

The $\mathcal{L}$ -dual of a Matsumoto space

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Abstract. In [HS1], [MHSS] the $\mathcal{L}$ -duals of a Randers and Kropina space were studied. In this paper we shall discuss the $\mathcal{L}$ -dual of a Matsumoto space. The metric of this $\mathcal{L}$ -dual space is completely new and it brings a new idea about $\mathcal{L}$ -duality because the $\mathcal{L}$ -dual of Matsumoto metric can be given by means of four quadratic forms and 1-forms on T^*M constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

1. Introduction

The study of $\mathcal{L}$ -duality of Lagrange and Finsler space was initiated by R. MIRON [Mi2] around 1980. Since then, many Finsler geometers studied this topic.

One of the remarkable results obtained are the concrete $\mathcal{L}$ -duals of Randers and Kropina metrics [HS2]. However, the importance of $\mathcal{L}$ -duality is by far limited to computing the dual of some Finsler fundamental functions.

Recently, in [BRS], the complicated problem of classifying Randers metrics of constant flag curvature was solved by means of duality. Other geometrical problems of (α, β) -metrics might be solved on future by considering not the metric itself, but its $\mathcal{L}$ -dual.

The concrete examples of $\mathcal{L}$ -dual metrics are quite few [HS1], [HS2]. In the present paper we succeeded to compute the dual of another well known (α, β) -metric, the Matsumoto metric. Surprisingly, despite of the quite complicated computations involved, we obtain the Hamiltonian function by means of four

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quadratic forms and a 1-form on T^*M . This metric is completely new and it brings a new idea about $\mathcal{L}$ -duality. The dual of an (α, β) -metric can be given by means of several quadratic forms and 1-forms on T^*M constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

2. The Legendre transformation

2.1. Definitions. Let $F^n = (M, F)$ be a n -dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β) -metric if F is homogeneous function of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j$, $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$.

A Finsler space with the fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y), \quad (2.1)$$

is called a *Randers space*.

A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}, \quad (2.2)$$

is called a *Kropina space*, and one with

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)}, \quad (2.3)$$

is called a *Matsumoto space*.

Let $C^n = (M, K)$ be an n -dimensional Cartan space having the fundamental function $K(x, p)$. We also consider Cartan spaces having the metric function of the following form:

$$K(x, p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i, \quad (2.4)$$

or

$$K(x, p) = \frac{a^{ij}(x)p_i p_j}{b^i(x)p_i}, \quad (2.5)$$

with $a_{ij}a^{jk} = \delta_i^k$ and we will again call these spaces Randers and Kropina spaces on the cotangent bundle T^*M , respectively.

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$.

It is known [MHSS] that if L is a differentiable function, we can consider the fiber derivative of L , locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$:

$$\varphi(x, y) = (x^i, \dot{\partial}_a L(x, y)) \quad (2.6)$$

which is called the *Legendre transformation*. We can define, in this case, the function $H : U^* \rightarrow R$:

$$H(x, p) = p_a y^a - L(x, y), \quad (2.7)$$

where $y = (y^a)$ is the solution of the equations:

$$p_a = \dot{\partial}_a L(x, y). \quad (2.8)$$

In the same manner, the fiber derivative is locally given by:

$$\psi(x, p) = (x^i, \dot{\partial}^a H(x, p)). \quad (2.9)$$

The function ψ is a diffeomorphism between the same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \rightarrow R$:

$$L(x, y) = p_a y^a - H(x, p), \quad (2.10)$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}^a H(x, p). \quad (2.11)$$

The Hamiltonian given by (2.7) is called the *Legendre transformation of the Lagrangian* L and the Lagrangian given by (2.10) is called the *Legendre transformation of the Hamiltonian* H .

If (M, K) is a Cartan space, then (M, H) is a Hamilton manifold [MHSS], where $H(x, p) = \frac{1}{2}K^2(x, p)$ is 2-homogeneous on a domain of T^*M . So, we get the following transformation of H on U :

$$L(x, y) = p_a y^a - H(x, p) = H(x, p). \quad (2.12)$$

Proposition 1 ([MHSS]). *The scalar field $L(x, y)$ defined by (2.12) is a positively 2-homogeneous regular Lagrangian on U .*

Therefore, we get the Finsler metric F of U , such that

$$L = \frac{1}{2}F^2 \quad (2.13)$$

Thus, for the Cartan space (M, K) one always can locally associate a Finsler space (M, F) which will be called the $\mathcal{L}$ -dual of a Cartan space $(M, K|_{U^*})$.

