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On the reduced height of a polynomial

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Abstract. Let P be a polynomial with real coefficients. We introduce its reduced height by the formula $\mathbb{H}(P) = \inf_{Q\in\mathbb{R}[x]-\text{monic}} H(PQ)$ and study its properties. This problem can be restated as a problem for power series vanishing at points of a finite symmetric set in $|z| < 1$ with prescribed multiplicities. Some relations of the reduced height of a polynomial to other measures of height are also given. Most of the results obtained in this paper show that the calculation of the reduced height of a polynomial is not straightforward. For example, we find that $\mathbb{H}(x^2-18x-82) = 63$ and $\mathbb{H}((x-8/5)^2) = 9216/8245$, but $\mathbb{H}(x^2 - 18x + 82) = 64.999999999999999999999999863\ldots$ which is the sum of a certain infinite series. It seems likely that this constant is a transcendental number.

1. Introduction

Let

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d(x-\alpha_1)\ldots(x-\alpha_d)$$

be a polynomial with real coefficients. If $a_d \neq 0$ then d is called the *degree* of P. There are also several measures of complexity (usually, called heights) of P which take into account the size of its coefficients. The best known and most useful are its *height* $H(P) := \max_{0\leqslant j\leqslant d}|a_j|$, its *length* $L(P) := \sum_{j=0}^{d}|a_j|$, its *Euclidean norm* $\|P\| := (\sum_{j=0}^{d}|a_j|^2)^{1/2}$, its *Mahler measure* $M(P) := |a_d|\prod_{j=1}^{d}\max\{1,|\alpha_j|\}$, etc.

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There are many diophantine applications when, for a given integer polynomial P, one needs either to find or to prove the existence of a nonzero integer polynomial G which is divisible by P and has the smallest possible height. In other words, one needs to evaluate $\min H(PQ)$ for a given $P(x) \in \mathbb{Z}[x]$, where the minimum is taken over every nonzero $Q(x) \in \mathbb{Z}[x]$. This problem is known as a special case of SIEGEL's lemma (see, for instance, [4], [11]). It is known that $\min H(PQ) \leqslant [M(P)]$, where $[\dots]$ stands for the integral part of a number. In particular, $\min H(PQ) = 1$ if $M(P) < 2$.

A similar quantity $\min \|PQ\|$, where $P(x) \in \mathbb{Z}[x]$ and where the minimum is taken over every nonzero $Q(x) \in \mathbb{Z}[x]$, was introduced and studied by FILASETA, ROBINSON and WHEELER [8].

In principle, one can study similar problems for polynomials with coefficients in an arbitrary subring of $\mathbb{C}$ (not just $\mathbb{Z}$), for example, for polynomials with real or complex coefficients. Of course, for $P(x) \in \mathbb{R}[x]$, one should allow Q to have real coefficients too, whereas, for $P(x) \in \mathbb{C}[x]$, it is natural to take Q in $\mathbb{C}[x]$. On the other hand, the normalization of the problem should be different. The requirement that Q is in $\mathbb{Z}[x]$ can be replaced by the requirement that Q is monic polynomial in $\mathbb{R}[x]$ or in $\mathbb{C}[x]$. So, for any given $P(x) \in \mathbb{R}[x]$ (or in $\mathbb{C}[x]$), we can study the quantities like

$$\inf \|PQ\|, \quad \inf L(PQ), \quad \inf H(PQ),$$

where the infimum is taken over every monic $Q(x) \in \mathbb{R}[x]$ (or in $\mathbb{C}[x]$, respectively).

The first of these three quantities can be calculated using an old result of SZEGÖ. For any $P(x) \in \mathbb{R}[x]$, SZEGÖ's theorem (see, e.g., [14]) implies that $\inf_{Q \in \mathbb{R}[x]-\text{monic}} \|PQ\| = M(P)$. This result was generalized to other L^p norms by DURAND [7]. LAWTON [10] noticed that it can be used for the practical calculation of $M(P)$ by introducing $M_n(P) := \min \|PQ\|$, where the minimum is taken over monic $Q(x) \in \mathbb{R}[x]$ of degree at most n. These minima $M_n(P)$ can be calculated without computing the roots of P. They tend to $M(P)$ as $n \to \infty$. See the papers of AMOROSO [1] and DÉGOT [5] for some further work on this problem.

Recently, in connection with the distribution of fractional parts of powers of an algebraic number the first named author introduced and started to study the second quantity $l(P) := \inf_{Q \in \mathbb{R}[x]-\text{monic}} L(PQ)$ (see [6]). We called $l(P)$ the *reduced length* of a polynomial. The reduced length was then investigated in detail by SCHINZEL [13]. In particular, he proved that, in principle, the reduced length of polynomials having no roots of modulus 1 can be calculated. SCHINZEL's results show that there is no hope that a simple formula for $l(P)$ can be found.

For example, the value of $l(2x^3 + 3x^2 + 4)$ is not known and is left as an open problem in [13].

In this paper, we shall study the third quantity, namely, the *reduced height* of $P(x) \in \mathbb{R}[x]$ defined by the formula

$$\mathbb{H}(P) := \inf_{Q \in \mathbb{R}[x] - \text{monic}} H(PQ). \tag{1}$$

We begin with the following basic properties of $\mathbb{H}(P)$:

Theorem 1. *Suppose that $P(x) \in \mathbb{R}[x]$, $c \in \mathbb{R}$, $w \in \mathbb{C}$, $k \in \mathbb{N}$. Then*

(i) $\mathbb{H}(cP) = |c|\mathbb{H}(P)$,

(ii) $\mathbb{H}((x-c)P(x)) = \mathbb{H}(P)$ *if* $|c| \leqslant 1$,

(iii) $\mathbb{H}((x-w)(x-\overline{w})P(x)) = \mathbb{H}(P)$ *if* $|w| \leqslant 1$,

(iv) $\mathbb{H}(x-c) = \max\{1, |c|-1\}$,

(v) $\mathbb{H}(\pm P(\pm x^k)) = \mathbb{H}(P(x))$.

Theorem 1 (i) shows that in the study of $\mathbb{H}(P)$ we can restrict ourselves to monic polynomials $P(x) \in \mathbb{R}[x]$. For monic $P(x) \in \mathbb{R}[x]$, we clearly have

$$1 \leqslant \mathbb{H}(P) \leqslant H(P).$$

It is evident that each monic polynomial $P(x) \in \mathbb{R}[x]$ is completely determined by the list (multiset) of its roots counted with multiplicities. Suppose $\mathcal{S}$ is such a list. Obviously, $\mathcal{S}$ must be closed under the map $z \to \overline{z}$. We shall call any such list a *symmetric set of order d* if $\mathcal{S} \subset \mathbb{C}$ satisfies $\mathcal{S} = \overline{\mathcal{S}}$ and contains d elements counted with multiplicities. For instance, $1, 2, 1+i, 1+i, 1-i, 1-i$ is a symmetric set of order 6. If $\mathcal{S}$ contains d distinct elements (so $\mathcal{S}$ itself is a set), then a corresponding polynomial P in $\mathbb{R}[x]$ is separable, i.e., P has no multiple roots.

Note that, by Theorem 1 (ii), (iii), we can restrict ourselves to the study of polynomials which have all their roots in $|z| > 1$. The next theorem shows that it is sufficient to consider separable polynomials.

Theorem 2. *Suppose $P, P_1, P_2, \cdots \in \mathbb{R}[x]$ are monic polynomials such that $\|P_N - P\| \to 0$ as $N \to \infty$. Then $\lim_{N\to\infty} \mathbb{H}(P_N) = \mathbb{H}(P)$.*

Indeed, each root α of P of multiplicity $m(\alpha) \geqslant 2$ can be replaced by $m(\alpha)$ distinct roots $\alpha, \alpha + 1/N, \ldots, \alpha + (m(\alpha)-1)/N$. For each N sufficiently large, say $N \geqslant N_0$, the polynomial P_N obtained in this way from P will be separable. The coefficients of a polynomial depend continuously on its roots. So $\|P_N - P\| \to 0$ as

$N \to \infty$. If we would know $\mathbb{H}(P_N)$ for $N \geqslant N_0$, then using Theorem 2 we would get $\mathbb{H}(P) = \lim_{N\to\infty} \mathbb{H}(P_N)$, where P is a polynomial having multiple roots.

Summarizing, we see that in evaluation of $\mathbb{H}(P)$ it is sufficient to consider monic separable polynomials $P(x) \in \mathbb{R}[x]$ whose roots all lie in $|z| > 1$. Also, if $\mathcal{S}$ is a symmetric set, we can define

$$\mathbb{H}(\mathcal{S}) := \inf H(G - x^{\deg G}), \tag{2}$$

where the infimum is taken over every nonzero monic polynomial $G(x) \in \mathbb{R}[x]$ vanishing at each $\alpha \in \mathcal{S}$ with multiplicity $\geqslant m(\alpha)$ if $\mathcal{S}$ contains $m(\alpha)$ copies of α. Of course, for $G(x) = x^n + g_{n-1}x^{n-1} + \cdots + g_0$, we have $H(G - x^{\deg G}) = \max_{0\leqslant j\leqslant n-1} |g_j|$. The problem of finding $\mathbb{H}(\mathcal{S})$ is thus the problem of finding the infimum over the heights of monic polynomials vanishing at $\mathcal{S}$ with prescribed multiplicities. If P is a monic polynomial corresponding to $\mathcal{S}$ then

$$\mathbb{H}(P) = \max\{1, \mathbb{H}(\mathcal{S})\}. \tag{3}$$

Note that (2) implies that $\mathbb{H}(\mathcal{S}) \leqslant \mathbb{H}(\mathcal{S}')$ if $\mathcal{S} \subset \mathcal{S}'$ are two symmetric sets. Combined with (3), this yields that

$$\mathbb{H}(P) \geqslant \mathbb{H}(Q) \tag{4}$$

if P, Q are two monic polynomials in $\mathbb{R}[x]$ such that $Q|P$.

