at $(0, 0)$ can be computed directly from the function f by the formula

$$\mathcal{R} = \frac{1}{f_x f_y} \left(\frac{f_{xxy}}{f_x} - \frac{f_{xyy}}{f_y} + \frac{f_{xy} f_{yy}}{f_y^2} - \frac{f_{xx} f_{xy}}{f_x^2} \right),$$

(cf. [1], p. 24). In this example we have $\mathcal{R}_{(0,0)} = -1$. The computation gives that $\text{Rad}(Q_1, \dots, Q_7) = s + 1$ on a neighborhood of $(0, 0)$. Thus the web is linearizable in a neighborhood of $(0, 0)$ and all the linearizations are projectively equivalent.

- (2) Let $\mathcal{W}$ be the web defined by $x = cte$, $y = cte$, $f(x, y) = cte$, where

$$f = \log(x) + \frac{1}{2} \log\left(\frac{x^2 + y^2}{x^2}\right) + \text{arctg}\left(\frac{y}{x}\right).$$

We have $\mathcal{R}_{(1,0)} = 2$, so $\mathcal{W}$ is not parallelizable at $(1, 0)$. On the other hand the resultant of the polynomials Q_2, Q_6 is not zero at $(1, 0)$. So this web is not linearizable at $(1, 0)$.

8. Appendix

$$\alpha = -\frac{3}{4R}(14R_2^2 - 16RR_{22} - 40R^3 - 56R_1R_2 + 40RR_{12} + 56R_1^2 - 40RR_{11}) + 30Rs^2 - 18R_2s,$$

$$\beta = 18(R_2 - R_1)s - 15Rs^2 - \frac{3}{4R}(70R_1R_2 - 44RR_{12} + 20RR_{11} + 20RR_{22} - 28R_1^2 - 28R_2^2 - 60R^3),$$

$$\gamma = -\frac{3}{4}(3R_2 - 6R_1)s^3 - \frac{3}{4R}(7R_2R_{12} + 12RR_{112} - 14R_1R_{12} - 7R_2R_{22} + 14R_1R_{11}$$

$$\begin{aligned}
& -8RR_{111} + 4RR_{222} - 47R_2R^2 + 14R_1R_{22} - 7R_{11}R_2 - 30R_1R^2 - 12RR_{122})s \\
& - \frac{3}{4R}(7R_2R_{122} - 7R_2R_{112} + 35R_1R_2R - 38R_1^2R - 2R_2^2R + 12R_{1122}R - 8R_{1112}R \\
& - 4R_{1222}R - 48R^2R_{12} + 8R^2R_{11} + 40R^2R_{22} + 8R^4 - 14R_1R_{122} + 14R_1R_{112}).
\end{aligned}$$

$$\begin{aligned}
\hat{\alpha} = & 18(R_2 - R_1)s - 15Rs^2 - \frac{3}{4R}(20RR_{22} + 70R_1R_2 - 28R_1^2 + 60R^3 - 28R_2^2 - 44RR_{21} \\
& + 20RR_{11}),
\end{aligned}$$

$$\begin{aligned}
\hat{\beta} = & 30Rs^2 + 18R_1s - \frac{3}{4R}(40R^3 - 16RR_{11} - 56R_1R_2 + 56R_2^2 + 14R_1^2 + 40RR_{21} \\
& - 40RR_{22}),
\end{aligned}$$

$$\begin{aligned}
\hat{\gamma} = & -\frac{3}{4}(3R_1 - 6R_2)s^3 - \frac{3}{4R}(12RR_{221} - 7R_1R_{22} - 14R_2R_{21} + 14R_{11}R_2 + 7R_1R_{21} \\
& - 12RR_{211} - 8RR_{222} + 14R_2R_{22} - 7R_1R_{11} + 4RR_{111} + 30R_2R^2 + 47R_1R^2)s \\
& - \frac{3}{4R}(35R_1R_2R - 7R_1R_{211} + 7R_1R_{221} + 8R_{2221}R + 4R_{2111}R - 12R_{2211}R \\
& + 8R^2R_{22} - 2R_1^2R + 40R^2R_{11} - 48R^2R_{21} - 38R_2^2R - 8R^4 + 14R_2R_{211} - 14R_2R_{221}),
\end{aligned}$$

