

Sufficient conditions for starlikeness of order α

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Abstract. In this paper we obtain some sufficient conditions for an analytic function to be starlike of order α , by using the differential operator recently introduced by F. Al-Oboudi. For such classes we also give some applications of a result due to M. Robertson.

1. Introduction and preliminaries

Let $H(U)$ be the space of all analytic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N} = \{1, 2, \dots\}$ let define the class of functions

$$A_n = \left\{ f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, \quad z \in U \right\}.$$

Let $A \equiv A_1$ and let S denotes the subclass of A consisting in those functions that are univalent in U .

A function $f \in A$ is called to be a *starlike function of order α* , if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in U,$$

where $\alpha < 1$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$; we write $S^* \equiv S^*(0)$ and, moreover $S^*(\alpha) \subset S$ for $0 \leq \alpha < 1$.

We mention that the class of all functions $f \in A_n$ that satisfy the above inequality is denoted by $S_n^*(\alpha)$, that is $S_n^*(\alpha) = S^*(\alpha) \cap A_n$.

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Recently, F. AL-BOUDI defined in [Ob04] the differential operator $D_\lambda^m : H(U) \rightarrow H(U)$ by

$$\begin{aligned} D_\lambda^0 f(z) &= f(z), \\ D_\lambda^1 f(z) &= (1 - \lambda)f(z) + \lambda z f'(z), \end{aligned} \quad (1.1)$$

$$D_\lambda^m f(z) = D_\lambda^1 (D_\lambda^{m-1} f(z)). \quad (1.2)$$

If $f \in A_n$, then from (1.1) and (1.2) we may easily deduce that

$$D_\lambda^m f(z) = z + \sum_{k=n+1}^{\infty} [1 + (k-1)\lambda]^m a_k z^k, \quad (1.3)$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$. Remark that, for $\lambda = 1$ we get the operator introduced by GR. ȘT. SĂLĂGEAN in [Sal83].

Definition 1.1. Let $S^m(n, \lambda, \alpha)$ denotes the class of functions $f \in A_n$ which satisfy the condition

$$\operatorname{Re} \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} > \alpha, \quad z \in U, \quad (1.4)$$

for some $\alpha < 1$, $\lambda \geq 0$ and $m \in \mathbb{N}$.

In order to prove that all the functions of $S^m(n, \lambda, \alpha)$ are univalent in U , first we will show an inclusion and a sharp inclusion relation between these classes.

To prove our main results we will need the next definition and lemmas.

If $f, g \in H(U)$ we say that the function f is *subordinate* to g , or g is *superordinate* to f , if there exists a function w analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. In such a case we write $f(z) \prec g(z)$.

Remark that, if g is univalent in U , then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

The next lemma represents a result concerning the generalized *Libera integral operator* introduced by S. D. BERNARDI in [Ber69], which shows that this operator preserves the starlikeness, the convexity and the close-to-convexity. We will give now only a part of the original form.

Lemma 1.1 ([LewMiZl76], [Pa79]). *If $L_c : A \rightarrow A$ is the integral operator defined by $L_c(f) = F$, where*

$$F(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt,$$

and $\operatorname{Re} c \geq 0$, then $L_c(S^) \subset S^*$.*

The next two lemmas deal with the so called *Briot–Bouquet* differential subordinations:

Lemma 1.2 ([EeMiMoRe83, Theorem 1]). *Let $\beta, \gamma \in \mathbb{C}$ and let h be a convex function in U with $\operatorname{Re} [\beta h(z) + \gamma] > 0$, $z \in U$. If p is analytic in U , with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 1.3 ([MiMo00, Theorem 3.3e]). *Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator $I_{\beta, \gamma}$ defined by*

$$I_{\beta, \gamma}(f)(z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma-1} dt \right]^{1/\beta}.$$

If $\alpha \in [-\frac{\gamma}{\beta}, 1)$ then the order of starlikeness of the class $I_{\beta, \gamma}(S_n^(\alpha))$, i.e. the largest number $\delta = \delta_n(\alpha; \beta, \gamma)$ such that $I_{\beta, \gamma}(S_n^*(\alpha)) \subset S_n^*(\delta)$, is given by the number $\delta_n(\alpha; \beta, \gamma) = \inf \{\operatorname{Re} q_n(z) : z \in U\}$, where*

$$q_n(z) = \frac{1}{\beta Q_n(z)} - \frac{\gamma}{\beta} \quad \text{and} \quad Q_n(z) = \frac{1}{n} \int_0^1 \left(\frac{1-z}{1-tz} \right)^{\frac{2\beta(1-\alpha)}{n}} t^{\frac{\beta+\gamma}{n}-1} dt.$$