Conversely, we can associate, locally, a Cartan space to every Finsler space which will be called the $\mathcal{L}$ -dual of a Finsler space $(M, F|_U)$.

3. The (α, β) Finsler – (α^*, β^*) Cartan $\mathcal{L}$ -duality

Let us recall some known results.

Theorem 3.1 ([HS1], [MHSS]). *Let (M, F) be a Randers space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i . Then:*

- (1) *If $b^2 = 1$, the $\mathcal{L}$ -dual of (M, F) is a Kropina space on T^*M with:*

$$H(x, p) = \frac{1}{2} \left(\frac{a^{ij}p_ip_j}{2b^ip_i} \right)^2. \quad (3.1)$$

- (2) *If $b^2 \neq 1$, the $\mathcal{L}$ -dual of (M, F) is a Randers space on T^*M with:*

$$H(x, p) = \frac{1}{2} \left(\sqrt{\tilde{a}^{ij}p_ip_j} \pm \tilde{b}^ip_i \right)^2, \quad (3.2)$$

where

$$\tilde{a}^{ij} = \frac{1}{1-b^2}a^{ij} + \frac{1}{(1-b^2)^2}b^ib^j; \quad \tilde{b}^i = \frac{1}{1-b^2}b^i,$$

(in (3.2) '–' corresponds to $b^2 < 1$ and '+ ' corresponds to $b^2 > 1$).

Theorem 3.2 ([HS1], [MHSS]). *The $\mathcal{L}$ -dual of a Kropina space is a Randers space on T^*M with the Hamiltonian:*

$$H(x, p) = \frac{1}{2} \left(\sqrt{\tilde{a}^{ij}p_ip_j} \pm \tilde{b}^ip_i \right)^2, \quad (3.3)$$

where

$$\tilde{a}^{ij} = \frac{b^2}{4}a^{ij}; \quad \tilde{b}^i = \frac{1}{2}b^i,$$

(in (3.3) '–' corresponds to $\beta < 0$ and '+ ' corresponds to $\beta > 0$).

In [HS1] the notation $\alpha^* = (a^{ij}(x)p_ip_j)^{\frac{1}{2}}$, $\beta^* = b^i(x)p_i$ are used, where $a^{ij}(x)$ are the reciprocal components of $a_{ij}(x)$ and $b^i(x)$ are the components of the vector field on M , $b^i(x) = a^{ij}(x)b_j(x)$. We can consider the metric functions $K = \alpha^* + \beta^*$ (Randers metric on T^*M) or $K = \frac{\alpha^{*2}}{\beta^*}$ (Kropina metric on T^*M) defined on a domain $D^* \subset T^*M$. So, one can easily rewrite the previous theorems:

Theorem 3.3. *Let (M, F) be a Randers space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i . Then:*

(1) If $b^2 = 1$, the $\mathcal{L}$ -dual of (M, F) is a Kropina space on T^*M with:

$$H(x, p) = \frac{1}{2} \left(\frac{\alpha^{*2}}{2\beta^*} \right)^2. \quad (3.4)$$

(2) If $b^2 \neq 1$, the $\mathcal{L}$ -dual of (M, F) is a Randers space on T^*M with:

$$H(x, p) = \frac{1}{2} \left(\alpha^* \pm \beta^* \right)^2, \quad (3.5)$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_i p_j}$ and $\beta^* = \tilde{b}^i p_i$ where

$$\tilde{a}^{ij} = \frac{1}{1-b^2} a^{ij} + \frac{1}{(1-b^2)^2} b^i b^j; \quad \tilde{b}^i = \frac{1}{1-b^2} b^i,$$

(in (3.5) $' - '$ corresponds to $b^2 < 1$ and $' + '$ corresponds to $b^2 > 1$).