If $\mathcal{S} \subset \{z \in \mathbb{C} : 0 < |z| < 1\}$ is a finite symmetric set then one can consider a power series of the form $1+\sum_{j=1}^{\infty} h_j x^j$ vanishing at the points of $\mathcal{S}$ with respective multiplicities. Let $\mathbb{H}_{\rm ser}(\mathcal{S})$ be the infimum over all $h > 0$ for which there exists a power series $1 + \sum_{j=1}^{\infty} h_j x^j$, where $h_j \in \mathbb{R}$, $|h_j| \leqslant h$, vanishing at each $\alpha \in \mathcal{S}$ with multiplicity $\geqslant m(\alpha)$. (Here, $m(\alpha)$ is the number of copies of α in $\mathcal{S}$.) Using a standard compactness argument, we shall derive the following lemma showing that the value $\mathbb{H}_{\rm ser}(\mathcal{S})$ is attained, the proof of which appears at the end of Section 5.

Lemma 3. *For any finite symmetric set $\mathcal{S} \subset \{z \in \mathbb{C} : 0 < |z| < 1\}$, there exists a series $1 + \sum_{j=1}^{\infty} h_j x^j$, where $h_j \in \mathbb{R}$, $|h_j| \leqslant \mathbb{H}_{\rm ser}(\mathcal{S})$, vanishing at each $\alpha \in \mathcal{S}$ with multiplicity $\geqslant m(\alpha)$.*

Evidently, $\mathcal{S}^{-1}$ (which contains elements reciprocal to those in $\mathcal{S}$) is a symmetric set if $\mathcal{S}$ is a symmetric set.

Theorem 4. *For any finite symmetric set $\mathcal{S}$, where $0 \notin \mathcal{S}$, we have $\mathbb{H}(\mathcal{S}) = \mathbb{H}_{\rm ser}(\mathcal{S}^*)$, where $\mathcal{S}^* := \mathcal{S}^{-1} \cap \{z \in \mathbb{C} : |z| < 1\}$.*

For example, if $\mathcal{S} = \{3, 4\}$ then $\mathcal{S}^* = \{1/4, 1/3\}$. We will show below (see (16)) that $\mathbb{H}((x-3)(x-4)) = 6$. Combined with (3) and Theorem 4 this yields $\mathbb{H}_{\rm ser}(\mathcal{S}^*) = 6$. The value 6 is attained for the power series $1-6x+6(x^2+x^3+\dots)$ vanishing at $1/4$ and $1/3$.

Although the value $\mathbb{H}_{\rm ser}(\mathcal{S})$ is attained by some coefficients of power series, this is not necessarily the case for $\mathbb{H}(\mathcal{S})$ and $\mathbb{H}(P)$. For example, taking $P(x) = x-2$ and $\mathcal{S} = \{2\}$, by Theorem 1 (iv), we have $\mathbb{H}(x-2) = \mathbb{H}(\mathcal{S}) = 1$. However, there is no monic polynomial G divisible by $x-2$ for which $H(G) = 1$. Indeed, for any $G(x) = x^n+g_{n-1}x^{n-1}+\dots+g_0$ satisfying $G(2) = 2^n+g_{n-1}2^{n-1}+\dots+g_0 = 0$, we have

$$H(G) \geqslant H(G-x^{\deg G}) = \max_{0\leqslant j\leqslant n-1} |g_j| \geqslant 2^n/(1+2+\dots+2^{n-1}) = 1+1/(2^n-1) > 1.$$

It follows that neither in (1) nor in (2) one can replace the infimum by the minimum.

Some problems for power series whose interpretation can be given in terms of $\mathbb{H}_{\rm ser}(\mathcal{S})$, where $\mathcal{S} \subset \{z \in \mathbb{C} : 0 < |z| < 1\}$, were considered by Beaucoup, Borwein, Boyd and Pinner in [2] and [3]. For instance, using (3) and Theorem 4 we can restate the main problem considered in [2] as follows: for any $d \in \mathbb{N}$, find the maximal $\kappa = \kappa(d) > 1$ for which $\mathbb{H}((x-\kappa)^d) = 1$. Another problem whose interpretation can be given in terms of the reduced height of a polynomial was considered in [3]: given $\varphi \in (0, \pi)$, find the largest $\varrho = \varrho(\varphi) > 1$ for which $\mathbb{H}((x-\varrho e^{i\varphi})(x-\varrho e^{-i\varphi})) = 1$. The results obtained in [2] and [3] combined with Theorem 4 serve as an additional motivation for the study of $\mathbb{H}(P)$.

We shall give the proofs of Theorems 1, 2, 4 in Section 6. In the next section we shall prove our main result which is based on Theorem 4. Its application to quadratic polynomials is given in Section 3 (see Section 6 for the proofs). Some practical computations and examples will be given in Section 4. Section 5 contains the proof of Lemma 3 and some auxiliary results from linear algebra which will be used in Section 6.

2. The main result

Let $P(x) = (x-\alpha_1)\dots(x-\alpha_d) \in \mathbb{R}[x]$ be a separable polynomial whose roots are all in $|z| > 1$. Put $\beta_1 := 1/\alpha_1, \dots, \beta_d := 1/\alpha_d$. Then $\mathcal{S} = \{\beta_1, \dots, \beta_d\}$ is a symmetric set with d distinct elements in $0 < |z| < 1$. We shall estimate $\mathbb{H}_{\rm ser}(\mathcal{S})$ from below.

Suppose that $J=\{k_1,\dots,k_{d-1}\}$ is a subset of $\mathbb{N}$ such that

$$D(J):=\begin{vmatrix}1 & \beta_1^{k_1} & \dots & \beta_1^{k_{d-1}}\\ \vdots & & & \\ 1 & \beta_d^{k_1} & \dots & \beta_d^{k_{d-1}}\end{vmatrix}\neq 0. \tag{5}$$

We define $S_n(J)$ by the formula

$$S_n(J):=\frac{1}{D(J)}\begin{vmatrix}\beta_1^n & \beta_1^{k_1} & \dots & \beta_1^{k_{d-2}} & \beta_1^{k_{d-1}}\\ \vdots & & & & \\ \beta_d^n & \beta_d^{k_1} & \dots & \beta_d^{k_{d-2}} & \beta_d^{k_{d-1}}\end{vmatrix}. \tag{6}$$

Clearly, from (5) and (6) we have

$$S_0(J)=1,\quad S_{k_1}(J)=\dots=S_{k_{d-1}}(J)=0. \tag{7}$$

We claim that

$$\mathbb{H}_{\mathrm{ser}}(\mathcal{S})\geqslant 1/\sum_{j\in\mathbb{N}\setminus J}|S_j(J)|. \tag{8}$$

Indeed, suppose that $1+\sum_{n=1}^{\infty}h_nx^n$ are arbitrary power series vanishing at $\mathcal{S}$. Put

$$\ell_j:=\frac{(-1)^{j-1}}{D(J)}\begin{vmatrix}\beta_1^{k_1} & \beta_1^{k_2} & \dots & \beta_1^{k_{d-2}} & \beta_1^{k_{d-1}}\\ \vdots & & & & \\ \beta_{j-1}^{k_1} & \beta_{j-1}^{k_2} & \dots & \beta_{j-1}^{k_{d-2}} & \beta_{j-1}^{k_{d-1}}\\ \beta_{j+1}^{k_1} & \beta_{j+1}^{k_2} & \dots & \beta_{j+1}^{k_{d-2}} & \beta_{j+1}^{k_{d-1}}\\ \vdots & & & & \\ \beta_d^{k_1} & \beta_d^{k_2} & \dots & \beta_d^{k_{d-2}} & \beta_d^{k_{d-1}}\end{vmatrix}.$$

By (6), we have $\ell_1\beta_1^n+\dots+\ell_d\beta_d^n=S_n(J)$. Hence, multiplying each equality $1+\sum_{n=1}^{\infty}h_n\beta_j^n=0$, where $j=1,\dots,d$, by ℓ_j and adding all d obtained equalities, we find that $S_0(J)+\sum_{n=1}^{\infty}h_nS_n(J)=0$. Using (7), we deduce that $1+\sum_{j\in\mathbb{N}\setminus J}h_jS_j(J)=0$. Hence

$$\sup_{n\in\mathbb{N}}|h_n|\geqslant\sup_{j\in\mathbb{N}\setminus J}|h_j|\geqslant 1/\sum_{j\in\mathbb{N}\setminus J}|S_j(J)|.$$

This proves (8).

Is there any chance that the inequality (8) by an appropriate choice of J becomes an equality? In order to describe some cases when this can happen we shall introduce the following notation. For each $n \in \mathbb{N}$, put

$$\begin{gathered}\delta_n = \delta_n(J) := S_n(J)/|S_n(J)| \in \{-1, 1\} \\ \text{and} \quad \phi(x) = \phi(J, x) := \sum_{j \in \mathbb{N} \setminus J}^{\infty} \delta_j x^j.\end{gathered} \tag{9}$$

Here, $\delta_n = 0$ in case $S_n(J) = 0$. Also, for each $j \in \{1, \dots, d, d+1\}$, let $D_j(J)$ denote the determinant of the matrix

$$\begin{pmatrix} \phi(\beta_1) & 1 & \beta_1^{k_1} & \dots & \beta_1^{k_{d-1}} \\ \vdots & & & & \\ \phi(\beta_d) & 1 & \beta_d^{k_1} & \dots & \beta_d^{k_{d-1}} \end{pmatrix} \tag{10}$$

whose jth column is omitted, so that $D_1(J) = D(J)$.