Moreover, if $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \max \left\{ \frac{\beta-\gamma-n}{2\beta}; -\frac{\gamma}{\beta} \right\}$ and $g = I_{\beta, \gamma}(f)$ with $f \in S_n^(\alpha)$, then*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > \delta_n(\alpha; \beta, \gamma) = \frac{\beta + \gamma}{{}_2F_1\left(1, \frac{2\beta(1-\alpha)}{n}, \frac{\beta+\gamma+n}{n}; \frac{1}{2}\right) \cdot \beta} - \frac{\gamma}{\beta}, \quad z \in U,$$

where ${}_2F_1$ represents the hypergeometric function.

Lemma 1.4 ([MiMo87]). *Let $\Omega \subset \mathbb{C}$, and suppose that the mapping $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for $z \in U$, and for all $x, y \in \mathbb{R}$ such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1 + c_n z^n + \dots$ is analytic in U and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$, then $\operatorname{Re} p(z) > 0$, for all $z \in U$.*

For the result presented in the last section, we will need the next lemma of M. Robertson.

Lemma 1.5 ([Rob61]). *Let $F : U \times [0, +\infty) \rightarrow \mathbb{C}$ be an analytic function in the unit disc U for all $0 \leq t \leq 1$, with $F(0, t) = 0$ for all $0 \leq t \leq 1$. Suppose that $F(\cdot, 0) = f \in S$, and let $p > 0$ a such number for which it exists*

$$F(z) = \lim_{t \rightarrow 0} \frac{F(z, t) - F(z, 0)}{zt^p}.$$

If $F(z, t) \prec f(z)$ for all $0 \leq t \leq 1$, then

$$\operatorname{Re} \frac{F(z)}{f'(z)} \leq 0, \quad z \in \mathbb{U}.$$

If in addition, F is also analytic in the unit disc $\mathbb{U}$ and $\operatorname{Re} F(0) \neq 0$, then

$$\operatorname{Re} \frac{F(z)}{f'(z)} < 0, \quad z \in \mathbb{U}.$$

2. Inclusion relations between the $S^m(n, \lambda, \alpha)$ subclasses

Theorem 2.1. 1) For all $0 \leq \lambda \leq 1$ and $1 - \lambda \leq \alpha < 1$, the inclusion

$$S^{m+1}(n, \lambda, \alpha) \subseteq S^m(n, \lambda, \alpha). \quad (2.1)$$

holds for all $m \in \mathbb{N}$.

2) If $0 < \lambda \leq 1$ and $1 - \lambda \leq \alpha < 1$, then the inclusion

$$S^{m+1}(n, \lambda, \alpha) \subseteq S^m(n, \lambda, \beta(n, \lambda, \alpha)), \quad (2.2)$$

where

$$\beta(n, \lambda, \alpha) = \frac{1}{{}_2F_1\left(1, \frac{2(1-\alpha)}{n\lambda}, \frac{1+n\lambda}{n\lambda}; \frac{1}{2}\right)},$$

is sharp and holds for all $m \in \mathbb{N}$.

PROOF. For the special case $\lambda = 0$, since $D_0^m f(z) = f(z)$, $z \in \mathbb{U}$, for all $m \in \mathbb{N}_0$, we have the equality $S^{m+1}(n, 0, \alpha) = S^m(n, 0, \alpha)$, $m \in \mathbb{N}$.

Let now consider the case $\lambda > 0$. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)}, \quad (2.3)$$

then $p(0) = 1$, and the first step of our proof is to show that $p \in H(\mathbb{U})$.

