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IP sets, Hilbert cubes

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Abstract. Given a subset E of the set of natural numbers, $FS(E)$ is defined as the collection of all sums of elements of finite subsets of E and any translation of $FS(E)$ is said to be a Hilbert cube. We estimate the rate of growth of E given that $FS(E)$ avoids a set of multiplies of a given infinite set of primes. The results are related to a result which states that there exists an infinite Hibert cube contained in the set of square-free numbers.

1. Introduction

There is an interesting and permanent dialog between ergodic theory and combinatorial number theory. Important configurations in ergodic theory and in the combinatorial number theory as well are sets called IP-sets and Hilbert cubes. Given a subset E of the set of natural numbers $FS(E)$ is defined as the collection of all sums of elements of finite subsets of E (sometimes the empty set is excluded). If E is infinite then the set $FS(E)$ is said to be an IP-set (which is an important notion in ergodic theory). A more general configuration is a Hilbert cube which is a translation of $FS(E)$, i.e. if $E = \{x_1, x_2, \dots\}$ is a finite or infinite set of integers with $x_i \neq x_j$ whenever $i \neq j$ and a_0 is a given integer, then

$$H = H(a_0, x_1, x_2, \dots) = \Big\{ a_0 + \sum \varepsilon_i x_i : \varepsilon_i \in \{0,1\} \text{ and } \sum \varepsilon_i < \infty \Big\}$$

called a Hilbert-cube (briefly H-cube) or a combinatorial cube.

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The existence of a "big" H-cube in a dense set was a crucial point of Szemerédi's proof of his theorem on arithmetic progressions,. Many authors have investigated infinite H-cubes in special sets (see e.g. [HI], [HE-SÁ], [BE-RU].)

A stronger version of an IP-set will be an $\overline{\text{IP}}$-set. We define $\overline{FS(A)}$ the set of all sums of the form $\sum x_i a_i$ where x_i is an integer and fulfils the condition $0 \le x_i \le i$. Furthermore we assume that $\sum x_i < \infty$. If a set contains an $\overline{FS(A)}$ then it is said to be $\overline{\text{IP}}$-set.

It is reasonable to distinguish the notion of IP-set ($\overline{\text{IP}}$-set) and Hilbert cube. Clearly the set of odd numbers contains an infinite Hilbert cube, but does not contain an IP-set. (In fact an $FS(E)$ is a subset of the set of odd numbers if and only if $E = \{2k-1\}$ for some natural number k). An interesting result of E. G. Strauss is that for every $\varepsilon > 0$ there exists a sequence A for which $d(A) \ge 1 - \varepsilon$ but A does not contain an infinite Hilbert cube (see in [HI]). Therefore it is a harder job to show that a set does not contain an infinite Hilbert cube than to prove that a given set contains an IP-set.

In [BE-RU] Bergelson and Ruzsa proved the following interesting fact: Let A be the sequence of squarefree numbers. The set $A - a$ contains an IP-set if and only if $a \in A$. (Consequently, for every $a \in A$ there exists an infinite Hilbert cube $H = H(a, x_1, x_2, \dots)$ in A.) They derived this result from the following

Theorem A. *Let $X \subseteq \mathbb{N}$ be a set such that $1 \notin X$, any two elements of X are coprime, and*

$$\sum_{x \in X} 1/x < \infty. \tag{1}$$

Define $B^c(X)$ as the set of natural numbers that are not divisible by any element of X. Then $B^c(X)$ contains an IP-set.

The aim of this note is to investigate related questions. We can ask a generalization of this result as follows:

Let S be an arbitrary semigroup, and let A be a subset of it. Assume that there exists an n, such that any j-element subset of A, $1 \le j \le n$ does not generate S. Let $B(A) = \bigcup_{i \in I} \langle A_i \rangle$, where $\langle A_i \rangle$ is the semigroup generated by A_i, for every i the cardinality of A_i is at most n, and $\{A_i : i \in I\}$ is a (disjoint) partition of A.