Using (6), (9) and (10) we have

$$D_2(J) = \begin{vmatrix} \phi(\beta_1) & \beta_1^{k_1} & \dots & \beta_1^{k_{d-1}} \\ \vdots & & & \\ \phi(\beta_d) & \beta_d^{k_1} & \dots & \beta_d^{k_{d-1}} \end{vmatrix} = \sum_{j=1}^{\infty} \delta_j S_j(J) D(J) = D(J) \sum_{j \in \mathbb{N} \setminus J}^{\infty} |S_j(J)|.$$

So if $D(J) \neq 0$ then $D_2(J) \neq 0$ and

$$D(J)/D_2(J) = 1/\sum_{j \in \mathbb{N} \setminus J}^{\infty} |S_j(J)|. \tag{11}$$

In order to show that $\mathbb{H}_{\mathrm{ser}}(\mathcal{S}) \leqslant 1/\sum_{j \in \mathbb{N} \setminus J} |S_j(J)|$ for certain $J = \{k_1, \dots, k_{d-1}\} \subset \mathbb{N}$ (which combined with (8) would imply the equality), we shall look into the series

$$1 + \sum_{j \in J} h_{k_j} x^{k_j} + \sum_{j \in \mathbb{N} \setminus J} \delta_j h_0 x^j = 1 + \sum_{j \in J} h_{k_j} x^{k_j} + h_0 \phi(x)$$

as a 'potential' candidate. By the definition of $D_j(J)$ (see (9) and (10)), we derive that the linear system

$$\begin{cases} h_0\phi(\beta_1) + h_{k_1}\beta_1^{k_1} + \dots + h_{k_{d-1}}\beta_1^{k_{d-1}} = -1, \\ h_0\phi(\beta_2) + h_{k_1}\beta_2^{k_1} + \dots + h_{k_{d-1}}\beta_2^{k_{d-1}} = -1, \\ \vdots \\ h_0\phi(\beta_d) + h_{k_1}\beta_d^{k_1} + \dots + h_{k_{d-1}}\beta_d^{k_{d-1}} = -1, \end{cases}$$

has a unique solution
$h_0 = -D_1(J)/D_2(J) = -D(J)/D_2(J)$, $h_{k_j} = (-1)^j D_{j+2}(J)/D_2(J)$, where $j = 1, \dots, d-1$. In particular, $|h_{k_j}| \leqslant |h_0|$ precisely when $|D_j(J)| \leqslant |D(J)|$ for each $j = 3, \dots, d+1$. By the definition of $\mathbb{H}_{\rm ser}(\mathcal{S})$ and (11), for $\mathcal{S} = \{\beta_1, \dots, \beta_d\} \subset \{z \in \mathbb{C} : \ 0 < |z| < 1\}$, we thus obtain that

$$\mathbb{H}_{\rm ser}(\mathcal{S}) \leqslant |h_0| = 1/\sum_{j \in \mathbb{N} \setminus J} |S_j(J)| = |D(J)/D_2(J)|$$

in case there is a $J \subset \mathbb{N}$ such that $D(J) \neq 0$ and $|D_j(J)| \leqslant |D(J)|$ for each $j = 3, \dots, d+1$. Combined with (8) this implies that

$$\mathbb{H}_{\rm ser}(\mathcal{S}) = 1/\sum_{j \in \mathbb{N} \setminus J} |S_j(J)| = |D(J)/D_2(J)| \tag{12}$$

when there is a $J \subset \mathbb{N}$ such that $|D_j(J)| \leqslant |D(J)|$ for each $j = 3, \dots, d+1$.

By Theorem 4, we have that $\mathbb{H}_{\rm ser}(\mathcal{S}) = \mathbb{H}(\{\alpha_1, \dots, \alpha_d\})$, so (3) and (12) yield the following theorem:

Theorem 5. *Suppose that $P(x) \in \mathbb{R}[x]$ is a monic separable polynomial whose roots all lie in $|z| > 1$. If $J \subset \mathbb{N}$ is such that $D(J) \neq 0$ then*

$$\mathbb{H}(P) \geqslant \frac{1}{\sum_{j \in \mathbb{N} \setminus J} |S_j(J)|} = \frac{|D(J)|}{|D_2(J)|}.$$

Furthermore, we have equality $\mathbb{H}(P) = \max\{1, |D(J)/D_2(J)|\}$ in case J is a subset of $\mathbb{N}$ such that $|D_j(J)| \leqslant |D(J)|$ for each $j = 3, \dots, d+1$.

In particular, taking $J_0 = \{1, \dots, d-1\}$, we have $D(J_0) = \prod_{1 \leqslant i < j \leqslant d} (\beta_j - \beta_i) \neq 0$. If

$$S_n = S_n(J_0) = \frac{1}{D(J_0)} \begin{vmatrix} \beta_1^n & \beta_1 & \dots & \beta_1^{d-2} & \beta_1^{d-1} \\ \vdots & & & & \\ \beta_d^n & \beta_d & \dots & \beta_d^{d-2} & \beta_d^{d-1} \end{vmatrix}, \tag{13}$$

where $n = d, d+1, \dots$, all have the same sign then $\phi(x) = \pm x^d/(1-x)$. Thus

$$D_2(J_0) = \pm \begin{vmatrix} \beta_1^d/(1-\beta_1) & \beta_1 & \dots & \beta_1^{d-1} \\ \vdots & & & \\ \beta_d^d/(1-\beta_d) & \beta_d & \dots & \beta_d^{d-1} \end{vmatrix} = \frac{\pm \beta_1 \dots \beta_d D(J_0)}{(1-\beta_1)\dots(1-\beta_d)}.$$

Hence

$$|D(J_0)/D_2(J_0)| = |(\beta_1^{-1} - 1)\dots(\beta_d^{-1} - 1)| = |(\alpha_1 - 1)\dots(\alpha_d - 1)| = |P(1)|.$$

It follows that

$$\mathbb{H}(P) \geqslant |P(1)|$$

provided that all $S_n = S_n(J_0)$, $n = d, d+1, \ldots$, have the same sign.

Put

$$\begin{aligned} R(x) &:= (x - 1/\alpha_1)\ldots(x - 1/\alpha_d) = (x - \beta_1)\ldots(x - \beta_d) \\ &= x^d + r_1 x^{d-1} + \cdots + r_d = P(1/x)x^d(-1)^d\beta_1\ldots\beta_d. \end{aligned}$$

Then $S_n = S_n(J_0)$ satisfy the linear recurrence relation

$$S_{n+d} + S_{n+d-1}r_1 + \cdots + S_n r_d = 0, \tag{14}$$

where $S_0 = 1$, $S_1 = \cdots = S_{d-1} = 0$. In Section 5, we shall prove the following lemma:

Lemma 6. *For each $j \in \{2, 3, \ldots, d+1\}$ we have*

$$\begin{vmatrix} \beta_1^d/(1-\beta_1) & 1 & \ldots & \beta_1^{j-3} & \beta_1^{j-1} & \ldots & \beta_1^{d-1} \\ \vdots & & & & & & \\ \beta_d^d/(1-\beta_d) & 1 & \ldots & \beta_d^{j-3} & \beta_d^{j-1} & \ldots & \beta_d^{d-1} \end{vmatrix} \tag{15}$$

$$= \pm\frac{|D(J_0)||r_d + r_{d-1} + \cdots + r_{d-j+2}|}{|R(1)|}.$$

If S_n, $n = d, d+1, \ldots$, all have the same sign then, by (10) and Lemma 6, we obtain that $|D_j(J_0)|/|D(J_0)| = |r_d + r_{d-1} + \cdots + r_{d-j+2}|/|R(1)|$ for $j \in \{2, 3, \ldots, d+1\}$. Combining this with Theorem 5 and using $|D(J_0)/D_2(J_0)| = |R(1)/r_d| = |P(1)|$ we derive the following corollary:

Corollary 7. *Let $P(x)$ be a separable polynomial with all roots in $|z| > 1$. Suppose $|r_d + \cdots + r_{d-j+1}| \leqslant |R(1)| = |r_d + \cdots + r_1 + 1|$ for each $j = 1, \ldots, d$, and suppose S_n, where $n = d, d+1, \ldots$, defined by (13) or (14) all have the same sign. Then $\mathbb{H}(P) = |P(1)|$.*

3. Quadratic polynomials

In this section, we give some corollaries of Theorem 5 to quadratic polynomials P. In particular, in the next two statements we compute explicitly the reduced length of a quadratic polynomial with two positive real roots. The proofs will be given in Section 6.

Corollary 8. *Let $u > v > 1$ be two real numbers, and let k be the largest positive integer for which $(1-2u^{1-k})/(u-1) \geqslant (1-2v^{1-k})/(v-1)$. Then*

$$\mathbb{H}((x-u)(x-v)) = \max\left\{1, \frac{u^k - v^k}{(u^k-2)/(u-1) - (v^k-2)/(v-1)}\right\}.$$

In particular, selecting $k = 1$ and combining this corollary with (12) (see also (3)), we obtain that

$$\mathbb{H}_{\mathrm{ser}}(\{1/v, 1/u\}) = (u-1)(v-1) \tag{16}$$

if $(1-1/v)(1-1/u) \geqslant 1/2$.