$$\tilde{\alpha} = 48\mathcal{R}s_2 - 48\mathcal{R}s_1 + \hat{\alpha}$$

$$\tilde{\beta} = 48\mathcal{R}s_2 - 48\mathcal{R}s_1 + \hat{\beta}$$

$$\begin{aligned}
a^1 = & \frac{363}{2}R_{12} - \frac{297}{2}Rs^3 + \left(\frac{441}{4}R_2 - 72R_1\right)s^2 + \frac{1}{R^2}\left(\frac{231}{2}R_1^3 - \frac{525}{4}R_1^2R_2 + \frac{273}{4}R_1R_2^2 - \frac{63}{4}R_2^3\right) \\
& + \left(\frac{825}{4}R_2 - 102R_1\right)R + 36R_{111} + \left(\frac{57}{4}R_{22} + \frac{189}{4}R_{11} - \frac{177}{4}R_{12}\right)R_2 \\
& + \frac{1}{R}(114R_{12} - 36R_{22} - 147R_{11})R_1 + \left(\frac{1}{R}\left(\frac{381}{2}R_1^2 - \frac{1023}{4}R_1R_2 + 96R_2^2\right)\right. \\
& \left.- \frac{1305}{2}R^2 - \frac{273}{2}R_{11} - \frac{165}{2}R_{22}\right)s + 15R_{122} - 33R_{112} - 3R_{222} \\
& + \frac{1}{R^2}\left(\frac{231}{2}R_1^3 - \frac{525}{4}R_1^2R_2 + \frac{273}{4}R_1R_2^2 - \frac{63}{4}R_2^3\right),
\end{aligned}$$

$$\begin{aligned}
a^2 = & -\frac{9}{2}Rs^3 + \left(18R_2 - \frac{45}{2}R_1\right)s^2 + \left(\frac{39}{2}R^2 - 3R_{22} - \frac{39}{2}R_{12} + \frac{39}{2}R_{11}\right. \\
& \left.+ \frac{1}{R}\left(42R_1R_2 - \frac{57}{2}R_1^2 - \frac{21}{8}R_2^2\right)\right)s + \left(\frac{417}{2}R_2 - \frac{342}{2}R_1\right)R - 42R_{122} + 12R_{222} \\
& + 42R_{112} + \frac{1}{R}\left((24R_{22} - \frac{45}{2}R_{11} - \frac{87}{2}R_{12})R_1 + \left(\frac{81}{2}R_{12} - \frac{39}{2}R_{22} - \frac{15}{2}R_{11}\right)R_2\right) \\
& + \frac{1}{R^2}\left(\frac{63}{2}R_1^3 - \frac{21}{2}R_1^2R_2 - \frac{105}{8}R_1R_2^2 + \frac{21}{4}R_2^3\right),
\end{aligned}$$

$$\begin{aligned}
a^3 = & -\frac{9}{4}Rs^3 + \left(\frac{117}{4}R_2 - 18R_1\right)s^2 + \left(\frac{999}{4}R^2 - \frac{129}{4}R_{12} + \frac{39}{4}R_{22} + \frac{39}{4}R_{11}\right. \\
& \left.+ \frac{1}{R}\left(\frac{429}{8}R_1R_2 - \frac{57}{4}R_1^2 - \frac{111}{4}R_2^2\right)\right)s + \left(-\frac{429}{2}R_1 + \frac{159}{4}R_2\right)R \\
& + 48R_{112} + 6R_{222} - 30R_{122} - 18R_{111} + \frac{1}{R}\left(\left(\frac{9}{2}R_{22} + \frac{75}{2}R_{11} - \frac{141}{2}R_{12}\right)R_1\right. \\
& \left.+ \left(\frac{39}{4}R_{22} - \frac{45}{4}R_{11} + \frac{39}{4}R_{12}\right)R_2\right) + \frac{1}{R^2}\left(\frac{21}{2}R_1^2R_2 - \frac{21}{2}R_1^3 + \frac{231}{8}R_1R_2^2 - \frac{63}{4}R_2^3\right),
\end{aligned}$$