Let $B^c(A) = S \setminus B(A)$. Our question is the following: for which A will contain $B^c(A)$ an infinite IP-set (Hilbert-cube)? If $S = \mathbb{N}$, and the A_i's are the one-element subsets of A then we obtain the original question of Bergelson and Ruzsa. In the present paper we are going to investigate the case when $S = \mathbb{N}^k$.

2. Notations

For $A \subseteq \mathbb{N}^k$ we define $B(A)$ as the set of all multiples of elements of A, i.e.,

$$B(A) = \bigcup_{a \in A} \langle a \rangle \cap \mathbb{N}^k = \{\lambda a : a \in A \text{ and } \lambda \in \mathbb{N}\}.$$

Let $B^c(A) = \mathbb{N}^k \setminus B(A)$ (the set of all vectors in $\mathbb{N}^k$ which are not a multiple of any element of A). $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{R}$ the set of real numbers. Let us denote by $d(P, Q)$ the Euclidean distance between $P, Q \in \mathbb{R}^k$. If the point Q is the origin, then we write $d(P)$. Given a set A, $|A|$ denotes the cardinality of A. The counting function $A(x)$ of $A \subseteq \mathbb{N}^k$ is defined by $A(x) = |\{a \in A : d(a) \leq x\}|$.

Let $A, B \subseteq \mathbb{R}^k$. The set addition of A and B is defined by $A + B = \{a + b : a \in A \text{ and } b \in B\}$. If $C \subseteq \mathbb{R}^k$ and $\lambda \in \mathbb{R}$, then let $\lambda C = \{\lambda c : c \in C\}$.

3. A result in $\mathbb{N}^k$

In the present section we prove a k-dimensional analogue of Bergelson–Ruzsa's result.

Theorem 1. *Let $A \subseteq \mathbb{N}^k$, let $k \geq 2$, and assume that*

$$\liminf_{x \to \infty} \frac{A(x)}{x^{k-1}} = 0. \tag{2}$$

Then $B^c(A)$ contains an $\overline{IP}$-set.

Proof of Theorem 1. Our task is to find an $\overline{\text{IP}}$-set containing in $B^c(A)$. By (2) we have that $B^c(A)$ is a non-empty set, therefore we can select a $\vec{x_1} \in B^c(A)$. Furthermore let us assume that the set $\overline{FS(X)}$ has been defined and we have $\overline{FS(X)} = \{\alpha_1\vec{x_1} + \alpha_2\vec{x_2} + \cdots + \alpha_n\vec{x_n} : \alpha_i \in \mathbb{N} \text{ and } \alpha_i \leq i\} \subseteq B^c(A)$.

Let $M = M(n) = \sum_{1 \leq i \leq n} i \cdot d(\vec{x_i})$. Consider a k-dimensional ball

$$G_k(M) = \{\vec{x} : \vec{x} \in \mathbb{R}^k \text{ and } d(\vec{x}) \leq M\},$$

By the definition of M we conclude that $\overline{FS(X)}$ is a subset of $G_k(M) \cap \mathbb{N}^k$. By (2) we have that there exists an $x > 2n \cdot M$ for which

$$A(x) \leq c_1 \left(\frac{x}{M}\right)^{k-1}$$

holds, where c_1 will be specified later. Let S be a $k-1$ dimensional surface of a sphere with radius x, i.e. $S=\{\vec{z}:\vec{z}\in\mathbb{R}^k \text{ and } d(\vec{z})=x\}$. Set a dense packing of S by $(k-1)$-dimensional balls with radius $2nM$, where packing means that any two different balls have at most one common point, and dense means that the volume of the union of the balls is a positive proportion of the volume of S, i.e. if $G_1,G_2,\dots,G_s$ are the balls then there is a constant $c(k)=c>0$ such that

$$c\cdot \mathrm{vol}(S)<\mathrm{vol}\,(\cup_{i=1}^s G_i)=\sum_{i=1}\mathrm{vol}(G_i)=s\cdot(c_2M^{k-1}),$$

where c_2 depends only on k. Hence we obtain

$$s>\frac{c\cdot \mathrm{vol}(S)}{c_2M^{k-1}}=c_3\left(\frac{x}{M}\right)^{k-1},$$

where c_3 depends only on k.