Theorem 3.4. The $\mathcal{L}$ -dual of a Kropina space is a Randers space on T^*M with the Hamiltonian:

$$H(x, p) = \frac{1}{2} \left(\alpha^* \pm \beta^* \right)^2, \quad (3.6)$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_i p_j}$ and $\beta^* = \tilde{b}^i p_i$ where

$$\tilde{a}^{ij} = \frac{b^2}{4} a^{ij}; \quad \tilde{b}^i = \frac{1}{2} b^i,$$

(in (3.6) $' - '$ corresponds to $\beta < 0$ and $' + '$ corresponds to $\beta > 0$).

We are going to compute now the dual of a Matsumoto metric. We obtain:

Theorem 3.5. Let (M, F) be a Matsumoto space and $b = (a_{ij} b^i b^j)^{\frac{1}{2}}$ the Riemannian length of b_i . Then

(1) If $b^2 = 1$, the $\mathcal{L}$ -dual of (M, F) is the space having the fundamental function:

$$H(x, p) = \frac{1}{2} \left(-\frac{b^i p_i}{2} \frac{\left(\sqrt[3]{a^{ij} p_i p_j} + \sqrt[3]{(b^i p_i + \sqrt{\tilde{a}^{ij} p_i p_j})^2} \right)^3}{a^{ij} p_i p_j + (b^i p_i + \sqrt{\tilde{a}^{ij} p_i p_j})^2} \right)^2, \quad (3.7)$$

where

$$\tilde{a}^{ij} = b^i b^j - a^{ij}.$$

- (2) If $b^2 \neq 1$, the $\mathcal{L}$ -dual of (M, F) is the space on T^*M having the fundamental function:

$$H(x, p) = \frac{1}{2} \left(-\frac{b^i p_i}{200} \frac{25 \left(2\sqrt{d_2^{ij} p_i p_j} + \sqrt{d_4^{ij} p_i p_j} \right)^2 + d_8^{ij} p_i p_j}{\sqrt{d_2^{ij} p_i p_j} \sqrt{d_4^{ij} p_i p_j} + d_9^{ij} p_i p_j} \right)^2, \quad (3.8)$$

where

$$c_1^{ij} = (b^i b^j + 2\varepsilon_1 a^{ij})^2 + (2a^{ij})^2 \varepsilon_3,$$

$$c_2^{ij} = a^{ij} (\theta_4^2 b^i b^j + a^{ij} \varepsilon_2),$$

$$c_3^{ij} = (2a^{ij})^2 \theta_5^3,$$

$$\sqrt[3]{\tilde{a}^{ij}}^2 = \sqrt[3]{c_1^{ij}} - 2\sqrt[3]{c_2^{ij}} + \sqrt[3]{c_3^{ij}},$$

$$d_1^{ij} = d_3^{ij} + 4m(a^{ij} b^2 - b^i b^j),$$

$$d_2^{ij} = \sqrt{d_3^{ij} a^{ij}} + 4\sqrt{d_1^{ij} a^{ij}} - d_3^{ij},$$

$$d_3^{ij} = 2\sqrt[3]{2a^{ij}(\tilde{a}^{ij})^2},$$

$$\sqrt{d_4^{ij}} = \sqrt{d_3^{ij}} + 3\sqrt{a^{ij}},$$

$$\sqrt{d_5^{ij}} = \sqrt{d_3^{ij} a^{ij}},$$

$$d_6^{ij} = d_1^{ij} a^{ij},$$

$$\sqrt{d_7^{ij}} = 2\sqrt{d_2^{ij}} + \sqrt{d_4^{ij}},$$

$$d_8^{ij} = 200 \left(\sqrt{d_6^{ij}} + 2na^{ij} \right) - 5 \left(4\sqrt{d_3^{ij}} + \sqrt{d_4^{ij}} \right),$$

