

On Hölder continuous solutions of functional equations

By ANTAL JÁRAI (Debrecen)

Abstract. In this work it is proved that the real solutions f of the functional equation

$$f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y))),$$

that are locally Hölder continuous with some exponent $0 < \alpha < 1$, are locally Hölder continuous with all exponent α , $0 < \alpha < 1$.

As it is treated in ACZÉL's classical book [1961], regularity is very important in the theory and practice of functional equations. The regularity problem of functional equations with two variables can be formulated as follows (see ACZÉL [1984] and JÁRAI [1986]):

Problem. Let T and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \rightarrow Z$, $g_i : D \rightarrow T$ ($i = 1, 2, \dots, n$) and $h : D \times Z^{n+1} \rightarrow Z$ be functions. Suppose that

- (1) $f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y)))$ whenever $(t, y) \in D$;
- (2) h is analytic;
- (3) g_i is analytic and for each $t \in T$ there exists a y for which $(t, y) \in D$ and $\frac{\partial g_i}{\partial y}(t, y)$ has rank s ($i = 1, 2, \dots, n$).

Is it true that every f , which is measurable or has the Baire property is analytic?

1991 *Mathematics Subject Classification*: 39B22.

This work is supported by Magyar Kereskedelmi és Hitelbank Rt Universitas Foundation and OTKA I/3 1652 grant.

The following steps may be used:

- (I) Measurability implies continuity.
- (II) Almost open solutions are continuous.
- (III) Continuous solutions are locally Lipschitz.
- (IV) Locally Lipschitz solutions are continuously differentiable.
- (V) All p times continuously differentiable solutions are $p + 1$ times continuously differentiable.
- (VI) Infinitely many times differentiable solutions are analytic.

The complete answer to this problem is unknown. The problems corresponding to (I), (II), (IV) and (V) are solved in JÁRAI [1986]. In the same paper, partial results in connection with (III) are treated. A partial result in connection with (VI) is treated in JÁRAI [1988] (in Hungarian).

In this paper we deal with locally Hölder continuous real solution. The result is a new step in (III). The main tool is the fundamental lemma of the theory of Campanato spaces (Lemma 1), which is a generalization of the famous classical Morrey lemma from the regularity theory of partial differential equations. For further references about this lemma see ZEIDLER's book [1990], II/A pp. 90–93.

A real function f is called locally Hölder continuous with exponent $0 < \alpha \leq 1$, if each point of its domain has a neighbourhood V such that

$$\sup_{x,y \in V} |f(x) - f(y)| / |x - y|^\alpha < \infty.$$

Any constant not less than this supremum is called a (local) Hölder constant for f . In the case $\alpha = 1$ Hölder continuous functions and Hölder constants are also called Lipschitz functions and Lipschitz constants, respectively. It is well-known, that continuously differentiable functions are locally Lipschitz.

Lemma 1. *Let G be a nonempty open set in $\mathbb{R}^n$. Let $\mathbf{B}_r(y)$ denote the closed ball with center y and radius r , and define the mean value $\bar{f}_{y,r}$ of the real valued function f by*

$$\bar{f}_{y,r} = \frac{1}{\text{meas } \mathbf{B}_r(y)} \int_{\mathbf{B}_r(y)} f(x) dx.$$

Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $r_0 > 0$ be given. Then the inequality

$$\int_{\mathbf{B}_r(y)} |f(x) - \bar{f}_{y,r}|^p dx \leq \text{const } r^{n+p\alpha}$$

for all $r < \min(r_0, \text{dist}(y, \partial G))$ and all $y \in G$ implies that f is locally Hölder continuous with exponent α on G .

Lemma 2. Let V , W and U be open real intervals, $r, R > 0$, $[t_0 - r, t_0 + r] \subset V$, $[y_0 - R, y_0 + R] \subset W$, $g : V \times W \rightarrow U$ a continuously differentiable function, and $f : U \rightarrow \mathbb{R}$ a continuous function. Suppose that all partial functions $y \mapsto g(t, y)$ are monotonic with inverse denoted by $x \mapsto G_t(x)$. If there exist constants B, B', L and L' such that $|f(x)| \leq B$, $|G'_t(x)| \leq B'$, $|g(t, y) - g(t', y')| \leq L(|t - t'| + |y - y'|)$ and $|G'_t(x) - G'_{t'}(x)| \leq L'|t - t'|$ whenever $|t - t_0| \leq r$, $|t' - t_0| \leq r$ and the left hand sides are defined, then the absolute value of the integral

$$\int_{t_0-r}^{t_0+r} \int_{y_0-R}^{y_0+R} f(g(t, y)) - f(g(t', y)) dy dt'$$

is bounded by $8LBB'r^2 + 8LBL'r^2(r + R)$ whenever $|t - t_0| \leq r$.