According to the definition (1.4), if $f \in S^{m+1}(n, \lambda, \alpha)$ then $f \in A_n$ and

$$\operatorname{Re} \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} > \alpha, \quad z \in \mathbb{U}. \quad (2.4)$$

If we denote by $H(z) = D_\lambda^m f(z)$ and using the definitions (1.1) and (1.2), the inequality (2.4) becomes

$$\operatorname{Re} \frac{zH'(z)}{H(z)} > \frac{\alpha + \lambda - 1}{\lambda}, \quad z \in \mathbb{U}. \quad (2.5)$$

From (1.3) we have $H(0) = H'(0) - 1 = 0$, and combining this with the inequality (2.5) we obtain that $H \in S^*$, whenever $1 - \lambda \leq \alpha < 1$.

Denoting by $h(z) = D_\lambda^{m-1}f(z)$, where $m \in \mathbb{N}$, then $h(0) = h'(0) - 1 = 0$, and from (1.1) and (1.2) we get

$$(1 - \lambda)h(z) + \lambda zh'(z) = H(z). \quad (2.6)$$

For $\lambda = 1$ the above differential equation has the solution

$$h(z) = \int_0^z \frac{H(t)}{t} dt,$$

where $H \in S^*$. From the well-known result concerning the *Alexander integral operator* we deduce that h is convex in $\mathbb{U}$, so is a univalent function in $\mathbb{U}$.

For $0 < \lambda < 1$, the relation (2.6) becomes

$$h(z) + \frac{\lambda}{1 - \lambda} zh'(z) = \frac{H(z)}{1 - \lambda}, \quad (2.7)$$

where $H \in S^*$. It is easy to see that the differential equation (2.7) has the solution

$$h(z) = \frac{1}{\lambda} \frac{1}{z^{\frac{1}{\lambda}-1}} \int_0^z H(t) t^{\frac{1}{\lambda}-2} dt = L_c(H)(z),$$

where $c = \frac{1}{\lambda} - 1$. Since $0 < \lambda < 1$, then $\operatorname{Re} c \geq 0$, and from Lemma 1.1 it follows that $h \in S^*$, so h is a univalent function in $\mathbb{U}$.

From the above results we conclude that h is a univalent function in $\mathbb{U}$ with the single zero $z_0 = 0$, i.e. $D_\lambda^{m-1}f(0) = 0$, $(D_\lambda^{m-1}f)'(0) = 1 \neq 0$ and $D_\lambda^{m-1}f(z) \neq 0$ for all $z \in \dot{\mathbb{U}} \equiv \mathbb{U} \setminus \{0\}$, hence we conclude that the function p defined by (2.3) is analytic in $\mathbb{U}$.

The inequality (2.4) together with (1.1) and (1.2) shows that

$$p(z) + \lambda \frac{zp'(z)}{p(z)} = \frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)} \prec h(z) = \frac{1 + (1 - 2\alpha)z}{1 + z}.$$

In the case $\lambda > 0$, according to Lemma 1.2 for $\beta := 1/\lambda$ and $\gamma := 0$, and using the fact that

$$\operatorname{Re} [\beta h(z) + \gamma] > \frac{\alpha}{\lambda} \geq 0, \quad z \in \mathbb{U},$$

we deduce that $p(z) \prec h(z)$, i.e. $f \in S^m(n, \lambda, \alpha)$.

To prove the second part of the theorem we will use Lemma 1.3 for the special case $\beta := 1/\lambda$ and $\gamma := 0$. We see that it is necessary to have $\alpha \in [\alpha_0, 1)$ and $1 - \lambda \leq \alpha < 1$, where $\alpha_0 \equiv \max \left\{ \frac{1-n\lambda}{2}; 0 \right\}$, hence

$$1 > \alpha \geq \max \left\{ \frac{1-n\lambda}{2}; 1-\lambda; 0 \right\} = 1-\lambda.$$

Since the conditions of Lemma 1.3 are satisfied, we obtain the sharp bound

$$\operatorname{Re} p(z) > \delta_n(\alpha; \beta, \gamma) = \beta(n, \lambda, \alpha) = \frac{1}{{}_2F_1\left(1, \frac{2(1-\alpha)}{n\lambda}, \frac{1+n\lambda}{n\lambda}; \frac{1}{2}\right)}, \quad z \in \mathbb{U},$$

that is $f \in S^m(n, \lambda, \beta(n, \lambda, \alpha))$. □

Considering in the above theorem the special case $n = 1$, for $\lambda = 1$ we need to have that $0 \leq \alpha < 1$. For $\alpha = 0$, since

$$\delta_1\left(\frac{\beta - \gamma - 1}{2\beta}; \beta, \gamma\right) = \frac{\beta - \gamma}{2\beta}$$

we get $\beta(1, 0, \lambda) = 1/2$. Taking in the relation (2.2) of Theorem 2.1 the special case $\alpha = 0$ we obtain the next result:

Corollary 2.1. *The inclusion*

$$S^{m+1}(1, 1, 0) \subseteq S^m\left(1, 1, \frac{1}{2}\right)$$

is sharp and holds for all $m \in \mathbb{N}$.