$$d_9^{ij} = 4\sqrt{d_6^{ij}} + 4a^{ij} p + 9\sqrt{d_5^{ij}},$$

and

$$m = 1 - b^2,$$

$$n = \frac{20b^2 - 29}{29},$$

$$p = \frac{1 - 2b^2}{2},$$

$$\theta_1 = -\frac{712b^6 - 452b^4 + 24b^2 + 1}{1728},$$

$$\theta_2 = \frac{576b^4 - 2232b^2 + 2628}{1728},$$

$$\begin{aligned}
\theta_3 &= -\left(\frac{8b^2 + 1}{12}\right)^2, \\
\theta_4 &= \frac{2b^2 + 1}{6}, \\
\theta_5 &= \frac{11b^2 + 1}{12}, \\
\varepsilon_1 &= 2(\theta_4^2 - \theta_2), \\
\varepsilon_2 &= 3\theta_3\theta_4^2 + \theta_2^2, \\
\varepsilon_3 &= 4\varepsilon_2 - 2\theta_1 - \varepsilon_1.
\end{aligned}$$

PROOF. By putting: $\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, we have $F = \frac{\alpha^2}{\alpha - \beta}$, and

$$p_i = \frac{1}{2} \dot{\partial}_i F^2 = \frac{y_i}{\alpha - \beta} + \frac{\alpha^2 b^i - y_i \beta}{(\alpha - \beta)^2}. \quad (3.9)$$

Contracting in (3.9) by p^i and b^i we get:

$$\begin{aligned}
\alpha^{*2} &= \frac{F}{(\alpha - \beta)^2} [F^2(\alpha - 2\beta) + \alpha^2 \beta^*] \\
\beta^* &= \frac{F}{(\alpha - \beta)^2} [\beta(\alpha - 2\beta) + \alpha^2 b^2].
\end{aligned} \quad (3.10)$$

In [Sh], for a Finsler (α, β) -metric F on a manifold M , one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0$, $\forall x \in M$.

The function ϕ satisfies: $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, ($|s| \leq b_0$).

A Matsumoto metric is a special (α, β) -metric with $\phi = \frac{1}{1-s}$.

Using SHEN's [Sh] notation $s = \frac{\beta}{\alpha}$, the formula (3.10) become:

$$\begin{aligned}
\alpha^{*2} &= F^2 \frac{1 - 2s}{(1 - s)^3} + F \frac{1}{(1 - s)^2} \beta^*, \\
\beta^* &= F s \frac{1 - 2s}{(1 - s)^2} + F \frac{1}{(1 - s)^2} b^2.
\end{aligned} \quad (3.11)$$

Now we put $1 - s = t$, i.e. $s = 1 - t$ and both equations become:

$$\alpha^{*2} = F^2 \frac{2t - 1}{t^3} + F \frac{1}{t^2} \beta^*, \quad (3.12)$$

$$\beta^* = F(1-t)\frac{2t-1}{t^2} + F\frac{1}{t^2}b^2. \quad (3.13)$$

We get

$$\beta^*t^2 = M(-2t^2 + 3t + b^2 - 1). \quad (3.14)$$

For $b^2 = 1$ from (3.13) we obtain:

$$F = -\frac{\beta^*t}{2t-3}, \quad (3.15)$$

and by substitution of F in (3.12), after some computations we get a cubic equation:

$$t^3 - 3t + \frac{9}{4}t - \frac{\beta^*}{2\alpha^{*2}} = 0. \quad (3.16)$$

Using Cardano's method for solving cubic equation [Wi], we get:

$$F = -\frac{\beta^*}{2} \frac{(2P-1)^2}{3P^2 + (P-1)^2}, \quad (3.17)$$

where for P we have:

$$P = \frac{1}{2} \sqrt[3]{\left(\frac{\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}}}{\alpha^*}\right)^2}. \quad (3.18)$$

After some computations, for F we get:

$$F = -\frac{\beta^*}{2} \frac{\left(\sqrt[3]{\alpha^{*2}} + \sqrt[3]{(\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}})^2}\right)^3}{\alpha^{*2} + (\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}})^2}. \quad (3.19)$$

Substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$ we can easily get (3.7).