PROOF. In the integral above the inner integral can be written as the difference of two integrals. Using the substitution $x = g(t, y)$ in the first, and the substitution $x = g(t', y)$ in the second integral respectively, we get

$$\int_{t_0-r}^{t_0+r} \left(\int_{g(t, y_0-R)}^{g(t, y_0+R)} f(x)G'_t(x) dx - \int_{g(t', y_0-R)}^{g(t', y_0+R)} f(x)G'_{t'}(x) dx \right) dt'.$$

The integrand of the outer integral can be rewritten as

$$\begin{aligned} \int_{g(t, y_0-R)}^{g(t', y_0-R)} f(x)G'_t(x) dx + \int_{g(t', y_0-R)}^{g(t, y_0+R)} f(x)(G'_t(x) - G'_{t'}(x)) dx \\ + \int_{g(t', y_0+R)}^{g(t, y_0+R)} f(x)G'_{t'}(x) dx. \end{aligned}$$

The first and the last term can be estimated by $L|t - t'|BB'$, and the middle term by $L(2r + 2R)BL'|t - t'|$. Using that $|t - t'| \leq 2r$, we get the stated result.

Theorem. Let $0 < \alpha < 1$. Let $T, Y, X_1, \dots, X_n$ and $Z_1, Z_2, \dots, Z_n$ be open subsets of $\mathbb{R}$, D an open subset of $T \times Y$. Consider the functions $f : T \rightarrow \mathbb{R}$, $f_i : X_i \rightarrow Z_i$ ($i = 1, \dots, n$), $g_i : D \rightarrow X_i$ ($i = 1, \dots, n$), $h : D \times Z_1 \times Z_2 \times \dots \times Z_n \rightarrow \mathbb{R}$. Suppose, that

(1) for each $(t, y) \in D$,

$$f(t) = h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y)));$$

(2) h is twice continuously differentiable;

(3) g_i is twice continuously differentiable on D and for each $t \in T$ there exists a y such that $(t, y) \in D$ and $\frac{\partial g_i}{\partial y}(t, y) \neq 0$ for $i = 1, \dots, n$;

(4) the functions $f_i, i = 1, \dots, n$ are locally Hölder continuous with exponent α .

Then f is locally H  lder continuous with exponent $2\alpha/(\alpha + 1)$.

PROOF. We have to prove that for each point $t_0 \in T$ the function f is H  lder continuous on a neighbourhood of t_0 with exponent $2\alpha/(1 + \alpha)$. Let us choose y_0 by (3) for t_0 . For an arbitrary set $V \subset \mathbb{R}$ let V_ε denote the ε -neighbourhood

$$V_\varepsilon = \{x : |x - y| < \varepsilon \text{ for some } y \in V\}$$

of V . Let V and W be open intervals containing t_0 and y_0 respectively, and $0 < \varepsilon \leq 1$ such that $V_\varepsilon \times W_\varepsilon \subset D$ and $\frac{\partial g_i}{\partial y}$ does not vanish on $V_\varepsilon \times W_\varepsilon$.

Hence the partial functions $y \mapsto g_i(t, y)$ have inverse on W_ε for all $t \in V_\varepsilon$ and $i = 1, 2, \dots, n$. Decreasing V , W and ε if necessary we may suppose that these inverses have derivatives bounded (in absolute value) by B' and are Lipschitz continuous with Lipschitz constant L' for $i = 1, 2, \dots, n$. Similarly, we may suppose that g_i is a Lipschitz function with Lipschitz constant L on $V_\varepsilon \times W_\varepsilon$, that f_i is H  lder continuous with exponent α and H  lder constant H and $|f_i|$ bounded by B on $g_i(V_\varepsilon \times W_\varepsilon)$ ($i = 1, 2, \dots, n$), moreover on

$$V_\varepsilon \times W_\varepsilon \times f_1(g_1(V_\varepsilon \times W_\varepsilon)) \times \dots \times f_n(g_n(V_\varepsilon \times W_\varepsilon))$$

the functions $\frac{\partial h}{\partial z_i}$ are Lipschitz continuous with Lipschitz constant L'_i ,

and the functions $\left| \frac{\partial h}{\partial t} \right|$ and $\left| \frac{\partial h}{\partial z_i} \right|$ are bounded by B'_0 and B'_i , respectively,