The convex hull of the origin and a G_i is said to be *cell* and denoted by C_i. The number of cells is at least

$$c_3 l\left(\frac{x}{M}\right)^{k-1}.$$

Since $A(x)\le c_1\left(\frac{x}{M}\right)^{k-1}$, if $c_1<c_3$ then there is a cell C which does not contain an element from A. We claim that the inside of C and $B(A)$ are disjoint sets. Assume now contrary to the assumption that there is a $\vec{b}\in B(A)$ and $\vec{b}$ lies in the inside of C. Since $\vec{b}\in B(A)$ we have that there is a positive integer t such that $\vec{b}/t\in A$. But C is a convex set thus $\vec{b}/t$ lies in C which contradicts to the fact that C and A are disjoint sets.

Dilate now C into $\frac{1}{n+1}C$. By the definition of C we infer that $\frac{1}{n+1}C$ contains a k-dimensional ball G^* with radius M and center from $\mathbb{N}^k$. Let x_{n+1} be the center of G^*.

Recall that

$$\overline{FS(X)}=\{\alpha_1\vec{x_1}+\alpha_2\vec{x_2}+\dots+\alpha_n\vec{x_n}:\alpha_i\in\mathbb{N};\ 0\le\alpha_i\le i\}$$

and clearly

$$\overline{FS(X\cup\{x_{n+1}\})}=\bigcup_{i=0}^{n+1}(\overline{FS(X)}+i\cdot x_{n+1}).$$

Consider the sequence of balls

$$G^*=G_1^*;G_2^*;\dots;G_{n+1}^*,$$

where the radius of G_i^* is M and the center is $i \cdot x_{n+1}$. Note that for all $G_i^* \subseteq C$. Furthermore observe that for each i, $1 \leq i \leq n+1$

$$i \cdot x_{n+1} + \overline{FS(X)} \subseteq G_i^*,$$

and hence the set of union of these sets also lies in C. It means that

$$\bigcup_{i=1}^{n+1} (\overline{FS(X)} + i \cdot x_{n+1}) \cap B(A) = \emptyset,$$

i.e.

$$\overline{FS(X \cup \{x_{n+1}\})} \cap B(A) = \emptyset,$$

as we wanted.

Hence the theorem. □

4. A result in $\mathbb{N}$

In the present paragraph we shall investigate how sharp is the Bergelson–Ruzsa's theorem. What happens in Theorem A if we replace a convergent series by a divergent one; what can we say about $B^c(A)$? Trivially for the sequence of primes, which forms a divergent series, the set $B^c(A)$ will not contain an infinite Hilbert cube since it will be the empty set. But on the other hand it is not completely clear that one can leave *infinitely* many primes from P, leaving P' such that $B^c(P')$ does not contain an Hilbert cube. First we shall prove:

Proposition 1. *There exists a subset P' of the set of primes P for which $|P \setminus P'| = \infty$, and $B^c(P')$ does not contain an infinite Hilbert cube.*

Furthermore we give an estimation that how "big" could be a Hilbert cube contained in $B^c(A)$, where A is a subsequence of all primes. So let $H = H(a_0, x_1, x_2, \dots)$ be a Hilbert cube in $B^c(A)$ and let

$$H_A(n) = H(n) = |\{x_1, x_2, \dots\} \cap [1, n]|,$$

We prove

Theorem 2. *Let $\langle p_i \rangle_{i=1}^{\infty}$ be an increasing sequence of primes and let $A = \{p_i : i \in \mathbb{N}\}$. Assume that there is a Hilbert cube $H(a_0, x_1, x_2, \dots) \subseteq B^c(A)$. Then for each $n \in \mathbb{N}$,*

$$H(n) < 8 \sum_{i=1}^{f(n)} p_i^{3/2}, \tag{3}$$

where $f(n)$ is the smallest s for which $p_1 p_2 \dots p_s \geq n$.

