

Hardy spaces and convergence of vector-valued Vilenkin–Fourier series

By FERENC WEISZ (Budapest)

Abstract. The atomic decomposition of a vector-valued martingale Hardy space is given. A classical inequality of Marcinkiewicz is generalized for UMD lattice valued (bounded) Vilenkin–Fourier series. It is proved that the Vilenkin–Fourier series of $f \in L_p(X)$ ($1 < p < \infty$) converges to f in $L_p(X)$ norm if and only if X is a UMD space. Moreover, a lacunary sequence of the UMD lattice valued Vilenkin–Fourier series of $f \in H_1(X)$ converges almost everywhere to f in X norm.

1. Introduction

For trigonometric and Walsh–Fourier series the partial sum operators are bounded on L_p ($1 < p < \infty$) spaces. An ℓ_r -valued version of this theorem is due to Marcinkiewicz and Zygmund for trigonometric Fourier series (see e.g. ZYGMUND [28, II. p. 225]), to SUNOUCHI [19] for Walsh–Fourier series and to YOUNG [27] for Vilenkin–Fourier series.

LADHAWALA and PANKRATZ [9] (see also WEISZ [24]) proved that if f is in the dyadic Hardy space H_1 and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers, then $s_{n_k}f$, the partial sums of the Walsh–Fourier series of f , converges a.e. to f . Moreover, SCHIPP and SIMON [17] verified that if $\Phi(u) = o(\log \log u)$ ($u \rightarrow \infty$) then there exists a function in $H_1\Phi(H_1)$ whose full sequence of partial sums

Mathematics Subject Classification: Primary: 42C10, 60G42, 60G46; Secondary: 46B42, 46E40.
Key words and phrases: Vector-valued Hardy spaces, atomic decomposition, UMD spaces, vector-valued Vilenkin–Fourier series.

This paper was written while the author was researching at University of Vienna (NuHAG) supported by Lise Meitner fellowship No M733-N04. This research was also supported by the Hungarian Scientific Research Funds (OTKA) No T043769, T047128, T047132.

diverges everywhere. Especially, if $\Phi(u) = 1$ ($u \geq 1$) then we get $H_1\Phi(H_1) = H_1$, i.e. it follows the existence of $f \in H_1$ such that $s_n f$ diverges everywhere (see LADHAWALA and PANKRATZ [9]). The analogous results for trigonometric Fourier series can be found in ZYGMUND [28, II. p. 235] and for Vilenkin–Fourier series in YOUNG [26].

In this paper we extend these results to vector-valued, more exactly to UMD space valued Walsh- and Vilenkin–Fourier series. The UMD (unconditionality property of martingale differences) Banach spaces were introduced by BURKHOLDER [2]. Since that time these spaces itself and their applications to Fourier analysis has been studied very intensively in the literature (e.g. BURKHOLDER [3], [4], RUBIO DE FRANCIA [15], [16], TOZONI [20], [21], MISHURA and WEISZ [13], [14], MARTINEZ and TORREA [12] and GIRARDI and WEIS [8]). Hardy spaces of scalar-valued martingales are investigated in the books LONG [11] and WEISZ [23].

Here we consider Walsh and Vilenkin martingales and give the atomic decomposition of a Banach space valued martingale Hardy space. We generalize the Marcinkiewicz inequality on partial sums for UMD space valued (bounded) Vilenkin–Fourier series. From this it follows that if X is a UMD space then the X valued Vilenkin–Fourier series of $f \in L_p(X)$ ($1 < p < \infty$) converges to f in norm. The converse is also true: if the Vilenkin–Fourier series converges in $L_p(X)$ norm then X is a UMD space. For Walsh–Fourier series this was proved in WENZEL [25] and TOZONI [20].

It is known that if $f \in L_p(X)$ ($1 < p < \infty$) and X is UMD then $s_n f \rightarrow f$ a.e. in X norm (see RUBIO DE FRANCIA [15] for trigonometric Fourier series and WEISZ [22] for Vilenkin–Fourier series). Finally, we extend this result to Hardy spaces, more exactly we prove that if f is in the Hardy space $H_1(X)$ and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers, then the partial sums of the Vilenkin–Fourier series $s_{n_k} f$ converge a.e. to f in X norm. In the proofs of these results martingale techniques are used.