$$\begin{aligned}
b^1 = & 144Rs^3 + \left(\frac{531}{4}R_1 - 63R_2\right)s^2 + (120R^2 + 156R_{22} - 156R_{12} + 57R_{11} \\
& + \frac{1}{R}\left(\frac{465}{2}R_1R_2 - \frac{183}{4}R_1^2 - 219R_2^2\right))s + (-99R_2 + \frac{279}{4}R_1)R + 39R_{112} - 21R_{122} \\
& - 15R_{111} + \frac{1}{R}\left(\left(\frac{291}{4}R_{11} + \frac{159}{4}R_{22} - \frac{423}{4}R_{12}\right)R_1 + (60R_{12} - 75R_{11} - 27R_{22})R_2\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{R^2} \left(\frac{609}{4} R_1^2 R_2 - \frac{231}{4} R_1^3 - \frac{357}{4} R_1 R_2^2 + \frac{63}{2} R_2^3 \right), \\
b^2 &= \frac{9}{4} R s^3 + \left(\frac{117}{4} R_1 - 18 R_2 \right) s^2 + \left(\frac{741}{4} R^2 - 3 \frac{9}{4} R_{11} - 3 \frac{9}{4} R_{22} + 12 \frac{9}{4} R_{12} + \frac{1}{R} \left(\frac{111}{4} R_1^2 \right. \right. \\
& \quad \left. \left. - \frac{429}{8} R_1 R_2 + \frac{57}{4} R_2^2 \right) \right) s + \left(\frac{603}{4} R_1 + \frac{39}{2} R_2 \right) R + 48 R_{122} - 30 R_{112} - 18 R_{222} \\
& \quad + 6 R_{111} + \frac{1}{R} \left(\left(\frac{39}{4} R_{11} + 3 \frac{9}{4} R_{12} - \frac{45}{4} R_{22} \right) R_1 + \left(\frac{9}{2} R_{11} - \frac{141}{2} R_{12} + \frac{75}{2} R_{22} \right) R_2 \right) \\
& \quad + \frac{1}{R^2} \left(\frac{231}{8} R_1^2 R_2 - \frac{63}{4} R_1^3 + \frac{21}{2} R_1 R_2^2 - \frac{21}{2} R_2^3 \right), \\
b^3 &= \frac{9}{2} R s^3 + \left(18 R_1 - \frac{45}{2} R_2 \right) s^2 + \left(3 R_{11} - \frac{39}{2} R^2 s + \frac{39}{2} R_{12} - \frac{39}{2} R_{22} \right. \\
& \quad \left. + \frac{1}{R} \left(\frac{21}{8} R_1^2 - 42 R_1 R_2 + \frac{57}{2} R_2^2 \right) \right) s + \left(\frac{243}{2} R_2 + \frac{9}{2} R_1 \right) R - 42 R_{112} + 42 R_{122} \\
& \quad + 12 R_{111} + \frac{1}{R} \left(R_1 \left(\frac{81}{2} R_{12} - \frac{39}{2} R_{11} - \frac{15}{2} R_{22} \right) + \left(24 R_{11} - \frac{87}{2} R_{12} - \frac{45}{2} R_{22} \right) R_2 \right) \\
& \quad + \frac{1}{R^2} \left(\frac{21}{4} R_1^3 - \frac{105}{8} R_1^2 R_2 - \frac{21}{2} R_1 R_2^2 + \frac{63}{2} R_2^3 \right), \\
c^1 &= 234 R s + 18 R_2 + 18 R_1, \\
c^2 &= 36 R_1 - 18 R_2, \\
c^3 &= 18 R_1 - 36 R_2, \\
d^1 &= \frac{45}{8} R s^5 + \left(\frac{45}{2} R_2 - \frac{171}{8} R_1 \right) s^4 + \left(\frac{9}{2} R_{11} - \frac{9}{2} R_{22} + \frac{1}{R} \left(-\frac{99}{8} R_1^2 + \frac{63}{8} R_1 R_2 - \frac{27}{8} R_2^2 \right) \right) s^3 \\
& \quad + \left(33 R_{222} - \left(\frac{21}{4} R_2 + 411 R_1 \right) R - \frac{423}{8} R_{122} + \frac{423}{8} R_{112} - \frac{51}{2} R_{111} \right. \\
& \quad \left. + \frac{1}{R} \left(\left(-\frac{111}{2} (R_{11} + R_{22} + \frac{375}{8} (R_{22} - R_{12} + R_{11})) R_1 + \frac{111}{2} R_{12} \right) R_2 \right) \right) s^2 \\