3. Sufficient conditions for starlikeness

Recently, LI and OWA [LiOw02] obtained the following result: if $f \in A$ satisfies

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > \frac{-\alpha}{2}, \quad z \in \mathbb{U},$$

for some $\alpha \geq 0$, then $f \in S^*$.

In fact, LEWANDOWSKI, MILLER and ZŁOTKIEWICZ in [LewMiZl76] and RAMESHA, KUMAR and PADMANABHAN in [RaKuPa95] have proved the next weaker form of this theorem: if $f \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho, \quad z \in \mathbb{U},$$

where $\rho = 2.2443697$, then $f \in S^*$.

The above result with $\rho = 3/2$ and $\rho = 1/6$ were earlier proved by LI and OWA in [LiOw98] and OBRADOVIĆ and RUSCHEWEYH in [ObRu92] respectively. Also, RAVICHANDRAN, SELVARAJ and RAJALAKSMI in [RaSeRa02] obtained some sufficient condition for functions in A_n to be starlike of order β .

We will obtain some other sufficient condition for functions to be starlike of order α , by using the differential operator D_λ^m already defined by (1.1) and (1.2).

Theorem 3.1. *Let $\alpha \geq 0$, $\beta < 1$, $m \in \mathbb{N}$ and $\lambda \geq 0$. If the function $f \in A_n$ satisfies*

$$\begin{aligned} \operatorname{Re} \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} \left[\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} + (1 - \alpha) \right] &> \alpha \beta \left(\beta + \frac{n\lambda}{2} - 1 \right) \\ &+ \left(\beta - \frac{\alpha\lambda n}{2} \right), \quad z \in \mathbb{U}, \end{aligned}$$

then $f \in S^m(n, \lambda, \beta)$.

PROOF. Let define the function p by

$$p(z) = \frac{1}{1 - \beta} \left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - \beta \right).$$

From the assumption it follows $p \in H(\mathbb{U})$ with $p(z) = 1 + c_n z^n + \dots$, and a simple computation shows that

$$\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} = \frac{\alpha\lambda(1 - \beta)zp'(z)}{(1 - \beta)p(z) + \beta} + \alpha[(1 - \beta)p(z) + \beta].$$

Hence

$$\begin{aligned} \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} \left[\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - (1 - \alpha) \right] &= \alpha(1 - \beta)\lambda zp'(z) + \alpha(1 - \beta)^2 p^2(z) \\ &+ (1 - \beta)(2\alpha\beta + 1 - \alpha)p(z) + \beta(\alpha\beta + 1 - \alpha) = \psi(p(z), zp'(z); z), \end{aligned}$$

where

$$\begin{aligned} \psi(r, s; t) &= \alpha\lambda(1 - \beta)s + \alpha(1 - \beta)^2 r^2 + (1 - \beta)(2\alpha\beta + 1 - \alpha)r^2 \\ &+ \beta(\alpha\beta + 1 - \alpha). \end{aligned}$$

For all $x, y \in \mathbb{R}$ satisfying $y \leq -n(1 + x^2)/2$ we have the inequalities

$$\begin{aligned} \operatorname{Re} \psi(ix, y; z) &= \alpha\lambda(1 - \beta)y - \alpha(1 - \beta)^2 x^2 + \beta(\alpha\beta + 1 - \alpha) \\ &\leq \frac{-n\lambda\alpha}{2}(1 - \beta) - \left[\frac{n\lambda\alpha}{2}(1 - \beta) + \alpha(1 - \beta)^2 \right] x^2 + \beta(\alpha\beta + 1 - \alpha) \end{aligned}$$

$$\leq \beta(\alpha\beta + 1 - \alpha) - \frac{n\lambda\alpha}{2}(1 - \beta) = \alpha\beta \left(\beta + \frac{n\lambda}{2} - 1 \right) + \left(\beta - \frac{n\lambda\alpha}{2} \right).$$