2. Vilenkin systems

In this paper we consider the unit interval $[0, 1)$, the σ -algebra $\mathcal{A}$ of the Borel sets and the Lebesgue measure λ . Let $(p_n, n \in \mathbb{N})$ be a sequence of natural numbers with entries at least 2. Introduce the notations $P_0 = 1$ and $P_{n+1} := \prod_{k=0}^n p_k$ ($n \in \mathbb{N}$). Every point $x \in [0, 1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \quad 0 \leq x_k < p_k, \quad x_k \in \mathbb{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k \rightarrow \infty} x_k = 0$.

The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \quad (n \in \mathbb{N})$$

are the *generalized Rademacher functions* where $\imath := \sqrt{-1}$. The product system generated by these functions is called a *Vilenkin system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \leq n_k < p_k$ and $n_k \in \mathbb{N}$. If $p_n = 2$ for every $n \in \mathbb{N}$ then it is called *Walsh system*. In this paper we suppose that the Vilenkin system is *bounded*, i.e. the sequence (p_n) is bounded. For a detailed investigation of the Walsh- and Vilenkin systems see SCHIPP, WADE, SIMON and PÁL [18].

Let $\mathcal{F}_n$ be the σ -algebra generated by $\{r_0, \dots, r_{n-1}\}$. It is easy to see that

$$\mathcal{F}_n = \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) : 0 \leq k < P_n\}$$

where $\sigma(\mathcal{H})$ denotes the σ -algebra generated by an arbitrary set system $\mathcal{H}$. By a *Vilenkin interval* we mean one of the form $[kP_n^{-1}, (k+1)P_n^{-1})$ for some $k, n \in \mathbb{N}$, $0 \leq k < P_n$.

For a Banach space X , the space $L_p(X)$ consists of all *strongly measurable functions* $f : [0, 1) \rightarrow X$ for which

$$\|f\|_{L_p(X)} := \left(\int_0^1 \|f\|_X^p d\lambda \right)^{1/p} \quad (0 < p \leq \infty).$$

If $f \in L_p(X)$ ($p \geq 1$) then the Bochner integral $\int_0^1 f d\lambda$ exists (see DIESTEL and UHL [6] and GARCIA-CUERVA and RUBIO DE FRANCIA [7]). The expectation and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E and E_n , respectively. We investigate the class of X -valued (Vilenkin) *martingales* $f = (f_n, \in \mathbb{N})$ with respect to $(\mathcal{F}_n, \in \mathbb{N})$. For a stopping time $\nu : [0, 1) \rightarrow \mathbb{N} \cup \{\infty\}$ the stopped martingale $(f_n^\nu, \in \mathbb{N})$ is defined by

$$f_n^\nu := \sum_{k=0}^n \mathbf{1}_{\{\nu \geq k\}} d_k f,$$

where $d_k f := f_k - f_{k-1}$, $f_{-1} := 0$.

If $f \in L_1(X)$ then $\hat{f}(n) := E(f\bar{w}_n)$ is said to be the n th *Vilenkin–Fourier coefficient* of f ($\in \mathbb{N}$). Denote by $s_n f$ the n th partial sum of the Vilenkin–Fourier series of f , namely,

$$s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k.$$

It is easy to see that $(s_{P_n} f, \in \mathbb{N})$ is an X -valued martingale.

We will suppose that X is a *Banach lattice*. As usual, $|\cdot|$ will denote the absolute value in X : $|x| := \sup\{x, -x\}$. For more about Banach lattices see LINDENSTRAUSS and TZAFRIRI [10]. A Banach lattice X is a UMD (unconditionality property for martingale differences) space, if for all $1 < p < \infty$, all X -valued martingale difference sequences $(d_1, d_2, \dots)$ and all numbers $\epsilon_1, \epsilon_2, \dots \in \{-1, 1\}$ there exists a positive real number C_p such that

$$\left\| \sum_{k=1}^n \epsilon_k d_k \right\|_{L_p(X)} \leq C_p \left\| \sum_{k=1}^n d_k \right\|_{L_p(X)} \quad (\in \mathbb{N}) \quad (1)$$

(see BURKHOLDER [2]). It is enough to assume (1) for some $1 < p < \infty$ and for all X -valued martingale difference sequences $(d_1, d_2, \dots)$ with respect to $(\mathcal{F}_n)$, because each $\mathcal{F}_n$ is atomic (see RUBIO DE FRANCIA [16] or GIRARDI and WEIS [8]).