We apply Theorem 2 in the following case:

Corolary. *Let $\alpha > 1$ and let $P' = \langle p_i \rangle_{i=1}^{\infty}$ be an increasing sequence of primes for which $\lim_{i\to\infty} \frac{p_i}{i^\alpha} = 1$. Then we have*

$$H_{P'}(n) < c(\alpha) \left(\frac{\log n}{\log\log n} \right)^{\frac{3\alpha+2}{2}}.$$

Proof of the Corollary. First let us note that for every $\alpha > 1$ there is a sequence P' of primes for which $\lim_{i\to\infty} \frac{p_i}{i^\alpha} = 1$. Indeed from the Prime Number Theorem we have that $q_i \sim i \cdot \log i$, where q_i is the i^{th} prime in the sequence of all primes. Now selecting an arbitrary sequence $i_j \sim \frac{j^\alpha}{\alpha \log j}$, we obtain

$$p_j = q_{i_j} \sim i_j \cdot \log i_j \sim j^\alpha.$$

Now using the Stirling's formula we get that $f(n) = c_1 \frac{\log n}{\log\log n}$, where c_1 depends only on α. By Theorem 2 we have

$$H_{P'}(n) < 8 \sum_{i=1}^{f(n)} p_i^{3/2} < c_1 \int_{i=1}^{f(n)} i^{3\alpha/2} < c(\alpha) \left(\frac{\log n}{\log\log n} \right)^{\frac{3\alpha+2}{2}}. \qquad \square$$

Remark. The related question of Bergelson and Ruzsa is the following: for a given sequence A of primes $B(A) = \{k \cdot p : \ p \in A \text{ and } k \in \mathbb{N}\}$ and the complement of $B(A)$ is the set of all integers which are composed solely of the primes not in A. In the Proposition we show the existence an infinite cube in $B^c(A)$.

We can ask an opposite question as well: Let A' be any sequence of prime numbers and let $\prod(A')$ be the set of all integers can be divided by primes only from A. Observe that $\prod(A')$ is a *generalized* multiplicative $\overline{\text{IP}}$-set. Then we ask: what is the maximal additive Hilbert cube which avoids $\prod(A')$? We shall return to this question in another paper.

Proof of Proposition 1. For the proof we need a lemma which is a special case of a deep result of R. Tijdeman (see in [T]).

Define by $U(P'')$ the set of all integers which are composed solely of the primes of P'', i.e. $U(P'') = \{n : p|n \Rightarrow p \in P''\}$ and let $\langle n_i \rangle_{i=1}^{\infty}$ enumerate $U(P'')$ in increasing order. $\square$

Lemma 1. *There exists an infinite sequence P'' of primes for which*

$$n_{i+1} - n_i > \sqrt{n_i}. \tag{4}$$

Note that Lemma 1 also implies that

$$n_i - n_{i-1} > \frac{\sqrt{n_i}}{2}. \tag{5}$$

Indeed if $n_i \le 4n_{i-1}$ then $n_i - n_{i-1} > \sqrt{n_{i-1}} \ge \frac{\sqrt{n_i}}{2}$ or $n_i > 4n_{i-1}$ and then $n_i - n_{i-1} > \frac{3}{4}n_i > \frac{\sqrt{n_i}}{2}$.

Now let $P' = P \setminus P''$. By Lemma 1 we get that $|P \setminus P'| = |P''| = \infty$. By the definition of $B(P')$, an integer m is an element of $B(P')$ if m has a prime divisor from the set P'. Hence $B^c(P')$ collects all integers composed solely of the primes P'', i.e. $m \in B^c(P')$ if and only if $m = \prod_{p\in P''} p^\alpha$ and hence $B^c(P') = U$.

Now assume contrary to the assertion that $B^c(P')$ contains an infinite Hilbert cube, say $H = H(a_0, x_1, x_2, \dots)$. Choose an element h from H, for which $h > 4x_1^2$. Since H is infinite such an element exists. By (4) and (5) we obtain

$$\min\{|m - h| : m \in B^c(P')\} > \frac{\sqrt{h}}{2} > x_1. \tag{6}$$

But h can be expressed as $h = h' + \varepsilon_1 x_1$, where $\varepsilon_1 = 0$ or $\varepsilon_1 = 1$, and x_1 is not a term of h'. If $\varepsilon_1 = 0$ then $h + x_1 \in H$, if $\varepsilon_1 = 1$ then $h - x_1 \in H$. Both cases contradict (6).