If we let

$$\Omega = \left\{ \omega \in \mathbb{C} : \operatorname{Re} \omega > \alpha\beta \left(\beta + \frac{n\lambda}{2} - 1 \right) + \left(\beta - \frac{\lambda n\alpha}{2} \right) \right\},$$

then $\psi(p(z), zp'(z); z) \in \Omega$ and $\psi(ix, y; z) \notin \Omega$, for all $x, y \in \mathbb{R}$ with $y \leq -n(1+x^2)/2$ and for all $z \in \mathbb{U}$, hence by applying Lemma 1.4 we obtain the required result. $\square$

Combining the above result together with the inclusion (2.1) of Theorem 2.1 we get the next corollary:

Corollary 3.1. *Let $0 \leq \lambda \leq 1$, $1 - \lambda \leq \alpha < 1$, $\beta < 1$ and $m \in \mathbb{N}$. If $f \in A_n$ satisfies*

$$\begin{aligned} \operatorname{Re} \left\{ \lambda \frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} \left[\alpha \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} + 1 - \alpha \right] + 1 - \lambda \right\} \\ > \lambda\alpha\beta \left(\beta + \frac{n\lambda}{2} - 1 \right) + \left(\lambda\beta - \frac{\lambda^2 n\alpha}{2} - \lambda + 1 \right), \quad z \in \mathbb{U}, \end{aligned}$$

then $f \in S^*(\rho)$, where $\rho = \frac{\beta - (1-\lambda)}{\lambda}$.

Taking $m = 1$ in Theorem 3.1, we obtain the following implication:

Corollary 3.2 ([RaSeRa02]). *Let $\alpha \geq 0$ and $\beta < 1$. If $f \in A_n$ satisfies*

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > \alpha\beta \left(\beta + \frac{n}{2} - 1 \right) + \left(\beta - \frac{\alpha n}{2} \right), \quad z \in \mathbb{U},$$

then $f \in S^*(\beta)$.

If we take in this corollary $\beta = \alpha/2$ and $n = 1$, we deduce the next result:

Corollary 3.3 ([LiOw02]). *Let $0 \leq \alpha < 2$. If $f \in A$ satisfies*

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > -\frac{\alpha^2}{4}(1 - \alpha), \quad z \in \mathbb{U},$$

then $f \in S^*\left(\frac{\alpha}{2}\right)$.

Now we shall prove another sufficient condition for a function $f \in A_n$ to be in the class $S^m(n, \lambda, \beta)$.

Theorem 3.2. Let $\lambda \geq 0$, $0 \leq \beta < 1$ and suppose that the numbers

$$a = \left(\frac{\lambda n}{2} + 1 - \beta \right)^2 \quad \text{and} \quad b = \left(\frac{\lambda n}{2} + \beta \right)^2 \quad (3.1)$$

satisfy the inequality

$$(a + b)\beta^2 < b(1 - 2\beta). \quad (3.2)$$

If t_0 is the positive root of the equation

$$2a(1 - \beta)^2 t^2 + [3a\beta^2 + b(1 - \beta)^2] t + [(\alpha + 2b)\beta^2 - (1 - \beta)^2 b] = 0,$$

let denote

$$\rho = \sqrt{\frac{(1 - \beta)^3 (1 + t_0)^2 (at_0 + b)}{\beta^2 + (1 - \beta)^2 t_0}}.$$

If $f \in A_n$ satisfies

$$\left| \left(\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - 1 \right) \left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - 1 \right) \right| < \rho, \quad z \in U, \quad (3.3)$$

then $f \in S^m(n, \lambda, \beta)$, where $m \in \mathbb{N}$.