The *maximal function* of an X -valued martingale $f = (f_n, \in \mathbb{N})$ is defined by

$$M_n f := \sup_{k \leq n} \|f_k\|_X, \quad Mf := \sup_{k \in \mathbb{N}} \|f_k\|_X.$$

The following theorem can be found in BOURGAIN [1], RUBIO DE FRANCIA [16] and TOZONI [21].

Theorem 1. *If X is a UMD lattice and $f \in L_p(X)$ then*

$$\rho \lambda(Mf > \rho) \leq C \|f\|_{L_1(X)}, \quad (\rho > 0)$$

and

$$\begin{aligned} \|f\|_{L_p(X)} &\sim \|Mf\|_{L_p(0,1)} \sim \left\| \left(\sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2} \right\|_{L_p(X)} \\ &\sim \left\| \left(\sum_{n=0}^{\infty} E_{n-1} |d_n f|^2 \right)^{1/2} \right\|_{L_p(X)} \end{aligned}$$

for all $1 < p < \infty$, where $\sim$ denotes the equivalence of the norms.

Note that the sequence $(\mathcal{F}_n)$ is regular. In this paper the positive constants C_p depend only on p and may denote different constants in different contexts.

3. Hardy spaces and atomic decomposition

The Hardy space $H_p(X)$ ($1 \leq p \leq \infty$) consists of all X -valued martingales f for which

$$\|f\|_{H_p(X)} := \|Mf\|_{L_p(\mathbb{R})} < \infty.$$

By Theorem 1, if X is UMD then $H_p(X) \sim L_p(X)$ for all $1 < p < \infty$. Moreover, if $(f_n) \in H_p(X)$ for some $1 \leq p < \infty$ then there exists $f \in L_p(X)$ such that $f = \lim_{n \rightarrow \infty} f_n$ in $L_p(X)$ norm and $f_n = E_n f$ (see e.g. DIESTEL and UHL [6]).

The atomic decomposition is a useful characterization of Hardy spaces (for scalar valued martingales see e.g. WEISZ [23]). Let us introduce first the concept of atoms. A function a is an *atom* if there exists a Vilenkin interval I such that

$$\int_I a d\lambda = 0, \quad \|a\|_{L_\infty(X)} \leq \lambda(I)^{-1}, \quad \{a \neq 0\} \subset I.$$

Though the proof of the next atomic decomposition is similar to the scalar valued case, for the sake of completeness we present a short proof.

Theorem 2. *Assume that X is a UMD lattice. Then $f \in H_1(X)$ if and only if there exist a sequence $(a^k, k \in \mathbb{N})$ of atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} \mu_k a^k = f \quad \text{a.e. in } X \text{ norm and} \quad \sum_{k=0}^{\infty} |\mu_k| < \infty. \quad (2)$$

Moreover,

$$\|f\|_{H_1(X)} \sim \inf \sum_{k=0}^{\infty} |\mu_k|$$

where the infimum is taken over all decompositions of f of the form (2).

PROOF. Assume that $f \in H_1(X)$. Define the stopping time ν_k by

$$\nu_k(x) := \inf\{n \in \mathbb{N} : E_n \mathbf{1}_{\{M_{n+1}f > 2^k\}}(x) \geq 1/d\}, \quad (k \in \mathbb{Z}),$$

where $d = \sup_n p_n$. From this it follows that $\nu_k \leq \nu_{k+1}$, ($k \in \mathbb{Z}$),

$$\{Mf > 2^k\} \subset \{\nu_k < \infty\}, \quad \lambda(\nu_k < \infty) \leq d\lambda(Mf > 2^k) \quad (3)$$

and $M_{\nu_k}f \leq 2^k$ for all $k \in \mathbb{Z}$, where $M_{\nu_k}f := M_n f$ if $\nu_k = n$. It is easy to see that

$$f = \sum_{k \in \mathbb{Z}} (f_{\nu_{k+1}} - f_{\nu_k}) \quad \text{a.e. in } X \text{ norm.}$$