($i = 1, 2, \dots, n$). Let us fix ε, V, W and y_0 . We shall prove that f is locally H  lder continuous on V with exponent $2\alpha/(1 + \alpha)$. Abusing notation let t_0 denote an arbitrary element of V and let $0 < r, R < \varepsilon$. Fixing t_0 let $\bar{f}$ denote the mean value of f on the interval with endpoints $t_0 - r, t_0 + r$. Let us integrate the two sides of the functional equation over the interval with endpoints $y_0 - R, y_0 + R$. We have

$$2Rf(t) = \int_{y_0-R}^{y_0+R} h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))) dy,$$

and

$$2R\bar{f} = \frac{1}{2r} \int_{t_0-r}^{t_0+r} \int_{y_0-R}^{y_0+R} h(t', y, f_1(g_1(t', y)), \dots, f_n(g_n(t', y))) dy dt'.$$

Hence

$$\begin{aligned} |f(t) - \bar{f}| &= \frac{1}{2R} \left| \int_{y_0-R}^{y_0+R} h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))) \right. \\ &\quad \left. - \frac{1}{2r} \int_{t_0-r}^{t_0+r} h(t', y, f_1(g_1(t', y)), \dots, f_n(g_n(t', y))) dt' dy \right|. \end{aligned}$$

To get a good upper estimate for the left hand side we need an upper estimate for the difference

$$h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))) - h(t', y, f_1(g_1(t', y)), \dots, f_n(g_n(t', y))).$$

We may apply the Taylor theorem for the function h with points

$$z = (t, y, z_1, \dots, z_n) \quad \text{and} \quad z' = (t', y, z'_1, \dots, z'_n)$$

where $t', t \in V$, $y \in W$, $z_i = f_i(g_i(t, y))$ and $z'_i = f_i(g_i(t', y))$ for $i = 1, \dots, n$. We have

$$\begin{aligned} h(z) - h(z') &= \int_0^1 \frac{\partial h}{\partial t}(\tau z + (1 - \tau)z')(t - t') d\tau \\ &\quad + \sum_{i=1}^n \int_0^1 \frac{\partial h}{\partial z_i}(\tau z + (1 - \tau)z')(z_i - z'_i) d\tau. \end{aligned}$$

Using this and omitting variables we have

$$\begin{aligned} 4rR|f(t) - \bar{f}| &= \left| \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} \left(\int_0^1 \frac{\partial h}{\partial t}(\tau z + (1 - \tau)z')(t - t') d\tau \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^n \int_0^1 \frac{\partial h}{\partial z_i}(\tau z + (1 - \tau)z')(z_i - z'_i) d\tau \right) dt' dy \right|. \end{aligned}$$

Using the triangle inequality, we get $n + 1$ terms on the right hand side. For the first term we get the trivial upper bound $4RrB'_0 2r$, where B'_0 is an

upper bound of $\left| \frac{\partial h}{\partial t} \right|$. If $\bar{h}'_i$ denotes the mean value of the partial derivative $\frac{\partial h}{\partial z_i}$, that is

$$\bar{h}'_i = \frac{1}{4rR} \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} \int_0^1 \frac{\partial h}{\partial z_i}(z) d\tau dt dy,$$

then the other terms can be rewritten in the form

$$\begin{aligned} \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} \int_0^1 \left(\frac{\partial h}{\partial z_i}(\tau z + (1 - \tau)z') - \bar{h}'_i \right) (z_i - z'_i) d\tau dt' dy \\ + \bar{h}'_i \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} (z_i - z'_i) dt' dy. \end{aligned}$$

First we give an upper estimate for the absolute value of the first term of this sum. An upper estimate of $|z_i - z'_i|$ is $H(L2r)^\alpha$, where H is a