Proof of Theorem 2. Assume that there exists an infinite cube $H = H(a_0, x_1, x_2, \dots) \subseteq B^c(A)$. We shall prove

$$H(n) < 8 \sum_{i=1}^{f(n)} p_i^{3/2}.$$

Now for every $k \le f(n)$ write $p_k = Mp + r,\ 0 < r < p$, where p is an arbitrary prime less than p_k. Clearly there exists a positive integer r' such that

$$r \cdot r' = p \cdot s + a_0,$$

for some $s \in \mathbb{N}$. Since $p_k \in A$ we obtain that for every $w \in \mathbb{N}$, $(pw + r')p_k \in B(A)$ and hence

$$(pw + r')p_k = p(p_k w + r'M + s) + a_0.$$

Define H_k the set of all elements of $\{x_1 < x_2 < \dots\} \cap [1, n]$ which are not divisible by p_k. Thus if

$$x_u \notin \bigcup_{k=1}^{f(n)} H_k$$

then we conclude

$$p_1p_2\cdots p_k \mid x_u \;\Rightarrow\; x_u > n,$$

i.e. all elements of $\{x_1, x_2, \dots\} \setminus \cup_{k=1}^{f(n)} H_k$ are bigger than n. Hence we have

$$H(n) \le \sum_{k=1}^{f(n)} |H_k|.$$

We give an upper estimation for H_k. Let $y_1 < y_2 < \dots y_u \le n$ be the elements of H_k. We shall prove $u \le 4p_k^{3/2}p$. Assume contrary to this assertion that $u > 4p_k^{3/2}p$. Split the set $\{y_1 < y_2 < \cdots < y_u\}$ into $2p$ many pairwise disjoint sets with cardinality at least $2p_k^{3/2}$. Denote them by $Y_1, Y_2, \dots Y_m, \dots Y_{2p}$. We claim that it is enough to prove that for each set Y_m there exists an n_m such that

$$p_k n_m + r'M + s \in FS(Y_m). \tag{7}$$

Indeed by the Erdős–Ginzburg–Ziv theorem (see [EGZ]) we can select p elements from the set $\{n_m\}_{1\le m\le 2p}$ having a sum $\equiv 0 \pmod p$ i.e.

$$\sum_{j=1}^{p}(p_k n_{m_j} + r'M + s) = p(p_k \cdot w + r'M + s),$$

with some integer w. But $(pw + r')p_k = p(p_k w + r'M + s) + a_0 \in B(A) \cap H(a_0, x_1, x_2 \dots)$ a contradiction.

Let $Y_m = Y = \{y_1, y_2, \dots, y_s\}$, $s \ge 2p^{3/2}$. We distinguish two cases; If there is an $r \not\equiv 0 \pmod{p_k}$ such that $y_i \equiv r \pmod{p_k}$ has at least p_k solution, then for some z, $z \cdot r \equiv r'M + s \pmod{p_k}$, which implies (5).

If for any $r \not\equiv 0 \pmod{p_k}$ $y_i \equiv r \pmod{p_k}$ has at most $p_k - 1$ solution, then we can select $Y' = \{y_{i_1}, \dots, y_{i_T}\} \subseteq Y$, for which the elements of Y' are pairwise incongruent and $T > 2\sqrt{p_k}$. □

Lemma 2. *Assume $Y' = \{y_{i_1}, \dots, y_{i_T}\}$ a set of integers for which the elements of Y' are pairwise incongruent and $T > 2\sqrt{p_k}$. Then $\sum(Y')$ intersects every residue classes $\pmod{p_k}$.*

This is a theorem of Olson [OL].

But it gives again (5).

Finally note that if $X \subseteq X'$ then $B^c(X') \subseteq B^c(X')$, hence without loss of generality $2 < p_1$.

We obtained $H_k \le 4p_k^{3/2}p$, and since this argument holds for all prime p for the smallest we get

$$H_k \le 8p_k^{3/2},$$

which proves the theorem.

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