Indeed, $\lambda(\nu_k < \infty) \rightarrow 0$ by (3) and so $f_{\nu_{k+1}} \rightarrow f$ a.e. in X norm as $k \rightarrow \infty$ and $\|f_{\nu_k}\|_X \leq 2^k \rightarrow 0$ as $k \rightarrow -\infty$. We decompose $\{\nu_k = l\} = \cup_n I_{k,n}^l$, where $I_{k,n}^l \in \mathcal{F}_l$ are Vilenkin intervals. If we define

$$\mu_{k,n}^l := 3 \cdot 2^k \lambda(I_{k,n}^l), \quad a_{k,n}^l := (\mu_{k,n}^l)^{-1} \mathbf{1}_{I_{k,n}^l} (f_{\nu_{k+1}} - f_{\nu_k})$$

then

$$f = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \sum_n \mu_{k,n}^l a_{k,n}^l \quad \text{a.e. in } X \text{ norm.} \quad (4)$$

Since $\nu_{k+1} \geq \nu_k = l$ on $I_{k,n}^l$, by the martingale property

$$\int_{I_{k,n}^l} (f_{\nu_{k+1}} - f_{\nu_k}) d\lambda = \int_{I_{k,n}^l} (f_{\nu_{k+1}} - f_l) d\lambda = 0.$$

This and

$$\|a_{k,n}^l\|_X \leq |\mu_{k,n}^l|^{-1} (\|f_{\nu_{k+1}}\|_X + \|f_{\nu_k}\|_X) \leq \lambda(I_{k,n}^l)^{-1}$$

imply that $a_{k,n}^l$ are atoms. By (3),

$$\sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \sum_n |\mu_k| = 3 \sum_{k \in \mathbb{Z}} 2^k \lambda(\nu_k < \infty) \leq 3d \sum_{k \in \mathbb{Z}} 2^k \lambda(Mf > 2^k) \leq CE(Mf).$$

Since $E(\|a\|_X) \leq 1$, the sum

$$\sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{N}} \sum_n |\mu_{k,n}^l| \|a_{k,n}^l\|_X$$

is convergent a.e. Thus the sum in (4) can be rearranged to get (2).

Conversely, suppose that f has a decomposition of the form (2). Since the sum in (2) converges in $L_1(X)$ norm, we have

$$E_n f = \sum_{k=0}^{\infty} \mu_k E_n a^k$$

and so we conclude

$$E(|Mf|) \leq \sum_{k=0}^{\infty} |\mu_k| E(\sup_n \|E_n a^k\|_X) \leq \sum_{k=0}^{\infty} |\mu_k| \int_{I_k} \sup_{n \geq n_k} E_n \|a^k\|_X d\lambda \leq \sum_{k=0}^{\infty} |\mu_k|,$$

where the Vilenkin interval $I_k \in \mathcal{F}_{n_k}$ is the support of a_k . $\square$

Note that the same proof works if we suppose only that X is a Banach space having the Radon–Nikodym property.

4. Marcinkiewicz inequality

Now we generalize the classical Marcinkiewicz inequality, mentioned in the Introduction, for UMD valued functions.

Theorem 3. *Assume that X is a UMD lattice and n_k is an arbitrary natural number for each $k \in \mathbb{N}$. If $(f^k, k \in \mathbb{N}) \in L_p(\ell_r(X))$ for some $1 < p, r < \infty$ then*

$$\left\| \left(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left(\sum_{k=0}^{\infty} \|f^k\|_X^r \right)^{1/r} \right\|_p. \quad (5)$$

If $(f^k, k \in \mathbb{N}) \in L_1(\ell_r(X))$ for some $1 < r < \infty$ then

$$\rho \lambda \left(\left(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \right)^{1/r} > \rho \right) \leq C_r \left\| \left(\sum_{k=0}^{\infty} \|f^k\|_X^r \right)^{1/r} \right\|_1, \quad (\rho > 0). \quad (6)$$