& \quad + \left(R \left(\frac{1233}{4} R_{22} - \frac{501}{4} R_{12} + \frac{87}{4} R_{11} - \frac{567}{2} R_{21} \right) - \frac{2205}{8} R^3 - \frac{363}{2} R_1^2 + \frac{903}{8} R_1 R_2 - \frac{417}{8} R_2^2 \right. \\
& \quad \left. - \frac{69}{2} R_{1112} - 36 R_{1222} + \frac{135}{2} R_{1122} + 6 R_{1111} + \frac{1}{R} \left(\left(\frac{567}{8} (R_{112} - R_{122}) + \frac{15}{2} R_{222} \right. \right. \right. \\
& \quad \left. \left. - \frac{33}{2} R_{111} \right) R_1 + \left(63 R_{122} - 63 R_{112} - \frac{9}{2} R_{222} + 3 R_{111} \right) R_2 - \frac{129}{8} R_{11}^2 \right. \\
& \quad \left. + \left(\frac{99}{4} R_{12} - \frac{69}{4} R_{22} \right) R_{11} + \frac{39}{4} R_{12} R_{22} - \frac{9}{8} R_{22}^2 - \frac{69}{8} R_{12}^2 \right) + \frac{1}{R^2} \left((R_{22} - R_{12} + R_{11}) \frac{231}{8} R_1^2 \right. \\
& \quad \left. + \frac{147}{8} (R_{12} - R_{22} - R_{11}) R_2 R_1 + \frac{63}{8} (R_{22} - R_{12} + R_{11}) R_2^2 \right) s + \left(\frac{39}{4} R_2 - \frac{303}{8} R_1 \right) R^2 \\
& \quad + \left(-6 R_{111} + \frac{138}{8} R_{112} - \frac{135}{8} R_{122} \right) R + \left(\frac{465}{8} R_{22} + \frac{741}{8} R_{11} - \frac{549}{8} R_{12} - 63 R_{21} \right) R_1 \\
& \quad + \left(21 R_{12} - \frac{207}{4} R_{22} - \frac{231}{4} R_{11} + \frac{315}{4} R_{21} \right) R_2 + 6 R_{11112} - 9 R_{11122} + 3 R_{11222} \\
& \quad + \frac{1}{R} \left(\frac{717}{8} R_1^2 R_2 - \frac{627}{8} R_1^3 + \left(24 R_{1122} - \frac{33}{2} R_{1112} - \frac{33}{8} R_2^2 - \frac{15}{2} R_{1222} \right) R_1 + \frac{9}{4} R_2^3 \right. \\
& \quad \left. + \left(3 R_{1112} + \frac{9}{2} R_{1222} - \frac{15}{2} R_{1122} \right) R_2 + \frac{129}{8} (R_{122} - R_{112}) R_{11} + \frac{69}{8} (R_{112} - R_{122}) R_{12} \right. \\
& \quad \left. + \frac{9}{8} R_{22} (R_{122} - R_{112}) \right) + \frac{1}{R^2} \left(\frac{231 R_1^2}{8} (R_{112} - R_{122}) \right) \\
& \quad + \frac{147 R_2 R_1}{8} (R_{122} - R_{112}) + \frac{63 R_2^2}{8} (R_{112} - R_{122}), \\
d^2 &= \left(\frac{9}{8} R_1 - \frac{9}{16} R_2 \right) s^4 + \left(9 R_{12} - 9 R^2 - \frac{9}{2} R_{22} + \frac{1}{R} \left(\frac{9}{8} R_2^2 \right) - \frac{27}{8} R_1^2 - \frac{9}{16} R_1 R_2 \right) s^3 + \left(\left(\frac{261}{8} R_1 \right. \right. \\
& \quad \left. \left. + \frac{573}{16} R_2 \right) R - \frac{45}{4} R_{112} - \frac{15}{4} R_{222} + \frac{45}{4} R_{122} + \frac{15}{2} R_{111} + \frac{1}{R} \left(\frac{69 R_1}{8} (R_{12} - R_{11} - \frac{69}{8} R_{22}) \right. \right. \\
& \quad \left. \left. + \left(\frac{69}{16} R_{11} + \frac{69}{16} R_{22} - \frac{69}{16} R_{12} \right) R_2 \right) \right) s^2 + \left(\left(135 R_{12} - \frac{165}{2} R_{11} - 36 R_{22} \right) R - \frac{165}{2} R^3 \right.
\end{aligned}$$