If $b^2 \neq 1$, the formula (3.15) is more complicated because:

$$F = \frac{\beta^*t^2}{-2t^2 + 3t + b^2 - 1}, \quad (3.20)$$

and by substituting this in (3.12) we obtain the quadric equation:

$$t^4 - 3t^3 + t^2 \frac{13 - 4b^2}{4} + t \frac{6\alpha^{*2}(b^2 - 1)}{4\alpha^{*2}} + \frac{\alpha^{*2}(b^2 - 1)^2 + \beta^{*2}(1 - b^2)}{4\alpha^{*2}} = 0. \quad (3.21)$$

After a quite long computation, formula (3.21) becomes a cubic equation (different from (3.16)) and solving it, we get:

$$\begin{aligned}
 F = & -\frac{\beta^*}{2} \left(\left(\sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\right)}} + \frac{A}{2} + \frac{3}{4} \right)^2 \right. \\
 & \left. + \sqrt{A^2 + m\left(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\right)} - \frac{5}{4} \left(A + \frac{3}{10}\right)^2 + n \right) \\
 & / \left(\left(\frac{3}{2} + 2A \right) \left(\sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\right)}} \right) \right. \\
 & \left. + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^{*2}}{\alpha^{*2}}\right)} + \frac{9}{2}A + p \right), \tag{3.22}
 \end{aligned}$$

where

$$A^2 = \sqrt[3]{\left(\frac{1}{2}\frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_1\right)^2 + \varepsilon_3} + \sqrt[3]{-4\left(\theta_4^3\frac{\beta^{*2}}{\alpha^{*2}} + \varepsilon_2\right)} + \theta_5. \tag{3.23}$$

By substituting now $\beta^* = b^i p_i$ and $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$, after some computations, from (3.23) we obtain (3.8). $\square$

3.1. Remarks.

- (1) It is easy to see that both relations, (3.7) and (3.8), are coming from (3.14). Indeed, substituting $b^2 = 1$ in (3.14) we get the cubic equation (3.16). As solution, we find (3.7). For $b^2 \neq 1$, from (3.14) we get the complicated quadric equation (3.21) with (3.8) as solution. If in (3.21) we would replace $b^2 = 1$ we would get $t^4 - 3t^3 + \frac{9}{4} = 0$ with $t_1 = t_2 = 0$ and $t_3 = t_4 = \frac{3}{2}$. It is impossible for these four solutions to exist in our proof. So, we can easily see that (3.7) and (3.8) are two different relations and we can't get (3.7) as a particular case of (3.8).
- (2) Using α^* and β^* we can get, for the $\mathcal{L}$ -dual of (M, F) , in the case $b^2 = 1$, the fundamental function:

$$H(x, p) = \frac{1}{2} \left(-\frac{\beta^*}{2} \frac{\left(\sqrt[3]{\alpha^{*2}} + \sqrt[3]{(\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}})^2} \right)^3}{\alpha^{*2} + (\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}})^2} \right)^2. \tag{3.24}$$

- (3) In (3.7) $\tilde{a}^{ij}$ is positive-definite and the Randers metric on T^*M $p_i b^i + \sqrt{p_i p_j \tilde{a}^{ij}}$ is positive-valued for any p .

4. Conclusions

Let's take a second look at formula (3.8). If we introduce the following quadratic forms:

$$\begin{aligned}\alpha_2^* &= \sqrt{d_2^{ij} p_i p_j}, & \alpha_4^* &= \sqrt{d_4^{ij} p_i p_j}, \\ \alpha_8^* &= \sqrt{d_8^{ij} p_i p_j}, & \alpha_9^* &= \sqrt{d_9^{ij} p_i p_j},\end{aligned}$$

defined on T^*M by the corresponding matrices, then (3.8) becomes:

$$H(x, p) = \frac{1}{2} \left(-\frac{\beta^*}{200} \frac{25(2\alpha_2^* + \alpha_4^*)^2 + (\alpha_8^*)^2}{\alpha_2^* \alpha_4^* + (\alpha_9^*)^2} \right)^2, \quad (4.1)$$

for $b^2 \neq 1$.

In other words, the $\mathcal{L}$ -duals of a Randers and Kropina metrics are expressed only with the duals α^* , β^* of α , β , respectively. However, the $\mathcal{L}$ -dual of a Matsumoto metric is given by means of four distinct quadratic forms on T^*M . Remark that the coefficients of the quadratic forms are constructed only from the Riemannian metric matrix element, a_{ij} and the 1-forms β 's coefficients $b_i(x)$.