PROOF. It is known that

$$\bar{w}_n s_n f = \sum_{j=0}^{\infty} \bar{w}_n T_j^n \left(w_n (E_{j+1}(f \bar{w}_n) - E_j(f \bar{w}_n)) \right) =: \sum_{j=0}^{\infty} d_j^n, \quad (7)$$

where the operator T_j^n is linear,

$$|T_j^n f|^2 \leq C E_j |f|^2, \quad (j, n \in \mathbb{N}) \quad (8)$$

and $(d_j^n, j \in \mathbb{N})$ is a martingale difference sequence with respect to $(\mathcal{F}_{j+1})$ (see WEISZ [23]). Note that (7) is a finite sum. Since ℓ_r ($1 < r < \infty$) is a UMD lattice, so is $\ell_r(X)$ (see RUBIO DE FRANCIA [16]). Then we may apply Theorem 1 and (8) to obtain

$$\begin{aligned} \left\| \left(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r \right)^{1/r} \right\|_p &= \left\| \left(\sum_{k=0}^{\infty} \left\| \sum_{j=0}^{\infty} d_j^{n_k} \right\|_X^r \right)^{1/r} \right\|_p \\ &\leq C_{p,r} \left\| \left(\sum_{k=0}^{\infty} \left\| \left(\sum_{j=0}^{\infty} |d_j^{n_k}|^2 \right)^{1/2} \right\|_X^r \right)^{1/r} \right\|_p \\ &\leq C_{p,r} \left\| \left(\sum_{k=0}^{\infty} \left\| \left(\sum_{j=0}^{\infty} E_j |E_{j+1}(f^k \bar{w}_{n_k}) - E_j(f^k \bar{w}_{n_k})|^2 \right)^{1/2} \right\|_X^r \right)^{1/r} \right\|_p \\ &\leq C_{p,r} \left\| \left(\sum_{k=0}^{\infty} \left\| \left(\sum_{j=0}^{\infty} |E_{j+1}(f^k \bar{w}_{n_k}) - E_j(f^k \bar{w}_{n_k})|^2 \right)^{1/2} \right\|_X^r \right)^{1/r} \right\|_p \end{aligned}$$

$$\leq C_{p,r} \left\| \left(\sum_{k=0}^{\infty} \left\| f^k \overline{w}_{n_k} \right\|_X^r \right)^{1/r} \right\|_p, \quad (9)$$

which proves (5). Note that in the last step we have used that

$$(E_{j+1}(f^k \overline{w}_{n_k}) - E_j(f^k \overline{w}_{n_k}), j \in \mathbb{N})$$

is a martingale difference sequence.

To prove (6) let us define the stopping time

$$\nu(x) := \inf\{n \in \mathbb{N} : E_n \mathbf{1}_{\{(\sum_{k=0}^{\infty} \|E_{n+1}(f^k \overline{w}_{n_k})\|_X^r)^{1/r} > \rho\}}(x) \geq 1/d\}.$$

Then

$$\lambda(\nu < \infty) \leq d\lambda\left(\sup_{n \in \mathbb{N}} \left(\sum_{k=0}^{\infty} \|E_n(f^k \overline{w}_{n_k})\|_X^r\right)^{1/r} > \rho\right) \quad (10)$$

and

$$\sup_{n \leq \nu} \left(\sum_{k=0}^{\infty} \|E_n(f^k \overline{w}_{n_k})\|_X^r\right)^{1/r} \leq \rho. \quad (11)$$

Obviously,

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \|s_{n_k} f^k\|_X^r\right)^{1/r} &= \left(\sum_{k=0}^{\infty} \left\| \sum_{j=0}^{\infty} d_j^{n_k} \mathbf{1}_{\{\nu \geq j+1\}} \right\|_X^r\right)^{1/r} \\ &\quad + \left(\sum_{k=0}^{\infty} \left\| \sum_{j=0}^{\infty} d_j^{n_k} \mathbf{1}_{\{\nu < j+1\}} \right\|_X^r\right)^{1/r} = (A) + (B). \end{aligned}$$