H  lder-constant for f_i and L is a Lipschitz-constant for g_i . Hence

$$\begin{aligned} & \left| \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} \int_0^1 \frac{\partial h}{\partial z_i} (\tau z + (1-\tau)z') - \bar{h}'_i(z_i - z'_i) d\tau dt' dy \right| \\ & \leq H(2rL)^\alpha \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} \int_0^1 \left| \frac{\partial h}{\partial z_i} (\tau z + (1-\tau)z') - \bar{h}'_i \right| d\tau dt' dy. \end{aligned}$$

Because the difference between the value and the mean value of a function is not greater than the difference between any two values, we need to estimate the difference $\left| \frac{\partial h}{\partial z_i} (\tau z + (1-\tau)z') - \frac{\partial h}{\partial z_i} (z'') \right|$. This is not greater than L'_i multiplied by the norm of $\tau z + (1-\tau)z' - z''$, that is, L'_i times the maximal distance between the vectors z and $z'' = (t'', y'', z''_1, \dots, z''_n)$, where $z''_i = f_i(g_i(t'', y''))$ and L'_i is a Lipschitz-constant for $\frac{\partial h}{\partial z_i}$. The maximal distance between z and z'' can be estimated by $r + R + nH(L(2r + 2R))^\alpha$. Hence we get the upper bound

$$4rRH(2rL)^\alpha L'_i(r + R + nH(L(2r + 2R))^\alpha)$$

for the first term.

To get an upper bound for the second term, we need an upper bound for the absolute value of

$$\begin{aligned} & \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} (z_i - z'_i) dt' dy = \\ & = \int_{y_0-R}^{y_0+R} \int_{t_0-r}^{t_0+r} f_i(g_i(t, y)) - f_i(g_i(t', y)) dt' dy, \end{aligned}$$

because $|\bar{h}'_i|$ is trivially bounded by the upper bound B'_i of $\left| \frac{\partial h}{\partial z_i} \right|$. From Lemma 2 we get the upper bound $8LBB'r^2 + 8LBL'r^2(r + R)$ for this integral.

Summing up all these estimates, we get

$$\begin{aligned} |f(t) - \bar{f}| & \leq 2B'_0 r + H(2rL)^\alpha \sum_{i=1}^n L'_i(r + R + nH(L(2r + 2R))^\alpha) \\ & + \sum_{i=1}^n B'_i(2LBB'r + 2LBL'r(r + R)) / R. \end{aligned}$$

If $r \leq R$ this can be rewritten as

$$|f(t) - \bar{f}| \leq C_0 r + C_1 r^\alpha R^\alpha + C_2 r / R,$$

where C_0 , C_1 and C_2 do not depend on t_0 , r and R . If we choose r and R such that they satisfy the condition $R = r^{(1-\alpha)/(1+\alpha)}$, then we have

$$|f(t) - \bar{f}| \leq (C_0 + C_1 + C_2)r^{2\alpha/(1+\alpha)}$$

whenever $0 < r < r_0 = \varepsilon^{(1+\alpha)/(1-\alpha)}$ and $|t - t_0| \leq r$. Integrating and using Lemma 1, we get that f is locally Hölder continuous on V which implies the theorem.

References

- [1961] J. ACZÉL, Vorlesungen über Funktionalgleichungen und ihre Anwendungen, *Birkhäuser Verlag*, 1961.
- [1984] J. ACZÉL, Some unsolved problems in the theory of functional equations II., *Aequationes Mathematicae*. **26** (1984), 255–260.
- [1986] A. JÁRAI, On regular solutions of functional equations, *Aequationes Mathematicae* **30** (1986), 21–54.
- [1988] A. JÁRAI, Függvényegyenletek regularitási tulajdonságai, Kandidátusi értekezés, 1988.
- [1990] E. ZEIDLER, Nonlinear functional analysis I–IV., *Springer-Verlag, New York*, 1990.

PERMANENT ADDRESS:

ANTAL JÁRAI

KOSSUTH LAJOS UNIVERSITY

H-4010 DEBRECEN, EGYETEM TÉR 1, PF. 12

HUNGARY

TEMPORARY ADDRESS:

UNIVERSITÄT GH PADERBORN, FB 17

D-33098 PADERBORN,

WARBURGER STR. 100

GERMANY

(Received May 4, 1992)