Inevitably, the following question occurs: if d_2^{ij} , d_4^{ij} , d_8^{ij} , d_9^{ij} are positively defined and therefore making sure that α_2^* , α_4^* , α_8^* , α_9^* exist.

The answer is not quite immediate and depends both on the value of b^2 and on a^{ij} , b^i , b^j . For example, if we take $b^2 < \frac{1}{2}$ and $a^{ij} > 2b^i b^j$ then, not only d_2^{ij} , d_4^{ij} , d_8^{ij} , d_9^{ij} are positively defined but also the four quadric forms are defined.

Certainly, there are many other values for b^2 , a^{ij} , b^i , b^j which give a certain positive answer, but the above values justify the existence of (4.1).

4.1. Remarks, examples.

Remark 4.1. For the $\mathcal{L}$ -dual of (4.1) we obtain the Matsumoto space with the fundamental function:

$$F = \frac{\tilde{a}_{ij} y^i y^j}{\sqrt{b^2 a_{ij} y^i y^j - \tilde{b}_i y^i}}, \quad (4.2)$$

where

$$\begin{aligned}\tilde{b}_i &= 4b^2 b_i, \\ \tilde{a}_{ij} &= a_{ij}^2 b_i b_j (7 + 8b^2) - \sqrt{a_{ij}} b_i [a_{ij} (1 + 2b^2) - 12b_i b_j] \\ &\quad \pm m [a_{ij}^2 b_i (7 + 8b^2) - \sqrt{a_{ij}} (a_{ij} - 12b_i b_j)],\end{aligned}$$

and

$$m = \sqrt{b_i b_j - b^2 a_{ij}}.$$

The other properties like curvature and the relation between geometrical properties of the $\mathcal{L}$ -dual metric (4.1) and the initial Matsumoto metric will be studied elsewhere.

Example 1. Let us consider a particular example and find its $\mathcal{L}$ -dual. For this, let us consider a surface S embedded in the usual Euclidian space R^3 , i.e.

$$S \hookrightarrow R^3, \quad (x, y) \in S \longrightarrow (x, y, z = f(x, y)) \in R^3.$$

It is known that the induced Riemannian metric on the surface S is given by:

$$(a_{ij}) = \begin{pmatrix} 1 + (f_x)^2 & f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix},$$

where f_x and f_y means partial derivative with respect to x and y , respectively.

If we consider now a coordinate system $(x, y, u, v) \in TM$ in the tangent bundle TM , then for α and β one can choose:

$$\alpha^2 = (1 + f_x^2)^2 u^2 + 2f_x f_y uv + (1 + f_y^2)^2 v^2,$$

and

$$\beta = f_x u + f_y v.$$

Now, for the induced Riemannian metric, we have:

$$\begin{aligned} \det \|a_{ij}\| &= 1 + f_x^2 + f_y^2, \\ (a^{ij}) &= \begin{pmatrix} \frac{1 + (f_y)^2}{1 + f_x^2 + f_y^2} & -\frac{f_x f_y}{1 + f_x^2 + f_y^2} \\ -\frac{f_x f_y}{1 + f_x^2 + f_y^2} & \frac{1 + (f_x)^2}{1 + f_x^2 + f_y^2} \end{pmatrix}, \\ \tilde{b}^1 &= \frac{f_x}{1 + f_x^2 + f_y^2}, \quad \tilde{b}^2 = \frac{f_y}{1 + f_x^2 + f_y^2}, \end{aligned}$$

and for the Riemannian length of $\tilde{b}_i$:

$$b^2 = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}, \quad 0 < b^2 < 1.$$