Using the definition of stopped martingales and (11) we get similarly to (9) that

$$\begin{aligned} \lambda((A) > \rho) &\leq \frac{1}{\rho^2} E((A)^2) \\ &\leq \frac{C_r}{\rho^2} E\left(\sum_{k=0}^{\infty} \left\| \sum_{j=0}^{\infty} (E_{j+1}(f^k \overline{w}_{n_k}) - E_j(f^k \overline{w}_{n_k})) \mathbf{1}_{\{\nu \geq j+1\}} \right\|_X^r\right)^{2/r} \\ &\leq \frac{C_r}{\rho^2} E\left(\sum_{k=0}^{\infty} \|E_{\nu}(f^k \overline{w}_{n_k})\|_X^r\right)^{2/r} \leq \frac{C_r}{\rho} E\left(\sum_{k=0}^{\infty} \|E_{\nu}(f^k \overline{w}_{n_k})\|_X^r\right)^{1/r} \\ &\leq \frac{C_r}{\rho} E\left(\sum_{k=0}^{\infty} \|f^k\|_X^r\right)^{1/r}. \end{aligned} \quad (12)$$

It is easy to see that $(B) = 0$ if $\nu = \infty$, and so $\{(B) > \rho\} \subset \{\nu < \infty\}$. Since $(\sum_{k=0}^{\infty} \|E_n(f^k \bar{w}_{n_k})\|_X^r)^{1/r}$ is a non-negative submartingale, we obtain

$$\lambda(\nu < \infty) \leq d\lambda\left(\sup_{\in \mathbb{N}} \left(\sum_{k=0}^{\infty} \|E_n(f^k \bar{w}_{n_k})\|_X^r\right)^{1/r} > \rho\right) \leq \frac{d}{\rho} E\left(\sum_{k=0}^{\infty} \|f^k\|_X^r\right)^{1/r}.$$

This together with (12) implies (6). $\square$

If we apply Theorem 3 for one k , only, then we get

Corollary 1. *If X is a UMD lattice and $f \in L_p(X)$ for some $1 < p < \infty$ then*

$$\|s_n f\|_{L_p(X)} \leq C_p \|f\|_{L_p(X)} \quad (\in \mathbb{N}) \quad (13)$$

and $s_n f \rightarrow f$ in $L_p(X)$ norm as $n \rightarrow \infty$.

The converse of this result easily follows from the proof of Theorem 3:

Theorem 4. *Assume that X is a Banach lattice. Inequality (13) holds for some (or equivalently for all) $1 < p < \infty$ if and only if X is UMD.*

PROOF. One can show that

$$T_j^n(w_n(E_{j+1}(f\bar{w}_n) - E_j(f\bar{w}_n))) = n_j(w_n(E_{j+1}(f\bar{w}_n) - E_j(f\bar{w}_n)))$$

if $n_j = 0$ or 1, where $n = \sum_{j=0}^{\infty} n_j P_j$, $0 \leq n_j < p_j$ (see WEISZ [23]). Consider only such numbers n for which $n_j = 0$ or 1 for each j . Then

$$\bar{w}_n s_n f = \sum_{j=0}^{\infty} n_j (E_{j+1}(f\bar{w}_n) - E_j(f\bar{w}_n)).$$

Inequality (13) implies

$$\left\| \sum_{j=0}^{\infty} n_j (E_{j+1}(f\bar{w}_n) - E_j(f\bar{w}_n)) \right\|_{L_p(X)} \leq C_p \|f\|_{L_p(X)}.$$

Writing $f\bar{w}_n$ instead of f we obtain

$$\left\| \sum_{j=0}^{\infty} n_j (E_{j+1}f - E_j f) \right\|_{L_p(X)} \leq C_p \|f\|_{L_p(X)}$$

and this implies that X is UMD (see (1)). $\square$

For other versions of this theorem see also WENZEL [25], TOZONI [20] and CLÉMENT et al. [5].

5. Almost everywhere convergence

It is known (see WEISZ [22]) that $s_n f \rightarrow f$ a.e. in X norm as $n \rightarrow \infty$, whenever $f \in L_p(X)$ for some $1 < p < \infty$. However, this does not hold for $L_1(X)$ or $H_1(X)$ even if $X = \mathbb{R}$ (see LADHAWALA and PANKRATZ [9] or SCHIPP and SIMON [17]). We say that an increasing sequence $(n_k, k \in \mathbb{N})$ of positive integers is *lacunary* if $n_{k+1}/n_k > \alpha > 1$ for all $k \in \mathbb{N}$. Now we are ready to prove our main result.