For $P(x) = (x-u)^2 \in \mathbb{R}[x]$ the result is as follows:

Corollary 9. *Let $u \geqslant 0$ be a real number. Then*

$$\mathbb{H}((x-u)^2) = \begin{cases} (u-1)^2 & \text{if } u \geqslant 2+\sqrt{2}, \\ 2u(u-1)^2/(u^2-2u+2) & \text{if } u \in [2, 2+\sqrt{2}\,], \\ 3u^2(u-1)^2/(2u^3-3u^2+2) & \text{if } u \in [\kappa_1, 2], \\ 4u^3(u-1)^2/(3u^4-4u^3+2) & \text{if } u \in [\kappa_2, \kappa_1], \\ 1 & \text{if } u \in [0, \kappa_2], \end{cases}$$

where $\kappa_1 := 1.6279\ldots$ and $\kappa_2 := 1.5405\ldots$ satisfy $\kappa_1^4 - 8\kappa_1 + 6 = 0$ and $4\kappa_2^5 - 11\kappa_2^4 + 8\kappa_2^3 - 2 = 0$, respectively.

The minimal polynomial of $1/\kappa_2$ is $2x^5 - 8x^2 + 11x - 4$. This polynomial was found in [2] with respect to the above mentioned problem: find the maximal $\kappa = \kappa(d)$ for which $\mathbb{H}((x-\kappa)^d) = 1$. We have $\kappa(2) = \kappa_2 = 1.5405\ldots$. See [2] for the minimal polynomials of $1/\kappa(3)$ and $1/\kappa(4)$. We remark that the fourth line of Corollary 9 applied to $u = 8/5 \in (\kappa_2, \kappa_1)$ yields the equality $\mathbb{H}((x-8/5)^2) = 9216/8245$ which was announced in the abstract.

Our final statement deals with quadratic polynomials having two complex conjugate roots.

Corollary 10. *Let $w = |w|e^{i\varphi}$ be a complex number. If $|w| \geqslant 2+\sqrt{2}$ then*

$$\mathbb{H}((x-w)(x-\overline{w})) = \frac{|w|^2}{1 + \sum_{j=1}^{\infty} |w|^{-j} |\sin((j+1)\varphi)/\sin(\varphi)|}. \tag{17}$$

This corollary is of interest in connection with the result of Schinzel, who proved in [13] that the reduced length $l(P)$ belongs to the field generated by the coefficients of P in the case when $P(x) \in \mathbb{R}[x]$ has all zeros outside the unit circle. It seems very likely that the value obtained at the right hand side of (17) can be transcendental for an algebraic integer w of degree 2 having a complex conjugate $\overline{w}$. In the next section we shall consider the example of $w = 9+i$ with minimal polynomial $P(x) = x^2 - 18x + 82$.

4. Practical computations

Let $P(x) \in \mathbb{R}[x]$ be a monic polynomial. Let us define

$$\mathbb{H}_n(P) := \min H(PQ), \tag{18}$$

where the minimum is taken through all monic polynomials $Q(x) \in \mathbb{R}[x]$ of degree at most n. Clearly, $\mathbb{H}_0(P) \geqslant \mathbb{H}_1(P) \geqslant \mathbb{H}_2(P) \geqslant \ldots \geqslant \mathbb{H}(P)$ and $\lim_{n\to\infty} \mathbb{H}_n(P) = \mathbb{H}(P)$.

Using the simplex method of linear programming we can calculate $\mathbb{H}_n(P)$ explicitly for small values of n, e.g., for $n = 20$.

For example, with the input $P(x) = x^2 - 18x - 82$ the output for $\mathbb{H}_{20}(P)$ is the polynomial $x^{22} - g_1x^{21} - g_2(x^{20} - x^{19} + \cdots - x + 1)$ with

$$g_1 = \frac{501374916424516058314281143579486 56}{263881534960299064058796703169185 1} = 18.999999999998023226693\ldots,$$

$$g_2 = \frac{151283283992847526085104101143692103 68}{240132196813872148293504999883958441}$$

$$= 63.000000000045005485157\ldots.$$

This suggests that

$$\mathbb{H}(x^2 - 18x - 82) = 63. \tag{19}$$

Indeed, using Theorem 1 (v) we have $\mathbb{H}(x^2 - 18x - 82) = \mathbb{H}(x^2 + 18x - 82)$. Note that $x^2 + 18x - 82 = (x - \alpha_1)(x - \alpha_2)$, where $\alpha_1 = -9 - \sqrt{163}$, $\alpha_2 = -9 + \sqrt{163}$. Setting $\beta_1 = 1/\alpha_1$ and $\beta_2 = 1/\alpha_2$, we get $R(x) = (x - \beta_1)(x - \beta_2) = x^2 - 9x/41 - 1/82$. The inequalities $1/82 \leqslant |R(1)| = 63/82$ and $1/82 + 9/41 \leqslant |R(1)| = 63/82$ of Corollary 7 hold. Moreover, by (13),

$$S_n = (\beta_1^n\beta_2 - \beta_2^n\beta_1)/(\beta_2 - \beta_1) = (\beta_2^{n-1} - \beta_1^{n-1})/2\sqrt{163}$$

are positive for every $n \geqslant 2$. Hence, by Corollary 7, we find that $\mathbb{H}(x^2+18x-82) = |1 + 18 - 82| = 63$. This proves (19).

Another example is more interesting. For $P(x) = x^2 - 18x + 82$, the output for $\mathbb{H}_{20}(P)$ is the polynomial $x^{22} - g_3x^{21} + g_4(x^{20} + \cdots + x + 1)$, where

$$g_3 = \frac{39566397351975506821506664017372395 52}{232743513835150040121292997351084209}$$

$$= 17.000000000000000000381\ldots,$$

$$g_4 = \frac{151283283992847526085104101143692103 68}{232743513835150040121292997351084209}$$
$$= 65.000000000000000002691\ldots.$$

This suggests that the limit value is 65. However, this is not true! In fact, we have

$$\mathbb{H}(x^2 - 18x + 82) = 1/\sum_{j=2}^{\infty} |S_j| = 64.999999999999999999999999863\ldots,$$

where $S_0 = 1$, $S_1 = 0$ and $S_n = 9S_{n-1}/41 - S_{n-2}/82$ for $n = 2, 3, \ldots$. Indeed, since $|w| = |9 + i| \geqslant 2 + \sqrt{2}$, the condition of Corollary 10 is satisfied. By (25) (see the proof of Corollary 10 below, where we took $J = \{1\}$), the right hand side of (17) is equal to $1/\sum_{j=2}^{\infty} |S_j|$. It seems likely that the constant $1/\sum_{j=2}^{\infty} |S_j|$ is transcendental.

Finally, suppose that $P(x) = (x + 3)(x + 2)(x + 1)x(x - 1)(x - 2)(x - 3)$. Then, by Theorem 1 (*ii*), (*v*), (3) and (16), we find that

$$\mathbb{H}(P) = \mathbb{H}((x^2 - 9)(x^2 - 4)) = \mathbb{H}((x - 9)(x - 4)) = 8 \cdot 3 = 24.$$

In other words, the minimal height of a monic polynomial with real coefficients vanishing at $-3, -2, -1, 0, 1, 2$ and 3 is equal to 24.

5. Auxiliary lemmas from linear algebra

Throughout, we shall write the linear system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_d = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_d = b_2, \\ \vdots \\ a_{d,1}x_1 + a_{d,2}x_2 + \cdots + a_{d,d}x_d = b_d \end{cases}$$

in the matrix form $A\mathbf{x} = \mathbf{b}$, where $A = \|a_{i,j}\|_{1 \leqslant i,j \leqslant d}$ is a $d \times d$ matrix, $\mathbf{x} := (x_1, \ldots, x_d)^T$, $\mathbf{b} := (b_1, \ldots, b_d)^T$. Here and below, T stands for the transpose.

Lemma 11. *Let $A = \|a_{i,j}\|_{1 \leqslant i,j \leqslant d}$ be a $d \times d$ matrix with complex entries, $b_1, \ldots, b_d \in \mathbb{C}$, and $\varepsilon > 0$. Suppose that the linear system $A\mathbf{x} = \mathbf{b}$ has at least one real solution, and that there exist $y_1, \ldots, y_d \in \mathbb{R}$ such that $|a_{i,1}y_1 + a_{i,2}y_2 + \cdots + a_{i,d}y_d - b_i| < \varepsilon$ for each $i = 1, \ldots, d$. Then there is a constant $c = c(A) > 0$ and a real vector $\mathbf{x} = (x_1, \ldots, x_d)^T$, where $x_j \in (y_j - \varepsilon c, y_j + \varepsilon c)$ for each $j = 1, \ldots, d$, such that $A\mathbf{x} = \mathbf{b}$.*

PROOF. By the condition of the lemma, there exist $\varepsilon_1, \dots, \varepsilon_d \in \mathbb{C}$ of moduli $< \varepsilon$ such that $a_{i,1}y_1 + a_{i,2}y_2 + \cdots + a_{i,d}y_d = b_i + \varepsilon_i$ for $i = 1, \dots, d$. Take any real vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{z} := (y_1 - x_1, \dots, y_d - x_d)^T$ is a real solution of $A\mathbf{z} = (\varepsilon_1, \dots, \varepsilon_d)^T$. If the matrix A is non-singular, namely, $\det A \neq 0$, then $A\mathbf{z} = (\varepsilon_1, \dots, \varepsilon_d)^T$ has a unique solution $z_j = \det A_j / \det A$ $(j = 1, 2, \dots, d)$, where A_j is the matrix A whose jth column is replaced by $(\varepsilon_1, \dots, \varepsilon_d)^T$. This yields that for each $j \in \{1, \dots, d\}$ we have $|y_j - x_j| < \varepsilon c$, where c depends on A only (and not on $b_1, \dots, b_d$). Since $y_j - x_j \in \mathbb{R}$, the proof of the lemma in this (non-singular) case is completed.