Using these and following step by step the second case of Theorem 3.5, we find:

$$d_2^{11} = M(A + 4B) - A^2,$$

$$\begin{aligned}
d_2^{12} &= d_2^{21} = P^2[E(1 - E) + 4F], \\
d_2^{22} &= N(C + 4D) - C^2, \\
d_4^{11} &= A + 3M, \\
d_4^{12} &= d_4^{21} = P(E + 3), \\
d_4^{22} &= C + 3N, \\
d_8^{11} &= 5M[40(B + 2nM) - 3] - 25A, \\
d_8^{12} &= d_8^{21} = 5P[40P(F + 2n) - 5E - 3], \\
d_8^{22} &= 5N[40(D + 2nN) - 3] - 25C, \\
d_9^{11} &= M(4B + 4p + 9A), \\
d_9^{12} &= d_9^{21} = P^2(4F - 4p + 9E), \\
d_9^{22} &= N(4D + 4p + 9C),
\end{aligned}$$

where

$$M = \sqrt{\frac{1 + (f_y)^2}{1 + f_x^2 + f_y^2}}, \quad N = \sqrt{\frac{1 + (f_x)^2}{1 + f_x^2 + f_y^2}}, \quad P = \sqrt{-\frac{f_x f_y}{1 + f_x^2 + f_y^2}},$$

and

$$\begin{aligned}
A &= \sqrt{R_1 - R_2 + 2M^2\theta_5}, & B &= \sqrt{R_1 - R_2 + M^2\theta_6}, \\
C &= \sqrt{R_3 - R_4 + 2N^2\theta_5}, & D &= \sqrt{R_3 - R_4 + N^2\theta_6},
\end{aligned}$$

and

$$E = \sqrt{R_5}, \quad F = \sqrt{R_5 + \frac{4}{c}},$$

where

$$\begin{aligned}
R_1 &= 2\sqrt[3]{2\frac{f_x^4(1 + f_y^2)}{(1 + f_x^2 + f_y^2)^5} + 8\varepsilon_1\frac{f_x^2(1 + f_y^2)^2}{(1 + f_x^2 + f_y^2)^4} + 8\varepsilon_4\frac{(1 + f_y^2)^3}{(1 + f_x^2 + f_y^2)^3}}, \\
R_2 &= 4\sqrt[3]{2\varepsilon_2\frac{(1 + f_y^2)^3}{(1 + f_x^2 + f_y^2)^3} + \theta_4^2\frac{f_x^2(1 + f_y^2)^2}{(1 + f_x^2 + f_y^2)^4}}, \\
R_3 &= 2\sqrt[3]{2\frac{f_y^4(1 + f_x^2)}{(1 + f_x^2 + f_y^2)^5} + 8\varepsilon_1\frac{f_y^2(1 + f_x^2)^2}{(1 + f_x^2 + f_y^2)^4} + 8\varepsilon_4\frac{(1 + f_x^2)^3}{(1 + f_x^2 + f_y^2)^3}},
\end{aligned}$$

$$R_4 = 4\sqrt[3]{2\varepsilon_2 \frac{(1+f_x^2)^3}{(1+f_x^2+f_y^2)^3} + \theta_4^2 \frac{f_y^2(1+f_x^2)^2}{(1+f_x^2+f_y^2)^4}},$$

$$R_5 = 2\left(\sqrt[3]{2\left(\frac{1}{c} - 2\varepsilon_1\right)^2 + 8\varepsilon_2 + 2\theta_5} - 2\sqrt[3]{2\varepsilon_2 - \frac{2}{c}\theta_4^2}\right),$$

and

$$c = 1 + f_x^2 + f_y^2,$$

$$m = \frac{1}{1 + f_x^2 + f_y^2},$$

$$n = -\frac{29 + 9f_x^2 + 9f_y^2}{29(1 + f_x^2 + f_y^2)},$$

$$p = \frac{1 - f_x^2 - f_y^2}{2(1 + f_x^2 + f_y^2)},$$

$$\theta_1 = -\frac{258c^3 - 1256c^2 + 1684c - 712}{12^3c^3},$$

$$\theta_2 = \frac{81c^2 + 90c + 48}{12^2c^2},$$

$$\theta_3 = -\left(\frac{9c - 8}{12c}\right)^2,$$

$$\theta_4 = \frac{3c - 2}{6c},$$

$$\theta_5 = \frac{12c - 11}{12c},$$

$$\theta_6 = \frac{12c^2 + 13c - 24}{6c^2},$$

$$\varepsilon_1 = \frac{-45c^2 - 138c - 32}{12^2c^2},$$

$$\varepsilon_2 = \frac{-2187c^4 + 41796c^3 - 15660c^2 + 24768c - 768}{12^4c^4},$$