Theorem 5. *Assume that X is a UMD lattice and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers. If $f \in H_1(X)$ then*

$$\rho \lambda \left(\sup_k \|s_{n_k} f\|_X > \rho \right) \leq C \|f\|_{H_1(X)}, \quad (\rho > 0).$$

PROOF. It is well known that every lacunary sequence $(n_k, k \in \mathbb{N})$ can be split into a finite number of lacunary subsequences $(n_k^j, k \in \mathbb{N})$ with $n_{k+1}^j \geq d n_k^j$ ($k \in \mathbb{N}$). Thus we may assume that $P_k \leq n_k < P_{k+1}$. Then $s_{n_k} f = s_{P_k} f + s_{n_k}(d_k f)$, where $d_k f := s_{P_{k+1}} f - s_{P_k} f$. Since $(s_{P_k} f)$ is a martingale, Theorem 1 implies

$$\rho \lambda \left(\sup_k \|s_{P_k} f\|_X > \rho \right) \leq C \|f\|_{L_1(X)} \leq C \|f\|_{H_1(X)} \quad (\rho > 0). \quad (14)$$

On the other hand, by Theorem 3,

$$\begin{aligned} \rho \lambda \left(\sup_k \|s_{n_k}(d_k f)\|_X > \rho \right) &\leq \rho \lambda \left(\left(\sum_{k=0}^{\infty} \|s_{n_k}(d_k f)\|_X^q \right)^{1/q} > \rho \right) \\ &\leq C_q \left\| \left(\sum_{k=0}^{\infty} \|d_k f\|_X^q \right)^{1/q} \right\|_1 \end{aligned}$$

for all $\rho > 0$ and $1 < q < \infty$. If we take an atomic decomposition of f as in (2) then

$$d_k f = \sum_{j=0}^{\infty} \mu_j d_k a^j \quad \text{a.e. in } X \text{ norm.}$$

It is easy to show that

$$\rho \lambda \left(\sup_k \|s_{n_k}(d_k f)\|_X > \rho \right) \leq C_q \sum_{j=0}^{\infty} |\mu_j| E \left(\sum_{k=0}^{\infty} \|d_k a^j\|_X^q \right)^{1/q}. \quad (15)$$

Since every UMD lattice is superreflexive (see e.g. RUBIO DE FRANCIA [16]), X is q -concave for some $1 < q < \infty$. We may suppose that $q > 2$. Hence X has

cotype q (see LINDENSTRAUSS and TZAFRIRI [10]). This means that

$$\left(\sum_{k=0}^N \|d_k a^j\|_X^q \right)^{1/q} \leq C \int_0^1 \left\| \sum_{k=0}^N r_k(t) d_k a^j \right\|_X dt$$

for every $N, j \in \mathbb{N}$, where r_k denote now the original Rademacher functions with $p_n = 2$ ($\in \mathbb{N}$). If I_j denotes the support of the atom a^j , then we obtain by the UMD property and by the definition of the atom that for each fixed t ,

$$\begin{aligned} E \left\| \sum_{k=0}^N r_k(t) d_k a^j \right\|_X &= \int_{I_j} \left\| \sum_{k=0}^N r_k(t) d_k a^j \right\|_X d\lambda \\ &\leq \lambda(I_j)^{1/2} \left(\int_{I_j} \left\| \sum_{k=0}^N r_k(t) d_k a^j \right\|_X^2 d\lambda \right)^{1/2} \leq C \lambda(I_j)^{1/2} \left(\int_{I_j} \|a^j\|_X^2 d\lambda \right)^{1/2} \leq C. \end{aligned}$$

Now Theorem 2, (14) and (15) finishes the proof of the theorem. $\square$

By the usual density argument of Marcinkiewicz and Zygmund we obtain

Corollary 2. Assume that X is a UMD lattice and $(n_k, k \in \mathbb{N})$ is a lacunary sequence of positive integers. If $f \in H_1(X)$ then $\lim_{k \rightarrow \infty} s_{n_k} f = f$ a.e. in X norm.

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FERENC WEISZ
 DEPARTMENT OF NUMERICAL ANALYSIS
 EÖTVÖS L. UNIVERSITY
 PÁZMÁNY P. SÉTÁNY 1/C.
 H-1117 BUDAPEST
 HUNGARY
 E-mail: weisz@ludens.elte.hu

(Received May 3, 2006; revised November 13, 2006)