In the alternative case, when $\det A = 0$, the equation $A\mathbf{z} = (\varepsilon_1, \dots, \varepsilon_d)^T$ has infinitely many solutions. We may suppose without loss of generality that the largest nonzero minor corresponds to the matrix $A' = \|a_{i,j}\|_{1 \leqslant i,j \leqslant r}$. Selecting $x_j = y_j$ for $j = r+1, \dots, d$, by the above argument applied to the non-singular matrix A' and to the vector $(x_1, \dots, x_r)$ (instead of $(x_1, \dots, x_d)$), we derive that $x_j \in (y_j - \varepsilon c, y_j + \varepsilon c)$ for $i = 1, \dots, r$, where c depends on A' only. This completes the proof of the lemma. □

For $z \in \mathbb{C}$ we set

$$V(z) := (z, z^2, \dots, z^d).$$

Likewise, let $V^{(m)}(z)$ be the vector whose each entry is mth derivative of the respective entry in $V(z)$. Given a symmetric set $\mathcal{S}$ of order $d = m_1 + \cdots + m_s$ (which is, say, a list of m_1 copies of β_1, …, m_s copies of β_s), we define the matrix $A(\mathcal{S})$ by its d consecutive rows

$$V(\beta_1), \dots, V^{(m_1-1)}(\beta_1), V(\beta_2), \dots, V^{(m_2-1)}(\beta_2), \dots, V(\beta_s), \dots, V^{(m_s-1)}(\beta_s).$$

Its determinant is known as a version of confluent Vandermonde determinant. It is nonzero if the numbers $\beta_1, \dots, \beta_s$ are distinct and $\beta_j \neq 0$ for $j = 1, \dots, s$ (see, e.g., [9]).

Lemma 12. *Let $\mathcal{S} \subset \{z \in \mathbb{C} : 0 < |z| < 1\}$ be a finite symmetric set. Then $\mathbb{H}(\mathcal{S}^{-1}) \leqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S})$.*

PROOF. Suppose $f(x) = 1 + \sum_{j=1}^{\infty} h_j x^j$, where $h_1, h_2, \dots \in \mathbb{R}$ and $h := \sup_{j \geqslant 1} |h_j|$, satisfies $f(\beta_1) = \cdots = f^{(m_1-1)}(\beta_1) = \cdots = f(\beta_s) = \cdots = f^{(m_s-1)}(\beta_s) = 0$. Set $d = m_1 + \cdots + m_s$. By (2), we see that it suffices to show that, for each $\varepsilon > 0$, there is a monic polynomial $G(x) = x^n + g_{n-1}x^{n-1} + \cdots + g_0 \in \mathbb{R}[x]$ satisfying $H(G - x^{\deg G}) = \max_{0 \leqslant j \leqslant n-1} |g_j| \leqslant h + \varepsilon$ which vanishes at $1/\beta_j$ with multiplicity $\geqslant m_j$ $(j = 1, \dots, d)$. Here, $\beta_j \neq 0$.

Put $\beta := \max_{1\leqslant j\leqslant s}|\beta_j| < 1$ and $m := \max\{m_1,\dots,m_s\}$. Take n so large that

$$\sum_{j=n+1}^{\infty} |h_j| j^{m-1}\beta^{j-m+1} < \varepsilon.$$

Set $f_n(x) := 1 + h_{d+1}x^{d+1} + \cdots + h_n x^n$. Note that

$$h_1 x + \cdots + h_d x^d = f(x) - f_n(x) - \sum_{j=n+1}^{\infty} h_j x^j.$$

On applying Lemma 11 to the matrix $A = A(\mathcal{S})$, the real vector $(x_1,\dots,x_d) = (h_1,\dots,h_d)$ and

$$(b_1,\dots,b_d) = (-f_n(\beta_1),\dots,-f_n^{(m_1-1)}(\beta_1),\dots,-f_n(\beta_s),\dots,-f_n^{(m_s-1)}(\beta_s)),$$

we find that there is a constant c depending on $\mathcal{S}$ only and $g_j \in (h_j-\varepsilon c, h_j+\varepsilon c)$ $(j=1,\dots,d)$ such that

$$A(\mathcal{S})(g_1,\dots,g_d)^T = (b_1,\dots,b_d)^T.$$

This means that the polynomial $1 + g_1x + \cdots + g_dx^d + h_{d+1}x^{d+1} + \cdots + h_nx^n$ vanishes at β_j with multiplicity $\geqslant m_j$ $(j=1,\dots,s)$. Its reciprocal polynomial $G(x) = x^n + g_1x^{n-1} + \cdots + g_dx^{n-d} + h_{d+1}x^{n-d-1} + \cdots + h_n$ satisfies $H(G - x^{\deg G}) \leqslant h+\varepsilon$ and, for each $j\in\{1,\dots,s\}$, vanishes at $1/\beta_j$ with multiplicity $\geqslant m_j$. It follows that $\mathbb{H}(\mathcal{S}^{-1}) \leqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}) + \varepsilon$. Since ε can be taken arbitrarily small, this completes the proof. □

The next lemma shows that the minimum in (18) is attained.

Lemma 13. *Let $P(x)\in\mathbb{R}[x]$ be a monic polynomial of degree d, and let n be a nonnegative integer. Then there is a monic polynomial $Q_n(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$ such that $\mathbb{H}_n(P) = H(PQ_n)$.*

Proof. Let $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Each monic polynomial divisible by $P(x)$ has the form $(x^d + a_{d-1}x^{d-1} + \cdots + a_0)(x^n + b_{n-1}x^{n-1} + \cdots + b_0)$. We need to minimize the maximum of the following $n+d+1$ numbers $|a_0b_0|$, $|a_0b_1 + a_1b_0|$, …, $|a_{d-1} + b_{n-1}|$, 1. Writing each inequality $|a| \leqslant H$ as two inequalities $H - a \geqslant 0$ and $H + a \geqslant 0$, we get a system of $2(n+d)$ inequalities in $n+1$ unknowns $b_0,\dots,b_{n-1},H$. It is clear that the minimum H for which this system has a solution exists, so the quantity $\mathbb{H}_n(P)$ is equal to $\max\{1,H\}$. This completes the proof of the lemma. □

In fact, by the fundamental theorem of linear programming, if a linear programming problem has a solution, then at least one of the solutions always occurs at a corner point. We can thus find the polynomial $G(x) = P(x)Q_n(x) = x^{n+d} + g_{n+d-1}x^{n+d-1} + \cdots + g_0$, where Q_n is the polynomial of Lemma 13, such that at most $d-1$ of the numbers $|g_{n+d-1}|, \ldots, |g_0|$ are smaller than $\max\{|g_{n+d-1}|, \ldots, |g_0|\}$. In other words, $|g_j| = H(G - x^{n+d})$ for all but at most $d-1$ indices $j \in \{0, 1, \ldots, n+d-1\}$. This explains our strategy used in the proof of Theorem 5. We just need to find the 'correct' set of $d-1$ 'small' coefficients, because, for any symmetric set $\mathcal{S}$ of order d in $0 < |z| < 1$, there exist a series $h(x) = 1 + \sum_{j=1}^{\infty} h_j x^j$ vanishing at $\mathcal{S}$ with each (except for at most $d-1$ coefficients) h_j equal to $\pm\mathbb{H}_{\mathrm{ser}}(\mathcal{S})$.

Proof of Lemma 6. Let us multiply both sides of (15) by $R(1) = (1-\beta_1)\ldots(1-\beta_d)$. We need to show that

$$\begin{vmatrix} 1-\beta_1 & \beta_1-\beta_1^2 & \ldots & \beta_1^{j-3}-\beta_1^{j-2} & \beta_1^{j-1} & \ldots & \beta_1^d \\ & \vdots & & & & & \\ 1-\beta_d & \beta_d-\beta_d^2 & \ldots & \beta_d^{j-3}-\beta_d^{j-2} & \beta_d^{j-1} & \ldots & \beta_d^d \end{vmatrix} = \pm|D(J_0)\sum_{k=d-j+2}^{d} r_k|. \quad (20)$$

Notice that the left hand side of (20) is equal to

$$\begin{vmatrix} 1-\beta_1^{j-2} & \beta_1-\beta_1^{j-2} & \ldots & \beta_1^{j-3}-\beta_1^{j-2} & \beta_1^{j-1} & \ldots & \beta_1^d \\ & \vdots & & & & & \\ 1-\beta_d^{j-2} & \beta_d-\beta_d^{j-2} & \ldots & \beta_d^{j-3}-\beta_d^{j-2} & \beta_d^{j-1} & \ldots & \beta_d^d \end{vmatrix}$$

for each $j \in \{3, \ldots, d+1\}$. Using the next well-known formula (see, e.g., Problem 346 in [12])

$$\begin{vmatrix} 1 & \beta_1 & \ldots & \beta_1^{k-1} & \beta_1^{k+1} & \ldots & \beta_1^d \\ \vdots & & & & & & \\ 1 & \beta_d & \ldots & \beta_d^{k-1} & \beta_d^{k+1} & \ldots & \beta_d^d \end{vmatrix}$$

$$= (\beta_1\ldots\beta_{d-k} + \cdots + \beta_{k+1}\ldots\beta_d) \prod_{1\leqslant i<j\leqslant d} (\beta_j - \beta_i) = (-1)^{d-k} r_{d-k} D(J_0),$$

where $k = 0, \ldots, d-1$, we can expand the left hand side of the next determinant by its first $j-2$ columns:

$$\begin{vmatrix} 1-\beta_1^{j-2} & \beta_1-\beta_1^{j-2} & \dots & \beta_1^{j-3}-\beta_1^{j-2} & \beta_1^{j-1} & \dots & \beta_1^d \\ & \vdots & & & & & \\ 1-\beta_d^{j-2} & \beta_d-\beta_d^{j-2} & \dots & \beta_d^{j-3}-\beta_d^{j-2} & \beta_d^{j-1} & \dots & \beta_d^d \end{vmatrix}$$

$$= (-1)^{d-j+2} D(J_0) \sum_{k=d-j+2}^{d} r_k.$$

This implies (20) and completes the proof of the lemma. □

We conclude this section with the proof of Lemma 3:

By the definition of $\mathbb{H}_{\rm ser}(\mathcal{S})$, for any $N \in \mathbb{N}$, there exist a power series $1 + \sum_{j=1}^{\infty} h_{j,N} x^j$ vanishing at $\mathcal{S}$ such that $|h_{j,N}| \leqslant \mathbb{H}_{\rm ser}(\mathcal{S}) + 1/N$. Put $h := \mathbb{H}_{\rm ser}(\mathcal{S})$. We can choose a sequence $N_1 < N_2 < N_3 < \dots$ of positive integers such that $h_{1,N_k} \to h_1 \in [-h, h]$ as $k \to \infty$. Then, we choose its subsequence (denoted by $N_1 < N_2 < N_3 < \dots$ again) such that $h_{2,N_k} \to h_2 \in [-h, h]$ as $k \to \infty$ and so on. We claim that the series $1 + \sum_{j=1}^{\infty} h_j x^j$ vanishes at $\mathcal{S}$ with required multiplicities. Indeed, suppose that $\beta \in \mathcal{S}$ (so $|\beta| < 1$), but $B := 1 + \sum_{j=1}^{\infty} h_j \beta^j \neq 0$. Take $M \in \mathbb{N}$ so large that

$$3 \max\{1, h\} |\beta|^{M+1} / (1 - |\beta|) < |B|/2.$$

Next, take $N \in \mathbb{N}$ which is, e.g., an element of the above sequence $N_1 < N_2 < \dots$ after M steps are taken and is so large that

$$|h_1 - h_{1,N}| < |B|/2M, \ |h_2 - h_{2,N}| < |B|/2M, \ \dots, \ |h_M - h_{M,N}| < |B|/2M.$$

Since $1 + \sum_{j=1}^{\infty} h_{j,N} \beta^j = 0$ for any $N \in \mathbb{N}$, using $|h_j - h_{j,N}| \leqslant 3 \max\{1, h\}$, we obtain that

$$|B| = |\sum_{j=1}^{\infty} (h_j - h_{j,N}) \beta^j| < \sum_{j=1}^{M} |h_j - h_{j,N}| + \sum_{j=M+1}^{\infty} |(h_j - h_{j,N}) \beta^j| < |B|/2 + |B|/2 = |B|,$$

a contradiction. It follows that $B = 0$. The case when β is a multiple root can be treated in the same manner. This completes the proof of the lemma. □

6. Proofs

Proof of Theorem 1. Clearly, $H(cP) = |c|H(P)$ implies (i). Using (i) we can assume that P in (ii), (iii), (v) is monic. Hence (4) implies that $\mathbb{H}((x-c)P(x)) \geqslant \mathbb{H}(P)$. On the other hand, suppose that $Q(x) \in \mathbb{R}[x]$ is such that $H(PQ) < \mathbb{H}(P)+\varepsilon$. Take $n > \deg(PQ)$. Since $(x^n-c^n)P(x)Q(x)$ is a polynomial of height $H(PQ)$ divisible by $(x-c)P(x)$, we obtain that $\mathbb{H}((x-c)P(x)) \leqslant \mathbb{H}(P)$. This proves (ii).

Similarly, by (4), we have $\mathbb{H}((x-w)(x-\overline{w})P(x)) \geqslant \mathbb{H}(P)$ for any $w \in \mathbb{C}$. Suppose that $Q(x) \in \mathbb{R}[x]$ is a polynomial for which $H(PQ) < \mathbb{H}(P)+\varepsilon$. We claim that, for $|w| \leqslant 1$, there is $n > \deg(PQ)$ such that the polynomial $(x^n - w^n)(x^n - \overline{w}^n) = x^{2n} - 2\Re(w^n)x^n + |w|^2$ is of height 1. This would imply (iii) as above. Writing $w = |w|e^{i\varphi}$, where $\varphi \in (0,\pi)$, we have $2\Re(w^n) = 2|w|^n\cos(n\varphi)$. Its modulus is at most 1 if $|\cos(n\varphi)| \leqslant 1/2$ which is equivalent to $\cos(2n\varphi) \leqslant -1/2$ and to $\|n\varphi/\pi\| \geqslant 1/3$. (Here, $\|x\|$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer, whereas in the proof of Theorem 2 the same notation is used for the Euclidean norm of a polynomial.) The inequality $\|n\varphi/\pi\| \geqslant 1/3$ clearly holds for infinitely many $n \in \mathbb{N}$ if φ/π is irrational. For $\varphi/\pi \in \mathbb{Q}$, namely, $\varphi/\pi = u/v$ with integer $u < v$, where $v \geqslant 2$, by taking $n = vk + r$, where $ur \equiv [v/2] \pmod v$, we have $\|(vk+r)u/v\| = \|ru/v\| = [v/2]/v \geqslant 1/3$. This completes the proof of (iii). (The example $\varphi = \pi/3$ shows that the constant $1/3$ in $\|n\varphi/\pi\| \geqslant 1/3$ cannot be improved.)

Obviously, $\mathbb{H}(P(x)) = \mathbb{H}(P(-x))$, so in the proof of (iv) we can assume that $c > 0$. Since $x^n - c^n$ is divisible by $x - c$, we have $\mathbb{H}(x-c) = 1$ for $c \in (0,1]$. Suppose $c > 1$. Note that $x - c$ divides the polynomial $x^n - ((c-1)/(1-c^{-n}))(x^{n-1} + \cdots + x + 1)$ of height $\max\{1,(c-1)/(1-c^{-n})\}$. Since c^{-n} tends to zero as $n \to \infty$, we have $\mathbb{H}(x-c) \leqslant \max\{1, c-1\}$. On the other hand, each polynomial $x^n + c_{n-1}x^{n-1} + \cdots + c_0$ vanishing at c has at least one coefficient c_i whose modulus is greater than or equal to $(c-1)/(1-c^{-n})$, since otherwise $c^n \leqslant |c_0| + \cdots + |c_{n-1}|c^{n-1} < (c^n-1)(c-1)/(c-1)(1-c^{-n}) = c^n$, which is a contradiction. Thus $\mathbb{H}_n(x-c) \geqslant \max\{1,(c-1)/(1-c^{-n})\}$. This implies that $\mathbb{H}(x-c) \geqslant \max\{1, c-1\}$ and proves (iv).

The proof of (v) is exactly the same as the proof of Proposition (iv) in [13]. The upper bound $\mathbb{H}(P(x^k)) \leqslant \mathbb{H}(P(x))$ is trivial by (1). For the lower bound $\mathbb{H}(P(x^k)) \geqslant \mathbb{H}(P(x))$, we write $P(x^k)Q(x)$ in the form

$$P(x^k)Q_0(x^k) + xP(x^k)Q_1(x^k) + \cdots + x^{k-1}P(x^k)Q_{k-1}(x^k).$$

Here, $Q_j(x^k) = x^{-j}\sum_{i\equiv j \pmod k} q_i x^i$, where $Q(x) = \sum_{i=0}^n q_i x^i$, is a polynomial in x^k. Observing that $H(P(x^k)Q(x)) \geqslant H(P(x^k)Q_0(x^k))$ we obtain that

$\mathbb{H}(P(x^k)) \geqslant \mathbb{H}(P(x))$. So $\mathbb{H}(P(x^k)) = \mathbb{H}(P(x))$. Combined with $\mathbb{H}(\pm P(\pm x)) = \mathbb{H}(P(x))$ this completes the proof of (v). □

Proof of Theorem 2. Since $\|P_N - P\| \to 0$ as $N \to \infty$ and $P, P_1, P_2, \ldots$ are all monic, we can assume without loss of generality that $P_N(x) = x^d + a_{N,d-1}x^{d-1} + \cdots + a_{N,0}$ for $N = 1, 2, \ldots$ and $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$. It follows that $\lim_{N\to\infty} a_{N,j} = a_j$ as $N \to \infty$ for each $j \in \{0, 1, \ldots, d-1\}$.

Let ε be a fixed positive number. Since $\mathbb{H}(P) = \lim_{n\to\infty} \mathbb{H}_n(P)$, there is a positive integer n so large that $\mathbb{H}_n(P) \geqslant \mathbb{H}(P) \geqslant \mathbb{H}_n(P) - \varepsilon$. Lemma 13 implies that $\mathbb{H}_n(P) = H(PQ_n)$ for some monic polynomial Q_n of degree n, thus $\mathbb{H}(P) \geqslant H(PQ_n) - \varepsilon$. Next, take N so large that $H(P_NQ_n) - H(PQ_n) \leqslant \varepsilon$. These inequalities show that $H(P_NQ_n) \leqslant \mathbb{H}(P) + 2\varepsilon$. But $\mathbb{H}(P_N) \leqslant H(P_NQ_n)$, so $\mathbb{H}(P_N) \leqslant \mathbb{H}(P) + 2\varepsilon$. Hence

$$\limsup_{N\to\infty} \mathbb{H}(P_N) \leqslant \mathbb{H}(P).$$

It remains to show that, for any $\varepsilon > 0$, there is a positive integer N_0 such that $\mathbb{H}(P) \leqslant \mathbb{H}(P_N) + \varepsilon$ for each $N \geqslant N_0$. This would imply that $\liminf_{N\to\infty} \mathbb{H}(P_N) \geqslant \mathbb{H}(P)$. So combined with the upper bound for the largest limit point we will be able to conclude that $\lim_{N\to\infty} \mathbb{H}(P_N) = \mathbb{H}(P)$.

By Theorem 1 (ii), (iii), we can assume that the roots of P are all in $|z| \geqslant r > 1$. Since the roots of a monic polynomial depend continuously on its coefficients, there is no loss of generality to assume that the roots of P_N are all in $|z| \geqslant r_1 > 1$.