$$\begin{aligned}
& -3\frac{45}{4}R_1^2 + \frac{261}{4}R_1R_2 + \frac{51}{4}R_2^2 - 3R_{2222} + \frac{27}{2}R_{1112} + \frac{51}{4}R_{1222} - \frac{81}{4}R_{1122} \\
& + \frac{1}{R}\left(\left(\frac{3}{4}R_{222} + \frac{27}{8}R_{122} - \frac{27}{8}R_{112} - \frac{9}{2}R_{111}\right)R_1 + \left(\frac{3}{2}R_{222} - \frac{117}{16}R_{122} + \frac{117}{16}R_{112} \right. \right. \\
& \left. \left. - \frac{3}{2}R_{111}\right)R_2 + R_{11}\left(\frac{15}{2}R_{22} - 15R_{12}\right) + 15R_{12}^2 - \frac{45}{2}R_{12}R_{22} + \frac{15}{2}R_{22}^2\right) \\
& + \frac{1}{R^2}\left(\frac{63R_1^2}{8}(R_{22} - R_{12} + R_{11}) + \frac{21R_2R_1}{16}(R_{22} - R_{12} + R_{11}) \right. \\
& \left. + \frac{21R_2^2}{8}(R_{12} - R_{11} - R_{22})\right)s + (12\frac{3}{2}R_1 - 174R_2)R^2 + (111R_{122} - 72R_{112} \\
& - 45R_{222})R + (78R_{22} - 96R_{12} + \frac{9}{2}R_{11})R_1 + (42R_{12} - \frac{75}{2}R_{11} - 45R_{22})R_2 \\
& - 9R_{11222} + 6R_{11122} + 3R_{12222} + \frac{1}{R}\left(-\frac{171}{8}R_1^3 + \frac{303}{16}R_1^2R_2 + \left(\frac{21}{4}R_{1122} - \frac{3}{4}R_{1222} \right. \right. \\
& \left. \left. - \frac{9}{2}R_{1112} + 3R_2^2\right)R_1 - \frac{3}{4}R_2^3 + \left(3R_{1122} - \frac{3}{2}R_{1222} - \frac{3}{2}R_{1112}\right)R_2 \right. \\
& \left. + 15(R_{122} - R_{112})R_{12} + \frac{15R_{22}}{2}(R_{112} - R_{122})\right) + \frac{1}{R^2}\left(\frac{63R_1^2}{8}(R_{112} - R_{122}) \right. \\
& \left. + \frac{21R_2R_1}{16}(R_{112} - R_{122}) + \frac{21R_2^2}{8}(R_{122} - R_{112})\right), \\
d^3 = & \left(\frac{9}{16}R_1 - \frac{9}{8}R_2\right)s^4 + \left(9R_{12} - 9R^2 - \frac{9}{2}R_{11} + \frac{1}{R}\left(\frac{9}{8}R_1^2 - \frac{9}{16}R_1R_2 - \frac{27}{8}R_2^2\right)\right)s^3 \\
& + \left(\left(\frac{1695}{16}R_1 - \frac{231}{8}R_2\right)R - \frac{15}{2}R_{222} + \frac{45}{4}R_{122} - \frac{45}{4}R_{112} + \frac{15}{4}R_{111} \right. \\