PROOF. Let define the function $p \in H(U)$ by

$$p(z) = \frac{1}{1 - \beta} \left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - \beta \right).$$

With this notation it follows that

$$\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - 1 = \frac{(1 - \beta)\lambda z p'(z) + [(1 - \beta)p(z) + \beta]^2 - [(1 - \beta)p(z) + \beta]}{(1 - \beta)p(z) + \beta},$$

hence

$$\begin{aligned} & \left(\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - 1 \right) \left(\frac{D_\lambda^m f(z)}{D_\lambda^{m-1} f(z)} - 1 \right) = \frac{(1 - \beta)(p(z) - 1)}{(1 - \beta)p(z) + \beta} \\ & \cdot \{ (1 - \beta)\lambda z p'(z) + [(1 - \beta)p(z) + \beta]^2 - [(1 - \beta)p(z) + \beta] \} = \psi(p(z), zp'(z); z). \end{aligned}$$

Now, for all $x, y \in \mathbb{R}$ satisfying $y \leq -n(1 + x^2)/2$, we have

$$\begin{aligned} |\psi(ix, y; z)|^2 &= \frac{(1 - \beta)^2(1 + t)}{\beta^2 + (1 - \beta)^2 t} \cdot \{ [(1 - \beta)\lambda y - \beta + \beta^2 - (1 - \beta)^2 t]^2 \\ &+ [2\beta(1 - \beta) - (1 - \beta)]^2 t \} = g(t, y), \end{aligned}$$

where $t = x^2$ and $y \leq -n(1+t)/2$. If $\lambda \geq 0$ and $0 \leq \beta < 1$, since

$$\begin{aligned} \frac{\partial g(t, y)}{\partial y} &= \frac{2(1-\beta)^3(1+t)}{\beta^2 + (1-\beta)^2 t} [(1-\beta)\lambda y - \beta + \beta^2 - (1-\beta)^2 t] \\ &= \frac{2(1-\beta)^4(1+t)\lambda}{\beta^2 + (1-\beta)^2 t} [\lambda y - \beta - (1-\beta)t] < 0, \quad t \geq 0, \end{aligned}$$

then for all $y \leq -n(1+t)/2$ we have

$$g(t, y) \geq g\left(t, \frac{-n(1+t)}{2}\right) = h(t), \quad t \geq 0.$$

According to the above results, we need to determine the minimum of the function $h : [0, +\infty) \rightarrow \mathbb{R}$,

$$h(t) = \frac{(1-\beta)^3(1+t)^2}{\beta^2 + (1-\beta)^2 t} (at + b),$$

where a and b are defined by (3.1).

With these notations, the derivative $h'(t) = \frac{(\beta-1)^2(1+t)}{[\beta^2 + (\beta-1)^2 t]^2} H(t)$, where

$$H(t) = 2a(1-\beta)^2 t^2 + [3a\beta^2 + b(1-\beta)^2] t + [(a+2b)\beta^2 - t(1-\beta)^2 b].$$

We have that $h'(-1) = 0$ and the other two roots of $h'(t) = 0$ are given by $H(t) = 0$, i.e.

$$2a(1-\beta)^2 t^2 + [3a\beta^2 + b(1-\beta)^2] t + [(a+2b)\beta^2 - (1-\beta)^2 b] = 0.$$

If we denote the discriminant of H by $D(\beta, \lambda n)$, then

$$D(\beta, \lambda n) = \lambda n \left(\beta - \frac{1}{2} \right)^2 \left[\left(\beta - \frac{1}{2} \right)^2 - \frac{1 + \lambda n}{4} \right] R(\beta, \lambda n), \quad (3.4)$$

where

$$R(\beta, \lambda n) = 4(\lambda n - 8)\beta^2 + 4(7\lambda n + 16)\beta - (9\lambda^2 n^2 + 32\lambda n + 32).$$

First we see that

$$\left(\beta - \frac{1}{2} \right)^2 \left[\left(\beta - \frac{1}{2} \right)^2 - \frac{1 + \lambda n}{4} \right] \leq 0, \quad \text{for } \beta \in [0, 1), \quad \lambda \geq 0, \quad n \in \mathbb{N}. \quad (3.5)$$

Since for all $\beta \in [0, 1)$ and $\lambda n \geq 0$ we have

$$R(\beta, \lambda n) = -9\lambda^2 n^2 + 4(\beta + 8)(\beta - 1)\lambda n - 32(\beta - 1)^2 \leq 0,$$

if we combine this inequality together with (3.5), from (3.4) it follows that $D(\beta, \lambda n) \geq 0$ for all $\lambda n \geq 0$ and $\beta \in [0, 1)$, so the roots of H are real. If the roots of H are denoted by t_0 and t_1 , then from the assumption (3.2) we have $t_0 t_1 > 0$, hence the equation $h'(t) = 0$ has one positive root t_0 .