Let $\mathcal{S} = \{\beta_1, \ldots, \beta_d\}$ and $\mathcal{S}_N = \{\beta_{N,1}, \ldots, \beta_{N,d}\}$ be the multisets which are reciprocal to the multisets $\{\beta_1^{-1}, \ldots, \beta_d^{-1}\}$ and $\{\beta_{N,1}^{-1}, \ldots, \beta_{N,d}^{-1}\}$ of roots of P and P_N, respectively. It is clear that the multisets $\mathcal{S}_N$ tend pointwise to $\mathcal{S}$ as $N \to \infty$. By (3) and Theorem 4 (whose proof will be given below), it is sufficient to prove the inequality

$$\mathbb{H}_{\mathrm{ser}}(\mathcal{S}) \leqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) + \varepsilon$$

for $N \geqslant N_0(\varepsilon)$.

The argument is quite close to the one used in the proof of Lemma 12. Fix some constants $r_2 > 0$ and $r_3 < 1$ such that the elements of $\mathcal{S}$ and $\mathcal{S}_N$ all lie in the annulus $r_2 \leqslant |z| \leqslant r_3$.

Let $f_N(x) = 1 + \sum_{j=1}^{\infty} g_{N,j}x^j$ be the series vanishing at $\mathcal{S}_N$ for which $\sup_{j\geqslant 1} |g_{N,j}| = \mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N)$. (Lemma 3 implies their existence for any N.) We shall change the first d coefficients of f_N into $g_1, \ldots, g_d$ and consider the series $\bar{f}_N(x) = 1 + \sum_{j=1}^{d} g_j x^j + \sum_{j=d+1}^{\infty} g_{N,j}x^j$. These d coefficients will be chosen

in a way which guarantees that $\bar{f}_N$ vanishes at $\mathcal{S}$ with required multiplicities, namely, $\bar{f}_N(\beta_s) = \cdots = \bar{f}_N^{(m_s-1)}(\beta_s) = 0$ if β_s occurs in $\mathcal{S}$ with multiplicity m_s. A corresponding linear system of d equations in d unknowns $g_1, \dots, g_d$ gives

$$g_j = F_{j,0}(\mathcal{S}) + F_{j,d+1}(\mathcal{S})g_{N,d+1} + F_{j,d+2}(\mathcal{S})g_{N,d+2} + \dots$$

for $j = 1, \dots, d$. Here, setting

$$D(\mathcal{S}) := \begin{vmatrix} \beta_1 & \beta_1^2 & \dots & \beta_1^d \\ \vdots & & & \\ \beta_d & \beta_d^2 & \dots & \beta_d^d \end{vmatrix}$$

and $D_{j,m}(\mathcal{S})$ for $D(\mathcal{S})$ whose jth column is replaced by the column $(\beta_1^m, \dots, \beta_d^m)$, we have

$$F_{j,m}(\mathcal{S}) = -D_{j,m}(\mathcal{S})/D(\mathcal{S}) \tag{21}$$

for each $m \in \{0, d+1, d+2, \dots\}$. Note that, although $D(\mathcal{S}) = 0$ in the case when at least one β_s belongs to $\mathcal{S}$ is with multiplicity $\geqslant 2$, the functions $F_{j,m}(\mathcal{S})$ are well defined. Moreover, from (21) it is easily seen that, for each $m \geqslant d+1$, $F_{j,m}(\mathcal{S})$ is a symmetric polynomial in $\beta_1, \dots, \beta_d$ of degree $\leqslant m$ with at most m^{r_4} terms, where r_4 is a positive constant depending on d only. For example, for $d = 2$, we have

$$g_1 = -(\beta_1+\beta_2)(\beta_1\beta_2)^{-1} + \beta_1\beta_2 \sum_{j=3}^{\infty} g_{N,j}(\beta_1^{j-3} + \beta_1^{j-2}\beta_2 + \dots + \beta_2^{j-3})$$

and

$$g_2 = (\beta_1\beta_2)^{-1} - \sum_{j=3}^{\infty} g_{N,j}(\beta_1^{j-2} + \beta_1^{j-1}\beta_2 + \dots + \beta_2^{j-2}).$$

Of course, the fact that f_N vanishes at $\mathcal{S}_N$ implies that

$$g_{N,j} = F_{j,0}(\mathcal{S}_N) + F_{j,d+1}(\mathcal{S}_N)g_{N,d+1} + F_{j,d+2}(\mathcal{S}_N)g_{N,d+2} + \dots$$

for each $j = 1, \dots, d$. Here, $F_{j,m}(\mathcal{S}_N)$ are defined as in (21), where each β_j in the above determinants is replaced by $\beta_{N,j}$. Subtracting $g_{N,j}$ from g_j, we deduce that

$$g_j - g_{N,j} = F_{j,0}(\mathcal{S}) - F_{j,0}(\mathcal{S}_N) + \sum_{t=d+1}^{\infty} g_{N,t}(F_{j,t}(\mathcal{S}) - F_{j,t}(\mathcal{S}_N)). \tag{22}$$

Now, fix $\varepsilon > 0$. We will show that, for $N \geqslant N_0(\varepsilon)$, $|g_j - g_{N,j}| < \varepsilon$ for each $j = 1, \dots, d$. For this, we split the sum in (22) into two sums corresponding to $t \leqslant M$ and $t \geqslant M+1$, where M will be chosen later (M will be the same for all N).

We will first bound the sum over $t \geqslant M+1$. Note that, since β_j and $\beta_{N,j}$ all lie in the annulus $r_2 \leqslant |z| \leqslant r_3$ and since $F_{j,t}(\mathcal{S})$ and $F_{j,t}(\mathcal{S}_N)$ are symmetric polynomials in $\beta_1, \dots, \beta_d$ and $\beta_{N,1}, \dots, \beta_{N.d}$, respectively, of degree $\leqslant t$ with $\leqslant t^{r_4}$ terms, $|F_{j,t}(\mathcal{S})|$ and $|F_{j,t}(\mathcal{S}_N)|$ do not exceed $t^{r_4} r_3^t$. Thus

$$\sum_{t=M+1}^{\infty} |g_{N,t}||F_{j,t}(\mathcal{S}) - F_{j,t}(\mathcal{S}_N)| \leqslant 2\mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) \sum_{t=M+1}^{\infty} t^{r_4} r_3^t < \varepsilon/2$$

if M is large enough, say, $M \geqslant M(\varepsilon)$. Here, we bound each $\mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N)$ from above by an absolute constant, so that $M(\varepsilon)$ is independent of N. Fix one of such large M, say, $M = M(\varepsilon)$.

Let us order the points of $\mathcal{S}_N$ so that, for each $j \in \{1, \dots, d\}$, $\beta_{N,j}$ is 'close' to β_j, and put

$$\delta_N := \max_{1 \leqslant j \leqslant d} |\beta_{N,j} - \beta_j|.$$

Clearly, $\delta_N \to 0$ as $N \to \infty$. From the formula (21), we obtain that there exists some positive constant r_5 which depends on r_2, r_3, M and $\mathcal{S}$ only such that

$$|F_{j,t}(\mathcal{S}) - F_{j,t}(\mathcal{S}_N)| \leqslant r_5 \delta_N$$

for every $t \in \{0, d+1, d+2, \dots, M\}$ and $j \in \{1, \dots, d\}$. Then (22) implies that

$$|g_j - g_{N,j}| \leqslant r_5 \delta_N + (M-d)\mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) r_5 \delta_N + \varepsilon/2$$

for each $j \in \{1, \dots, d\}$, where r_5 is a positive constant. Taking $N_0(\varepsilon)$ so large that $r_5\delta_N + (M-d)\mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) r_5 \delta_N < \varepsilon/2$ for $N \geqslant N_0(\varepsilon)$, we obtain that $|g_j - g_{N,j}| < \varepsilon$ for all $N \geqslant N_0(\varepsilon)$.

It follows that, for $N \geqslant N_0(\varepsilon)$, the moduli of the coefficients of the series $\bar{f}_N$, namely, $|g_1|, \dots, |g_d|, |g_{N,d+1}|, |g_{N,d+2}|, \dots$ are all smaller than $\sup_{j \geqslant 1} |g_{N,j}| + \varepsilon = \mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) + \varepsilon$. But $\bar{f}_N$ vanishes at $\mathcal{S}$ with required multiplicities, so

$$\mathbb{H}_{\mathrm{ser}}(\mathcal{S}) \leqslant \sup\{|g_1|, \dots, |g_d|, |g_{N,d+1}|, |g_{N,d+2}|, \dots\} \leqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}_N) + \varepsilon,$$

as claimed. □

Proof of Theorem 4. Note that, as in Theorem 1 (ii), (iii), we have $\mathbb{H}(\mathcal{S}) = \mathbb{H}(\mathcal{S}')$, where $\mathcal{S}' = \mathcal{S} \cap \{z \in \mathbb{C} : |z| > 1\}$. Evidently, $\mathcal{S}^* = \mathcal{S}'^{-1}$, so for the proof of $\mathbb{H}(\mathcal{S}) = \mathbb{H}_{\mathrm{ser}}(\mathcal{S}^*)$ it suffices to show that $\mathbb{H}(\mathcal{S}') = \mathbb{H}_{\mathrm{ser}}(\mathcal{S}'^{-1})$. The bound $\mathbb{H}(\mathcal{S}') \leqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}'^{-1})$ follows immediately from Lemma 12.