& \left. + \frac{1}{R}\left(\left(6\frac{9}{16}R_{12} - \frac{69}{16}R_{22} - 6\frac{9}{16}R_{11}\right)R_1 + \frac{54}{8}(R_{12} + R_{22} + R_{11})R_2\right)\right)s^2 \\
& + \left(\left(\frac{177}{2}R_{12} - \frac{237}{4}R_{11} - 36R_{22}\right)R - 36R^3 + -\frac{897}{16}R_1^2 + \frac{1041}{16}R_1R_2 + \frac{249}{8}R_2^2 - 3R_{1111} \right. \\
& - \frac{81}{4}R_{1122} + \frac{27}{2}R_{1222} + \frac{51}{4}R_{1112} + \frac{1}{R}\left(\left(\frac{117}{16}(R_{122} - R_{112}) + \frac{3}{2}(R_{111} - R_{222})\right)R_1 \right. \\
& \left. + \left(\frac{27}{8}R_{112} - \frac{9}{2}R_{222} - \frac{27}{8}R_{122} + \frac{3}{4}R_{111}\right)R_2 + \frac{15}{2}R_{11}^2 + R_{11}\left(\frac{15}{2}R_{22} - \frac{45}{2}R_{12}\right) \right. \\
& \left. + 15R_{12}^2 - 15R_{12}R_{22}\right) + \frac{1}{R^2}\left(\frac{21R_1^2}{8}(R_{12} - R_{11} - R_{22}) + \frac{21R_2R_1}{16}(R_{22} - R_{12} + R_{11}) \right. \\
& \left. + \frac{63R_2^2}{8}(R_{22} - R_{12} + R_{11})\right)s + (147R_2 + 6R_1)R^2 + R(36R_{122} - 66R_{112} + 3R_{111}) \\
& + (30R_{22} - 27R_{12} - 39R_{11})R_1 + \left(\frac{33}{2}R_{12} - 45R_{22} - \frac{15}{4}R_{11}\right)R_2 + 9R_{1122} - 6R_{1222} \\
& - 3R_{1112} + \frac{1}{R}\left(\frac{57}{8}R_1^3 + \frac{3}{16}R_1^2R_2 + \left(15R_2^2 - 3R_{1122} + \frac{3}{2}R_{1112} + \frac{3}{2}R_{1222}\right)R_1 + \frac{9}{4}R_2^3 \right. \\
& \left. + \left(\frac{9}{2}R_{1222} - \frac{21}{4}R_{1122} + \frac{3}{4}R_{1112}\right)R_2 + \frac{15R_{11}}{2}(R_{112} - R_{122}) + 15R_{12}(R_{122} - R_{112})\right) \\
& + \frac{1}{R^2}\left(\left(\frac{21}{8}R_{122} - \frac{21}{8}R_{112}\right)R_1^2 + \left(\frac{21}{16}R_{112} - \frac{21}{16}R_{122}\right)R_2R_1 + \left(-\frac{63}{8}R_{122} + \frac{63}{8}R_{112}\right)R_2^2\right)
\end{aligned}$$

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