From the fact that $h'(t) \leq 0$ for $t \in [0, t_0]$ and $h'(t) \geq 0$ for $t \geq t_0$, we get that $h(t) \geq h(t_0)$ for all $t \geq 0$, and it follows that

$$|\psi(ix, y; z)|^2 \geq h(t_0),$$

for all $x, y \in \mathbb{R}$ such that $y \leq -n(1+x^2)/2$ and $z \in \mathbb{U}$.

If we define the set $\Omega = \{\omega \in \mathbb{C} : |\omega| < \rho\}$, then $\psi(p(z), zp'(z); z) \in \Omega$ and $\psi(ix, y; z) \notin \Omega$ for all $x, y \in \mathbb{R}$ with $y \leq -n(1+x^2)/2$ and for all $z \in \mathbb{U}$, hence by applying Lemma 1.4 we obtain our result. $\square$

Remarks 3.1. 1. For the special $m = 1$ and $\lambda = 1$, the result was studied in [RaSeRa02].

2. For the special case $n = 1$, $\beta = 0$, $m = 1$ and $\lambda = 1$, we may easily obtain $t_0 = (\sqrt{73} - 1)/36$ and therefore we have the following result from [LiOw02]: if $f \in A$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho, \quad z \in \mathbb{U},$$

where $\rho = \sqrt{\frac{827+73\sqrt{73}}{288}}$, then $f \in S^*$.

4. Some applications of a result of M. Robertson

Now, by using Lemma 1.5, we will obtain a sufficient condition such that a function $f \in A$ belongs to $S^m(1, \lambda, \rho)$.

Theorem 4.1. *Let $\alpha < 1$, $\lambda \geq 0$ and $m \in \mathbb{N}$. Let $f \in A$, and suppose that the next two relations hold for all $0 \leq t \leq 1$:*

$$g(z) = \frac{1}{1-\alpha} [D_\lambda^m f(z) - \alpha D_\lambda^{m-1} f(z)] \in S,$$

and

$$G(z, t) = \frac{1}{1-\alpha} [(1-t)D_\lambda^m f(z) - \alpha(1-t^2)D_\lambda^{m-1} f(z)] \prec g(z).$$

Then $f \in S^m(1, \lambda, \rho(\lambda, \alpha, m))$, where $\rho(\lambda, \alpha, m) = \alpha + 1 - \lambda + \mu(\lambda, \alpha, m)$ and

$$\mu(\lambda, \alpha, m) = \inf \left\{ \alpha(\lambda - 1) \operatorname{Re} \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} : z \in \mathbb{U} \right\}. \quad (4.1)$$

PROOF. It is easy to see that

$$G(z) = \lim_{t \rightarrow 0^+} \frac{G(z, t) - G(z, 0)}{zt} = \frac{-D_\lambda^m f(z)}{(1 - \alpha)z}$$

and

$$g'(z) = \frac{1}{1 - \alpha} \left[(D_\lambda^m f(z))' - \alpha (D_\lambda^{m-1} f(z))' \right].$$

Furthermore, it follows that $G \in H(U)$ and $\operatorname{Re} G(0) = -1/(1 - \alpha) \neq 0$.

Consequently, by using Lemma 1.5 for the special case $p = 1$, together with the definitions (1.1) and (1.2), we obtain

$$\operatorname{Re} \frac{g'(z)}{G(z)} = \operatorname{Re} \left[\alpha + 1 - \lambda + \alpha(\lambda - 1) \frac{D_\lambda^{m-1} f(z)}{D_\lambda^m f(z)} - \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \right] < 0, \quad z \in U,$$

and multiplying by $\lambda \geq 0$ we get

$$\operatorname{Re} \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \geq \alpha + 1 - \lambda + \alpha(\lambda - 1) \operatorname{Re} \frac{D_\lambda^{m-1} f(z)}{D_\lambda^m f(z)}, \quad z \in U.$$

If $\mu(\lambda, \alpha, m)$ is given by (4.1), the above inequality shows that

$f \in S^m(1, \lambda, \rho(\lambda, \alpha, m))$, which completes the proof of the theorem. $\square$

Remark 4.1. If we take in the above theorem $\lambda = 1$ we have the result of OWA, OBRADOVIĆ and LEE from [OwObLe86], while for $\lambda = 1$ and $m = 0$ we have the result of OBRADOVIĆ obtained in [Ob83].