$$\varepsilon_3 = \frac{921c^4 + 14732c^3 - 1084c^2 + 6832c - 256}{12^3c^4},$$

$$\varepsilon_4 = \frac{13077c^4 + 189204c^3 + 8916c^2 + 90816c - 2048}{12^4c^4},$$

getting in this way all the four quadric form which allow us to find, in T^*M , using (4.1), the $\mathcal{L}$ -dual of our particular Matsumoto space from above.

For the above construction, we need to analyze the existence of the expressions under the radicals. M , N allways exist.

First of all, because of the radical in the expression of P we must have $f_x f_y \leq 0$. If $f_x f_y = 0$ we get $d_2^{12} = d_2^{21} = 0$ and $d_4^{12} = d_4^{21} = 0$, $d_8^{12} = d_8^{21} = 0$, $d_9^{12} = d_9^{21} = 0$.

Let us put $\Delta = (\varepsilon_1 - \theta_4^2)^2 - 4(\varepsilon_4 - 2\varepsilon_2)$ and $S = 4(\varepsilon_4 - 2\varepsilon_2)$. Therefore, we have:

If $\Delta < 0$ then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ for any value of c . This allows us to conclude that A , B , C , D always exist.

If $\Delta \geq 0$ and $c \in [1, \frac{4}{3}]$ or $\Delta \geq 0$ and $S \geq 0$, then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ proving the existence of A , B , C , D .

We also need to have $R_5 \geq 0$. But this depends on the value of $c \geq 1$. For example, if $c \in [1, \frac{4}{3}]$ we have $R_5 \in [-0,0701; 2,1898]$.

To complete our discussion, we mention here the following result [SS1]: if $f_x^2 + f_y^2 \leq \frac{1}{3}$ i.e. $1 \leq c \leq \frac{4}{3}$, then $\frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2} \leq \frac{1}{4}$ and the fundamental tensor g^{ij} of Matsomoto space $F = \frac{\alpha^2}{\alpha - \beta}$ with α and β defined above is positively defined, or equivalently, the indicatrix is convex.

Example 2. Let us consider the surface S to be a plane, $z = f(x, y) = \frac{1}{2}x$.

The convexity condition for the indicatrix is satisfied, i.e.: $f_x^2 + f_y^2 = \frac{1}{4} < \frac{1}{3}$. Now, $f_x = \frac{1}{2}$, $f_y = 0$,

$$(a_{ij}) = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & 1 \end{pmatrix}, \quad \det \|a_{ij}\| = \frac{5}{4}, \quad (a^{ij}) = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & 1 \end{pmatrix},$$

and $\tilde{b}^1 = \frac{2}{5}$, $\tilde{b}^2 = 0$ and $b^2 = \frac{1}{5}$.

Following the calculus from above, we get:

$$d_2^{11} = 10.7621695,$$

$$d_2^{12} = d_2^{21} = 0,$$

$$d_2^{22} = 18.5916118,$$

$$d_4^{11} = 4.1619406,$$

$$d_4^{12} = d_4^{21} = 0,$$

$$d_4^{22} = 3.3692342,$$

$$d_8^{11} = 255.0575035,$$

$$d_8^{12} = d_8^{21} = 0,$$

$$d_8^{22} = 185.6868118,$$

$$d_9^{11} = 24.6023378,$$

$$d_9^{12} = d_9^{21} = 0,$$

$$d_9^{22} = 23.1147203,$$

and for the four quadratic forms and β^* we get:

$$\alpha_2^{*2} = 10.7621695t^2 + 18.5916118s^2,$$

$$\alpha_4^{*2} = 4.1619406t^2 + 3.3692342s^2,$$

$$\alpha_8^{*2} = 255.0575035t^2 + 185.6868118s^2,$$

$$\alpha_9^{*2} = 24.6023378t^2 + 23.1147203s^2,$$

$$\beta^* = 0.4t.$$

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