As for the inequality $\mathbb{H}(\mathcal{S}') \geqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}'^{-1})$, observe first that for each $\varepsilon > 0$ there is a polynomial $G(x)$ vanishing at the points of $\mathcal{S}'$ with prescribed multiplicities that satisfies $H(G - x^{\deg G}) < \mathbb{H}(\mathcal{S}') + \varepsilon$. On replacing $G(x)$ by its reciprocal and adding zero terms we obtain a series that vanish at $\mathcal{S}'^{-1}$ with prescribed multiplicities. It follows that $\mathbb{H}_{\mathrm{ser}}(\mathcal{S}'^{-1}) \leqslant H(G - x^{\deg G}) + \varepsilon$. This, by (2) and (3), implies that $\mathbb{H}(\mathcal{S}') \geqslant \mathbb{H}_{\mathrm{ser}}(\mathcal{S}'^{-1})$ and completes the proof of $\mathbb{H}(\mathcal{S}) = \mathbb{H}_{\mathrm{ser}}(\mathcal{S}^*)$. □

Proof of Corollary 8. Let k be the largest positive integer for which $(1 - 2u^{1-k})/(u-1) \geqslant (1 - 2v^{1-k})/(v-1)$. Take $J = \{k\}$ and apply Theorem 5 to $\beta_1 = 1/u$, $\beta_2 = 1/v$. By (5) and (6), we have $D(J) = v^{-k} - u^{-k} > 0$ and $S_n(J) = (uv)^{-k}(u^{k-n} - v^{k-n})/D(J)$ which is positive for $n < k$ and negative for $n > k$. Hence, by (9),

$$\phi(x) = x + \cdots + x^{k-1} - x^{k+1} - x^{k+2} - \cdots = (x - x^k - x^{k+1})/(1-x).$$

Next, by (10), we obtain that

$$\begin{aligned} D_2(J) &= \frac{\phi(u^{-1})}{v^k} - \frac{\phi(v^{-1})}{u^k} = \frac{u^{-1} - u^{-k} - u^{-k-1}}{(1-u^{-1})v^k} - \frac{v^{-1} - v^{-k} - v^{-k-1}}{(1-v^{-1})u^k} \\ &= (uv)^{-k}\left(\frac{u^k - u - 1}{u-1} - \frac{v^k - v - 1}{v-1}\right) = (uv)^{-k}\left(\frac{u^k - 2}{u-1} - \frac{v^k - 2}{v-1}\right). \end{aligned}$$

Thus

$$\frac{D(J)}{D_2(J)} = \frac{u^k - v^k}{(u^k-2)/(u-1) - (v^k-2)/(v-1)}. \tag{23}$$

Similarly, by (10), we have

$$D_3(J) = \phi(u^{-1}) - \phi(v^{-1}) = \frac{1 - u^{-k+1} - u^{-k}}{u-1} - \frac{1 - v^{-k+1} - v^{-k}}{v-1}. \tag{24}$$

By (23), (24) and Theorem 5, we see that the proof of the corollary will be completed if we will show that $|D_3(J)| \leqslant D(J) = v^{-k} - u^{-k}$, namely,

$$\left|\frac{1 - u^{-k+1} - u^{-k}}{u-1} - \frac{1 - v^{-k+1} - v^{-k}}{v-1}\right| \leqslant v^{-k} - u^{-k}.$$

This inequality is equivalent to the system of two inequalities

$$\begin{cases} (1 - 2u^{-k})/(u-1) \leqslant (1 - v^{-k})/(v-1), \\ (1 - 2u^{1-k})/(u-1) \geqslant (1 - v^{1-k})/(v-1). \end{cases}$$

Clearly, both these inequalities hold if k is defined as above. This completes the proof of Corollary 8. □

Proof of Corollary 9. Let $\mathcal{S}_{u,v}=\{u,v\}$ and let $\mathcal{S}_{u,u}$ be the symmetric set u, u. Clearly,

$$\lim_{v\to u}\mathbb{H}(\mathcal{S}_{u,v})=\mathbb{H}(\mathcal{S}_{u,u}).$$

(This follows from Theorem 2, where we proved that $\mathbb{H}(\mathcal{S}_N)\to\mathbb{H}(\mathcal{S})$ if $\mathcal{S}_N$ as a vector tends to $\mathcal{S}$ as $N\to\infty$.) By a standard computation, we find that

$$\lim_{v\to u}\frac{u^k-v^k}{(u^k-2)/(u-1)-(v^k-2)/(v-1)}=\frac{ku^{k-1}(u-1)^2}{(k-1)u^k-ku^{k-1}+2}$$

and

$$\lim_{v\to u}\frac{(1-2u^{1-k})/(u-1)-(1-2v^{1-k})/(v-1)}{u-v}=-\frac{u^k-2ku+2(k-1)}{u^k(u-1)^2}.$$

Set $u_1:=\infty$, and suppose $u_k>1$, where $k=2,3,\ldots$, is the largest real root of the equation $x^k-2kx+2(k-1)=0$. Clearly, $u_1>u_2>u_3>\ldots$, $\lim_{k\to\infty}u_k=1$, and the condition 'the largest k for which $(1-2u^{1-k})/(u-1)\geqslant(1-2v^{1-k})/(v-1)$ holds' becomes '$u\in[u_{k+1},u_k)$'. So Corollary 8 implies that

$$\mathbb{H}(\mathcal{S}_{u,u})=\frac{ku^{k-1}(u-1)^2}{(k-1)u^k-ku^{k-1}+2}$$

for each $u\in[u_{k+1},u_k)$. Observing that $u_2=2+\sqrt{2}$, $u_3=2$, $u_4=\kappa_1$ and that at $u=\kappa_2$ the equality $4u^3(u-1)^2=3u^4-4u^3+2$ holds, we conclude the proof of Corollary 9 via (3). □

Proof of Corollary 10. Take $J=J_0=\{1\}$ in Theorem 5. Suppose that $\mathcal{S}=\{w,\overline{w}\}=\{|w|e^{i\varphi},|w|e^{-i\varphi}\}$. Then $\beta_1=\beta=|w|^{-1}e^{i\varphi}$, $\beta_2=\overline{\beta}=|w|^{-1}e^{-i\varphi}$. We now find that

$$S_n=S_n(J_0)=S_n=(\beta^n\overline{\beta}-\overline{\beta}^n\beta)/(\overline{\beta}-\beta)=-|w|^{-n}\sin((n-1)\varphi)/\sin(\varphi).$$

Since

$$\sum_{n=2}^{\infty}|S_n|=|w|^{-2}(1+\sum_{j=1}^{\infty}|w|^{-j}|\sin((j+1)\varphi)/\sin(\varphi)|),\tag{25}$$

we find from Theorem 5 that $\mathbb{H}(\mathcal{S})\geqslant|w|^2/(1+\sum_{j=1}^{\infty}|w|^{-j}|\sin((j+1)\varphi)/\sin(\varphi)|)$. In order to show that equality holds it suffices to prove that $|\phi(\beta)\overline{\beta}-\phi(\overline{\beta})\beta|$ and $|\phi(\beta)-\phi(\overline{\beta})|$ are both smaller than or equal to $|\beta-\overline{\beta}|$. Here, $\phi(x)=\sum_{j=2}^{\infty}\delta_jx^j$, $\delta_j\in\{-1,1\}$.

Since $|\beta^j\overline{\beta}-\overline{\beta}^j\beta|/|\beta-\overline{\beta}| \leqslant (j-1)|\beta|^j = (j-1)|w|^{-j}$ and $|\beta^j-\overline{\beta}^j|/|\beta-\overline{\beta}| \leqslant j|\beta|^{j-1} = j|w|^{-j+1}$, we obtain that

$$\frac{|\phi(\beta)\overline{\beta}-\phi(\overline{\beta})\beta|}{|\beta-\overline{\beta}|} \leqslant \sum_{j=2}^{\infty}(j-1)|w|^{-j} = \frac{1}{(|w|-1)^2},$$

and

$$\frac{|\phi(\beta)-\phi(\overline{\beta})|}{|\beta-\overline{\beta}|} \leqslant \sum_{j=2}^{\infty} j|w|^{-j+1} = \frac{2|w|-1}{(|w|-1)^2}.$$

Clearly, both right hand sides $1/(|w|-1)^2$ and $(2|w|-1)/(|w|-1)^2$ are smaller than or equal to 1, because $|w| \geqslant 2+\sqrt{2}$. This completes the proof of the corollary. □

Finally, we remark that, by Theorem 5, for any $d \in \mathbb{N}$ there is a constant $\eta(d)$ such that the reduced height of $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ which has all its roots in $|z| \geqslant \eta(d)$ can be evaluated by the formula

$$\mathbb{H}(P) = \left(\sum_{j=d}^{\infty}|S_j|\right)^{-1},$$

where S_n, $n = 0, 1, 2, \ldots$, satisfy the linear recurrence relation (14). Indeed, then the roots $\beta_i = 1/\alpha_i$ of the polynomial

$$R(x) = (x-\beta_1)\ldots(x-\beta_d) = x^d + r_1x^{d-1} + \cdots + r_d = P(1/x)x^d(-1)^d\beta_1\ldots\beta_d$$

are so small that the conditions of Theorem 5 are satisfied for the set $J = J_0 = \{1, \ldots, d-1\}$.

The value $\eta(2)$ is equal to $2+\sqrt{2}$. (See Corollaries 8–10.) It seems that, for each $d \geqslant 2$, the formulae $\eta(d) = 1/(1-(1+1/(d-1)!)^{-1/d})$ is true. Some evidence towards this formulae can be given as follows. For $|z| \leqslant 1-(1+1/(d-1)!)^{-1/d}$, the inequality $|\phi^{(d-1)}(z)| \leqslant 1$, where $\phi(z) = \sum_{j=d}^{\infty} \pm z^j$, holds for any distribution of signs $\pm$, giving $|D_{d+1}(J_0)| \leqslant |D(J_0)|$ (see (10) and Theorem 5).

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