Theorem 4.2. Let $\lambda > 0$, $\alpha < 1$ and $m \in \mathbb{N}_0$. If the function $f \in S^{m+1}(n, \lambda, \alpha)$, then

$$\operatorname{Re} \left[\frac{D_\lambda^m f(z)}{z} \right]^\beta > \frac{n\lambda}{2\beta(1 - \alpha) + n\lambda}, \quad z \in U, \quad (4.2)$$

whenever $0 < 2\beta(1 - \alpha) \leq n\lambda$. (The power in (4.2) is the principal one)

PROOF. If $f \in S^{m+1}(n, \lambda, \alpha)$, according to the definition (1.4) and using (1.1) and (1.2), we have

$$1 - \lambda + \lambda \operatorname{Re} \frac{z (D_\lambda^m f(z))'}{D_\lambda^m f(z)} > \alpha, \quad z \in U.$$

It follows that $D_\lambda^m f(z) \neq 0$ for all $z \in \dot{U} \equiv U \setminus \{0\}$, and combining this together with (1.3) we deduce that

$$\frac{D_\lambda^m f(z)}{z} \neq 0, \quad z \in U.$$

Let now define the function p by

$$\left[\frac{D_\lambda^m f(z)}{z} \right]^\beta = (1 - \mu)p(z) + \mu, \quad (4.3)$$

where

$$\frac{1}{2} \leq \mu = \frac{n\lambda}{2\beta(1 - \alpha) + n\lambda} < 1, \quad (4.4)$$

whenever $0 < 2\beta(1 - \alpha) \leq \lambda n$, $\lambda > 0$ and $\alpha < 1$. Then $p \in H(\mathbb{U})$ with $p(0) = 1$, and differentiating logarithmically both sides of (4.3) we obtain

$$\frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} - \alpha = \frac{\lambda z p'(z)}{\beta[p(z) + \frac{\mu}{1-\mu}]} + 1 - \alpha.$$

Using the fact $f \in S^{m+1}(n, \lambda, \alpha)$, this above relation shows that

$$\operatorname{Re} \frac{\lambda z p'(z)}{\beta[p(z) + \frac{\mu}{1-\mu}]} + 1 - \alpha > 0, \quad z \in \mathbb{U}, \quad (4.5)$$

and if define the function $\psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ by

$$\psi(u, v; z) = \frac{\lambda v}{\beta(u + \frac{\mu}{1-\mu})} + 1 - \alpha, \quad (4.6)$$

then (4.5) may be rewritten as $\operatorname{Re} \psi(p(z), zp'(z); z) > 0$, $z \in \mathbb{U}$.

From (4.6) it follows that ψ is continuous on the domain $D = (\mathbb{C} \setminus (-\frac{\mu}{1-\mu})) \times \mathbb{C} \times \mathbb{U}$, $(1, 0; z) \in D$ and $\operatorname{Re} \psi(1, 0; z) = 1 - \alpha > 0$, for all $z \in \mathbb{U}$. Moreover, for all $(ix, y; z) \in D$ such that $x, y \in \mathbb{R}$ and $y \leq -n(1 + x^2)/2$, a simple calculus combined with (4.4) shows that

$$\operatorname{Re} \psi(ix, y; z) \leq -\frac{\lambda n}{2\beta} \cdot \frac{\mu}{1-\mu} \cdot \frac{x^2 + 1}{x^2 + (\frac{\mu}{1-\mu})^2} + 1 - \alpha \leq 0, \quad z \in \mathbb{U},$$

provided $0 < 2\beta(1 - \alpha) \leq \lambda n$, $\lambda > 0$ and $\alpha < 1$.

Consequently, the function ψ satisfies the conditions of Lemma 1.4 with $\Omega = \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$, and thus we deduce

$$\operatorname{Re} p(z) > 0, \quad z \in \mathbb{U}.$$

This inequality together with the relation (4.3) implies (4.2), and the proof is complete. $\square$

Remark 4.2. Taking in this theorem $\lambda = 1$ and $n = 1$ we obtain the result of OWA, OBRADOVIĆ and LEE from [OwObLe86], and letting $\lambda = 1$, $m = 0$ and $n = 1$ we obtain the result of OBRADOVIĆ [Ob83].

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