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Combinatorial Diophantine equations – the genus 1 case

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Abstract. In this paper some Diophantine equations concerning binomial coefficients, power sums and product of consecutive integers are solved. The equations are reduced to genus 1 equations or Mordell-type equations and solved with the so-called *Ellog* method of Stroeker and Tzanakis and with the Magma computational algebra system.

1. Introduction

Many Diophantine equations possess combinatorial background. A lot of deep finiteness (both effective and ineffective) results are known about the solutions of such equations. We refer to the papers [4], [5], [7], [8], [23], [24] and the references given there. One of the first results giving all integer solutions of a combinatorial Diophantine equation is a theorem of Mordell [19], which provides all integer solutions of the equation $y(y+1) = x(x+1)(x+2)$. Other scattered equations have been investigated by several authors, see for example [1], [2], [6], [17], [22], [27], [35], [36], [40]. Hajdu and Pintér [15] systematically collected and solved those combinatorial equations that can be reduced to Mordell-type equations. The purpose of this note is to extend this result to more general combinatorial equations that can be reduced to general elliptic equations. Namely, we collect those equations that can be reduced to equations of genus 1. We mention that beside a lot of sparse results (see e.g. [20], [21], [22], [30] and

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[39]), Stroeker and de Weger [31] solved all such equations involving binomial coefficients.

Our equations can be reduced to elliptic Diophantine equations. By a classical result of Baker [3] it is known that such an equation can have only finitely many integer solutions and for the size of the solutions an effective upper bound can be given. The method applied in the paper is based on a different approach, which uses the algebraic and geometric features of elliptic curves. The theory of elliptic curves is a very intensively investigated field of number theory. As the literature of this topic is extremely rich, we only refer to the book of Silverman and Tate [26] and the references given there. The concrete algorithm we use is based upon a theorem obtained by Gebel, Pethő and Zimmer [12] and Stroeker and Tzanakis [27], later extended by Stroeker and Tzanakis [29]. We outline the algorithm in the third section and illustrate it with an example.

2. New results

As we mentioned in the introduction, we systematically collect and solve those unsolved combinatorial Diophantine equations which can be reduced to equations of genus 1 or to Mordell-type equations (see the details later). We need some notation to formulate our results. For all $n, x \in \mathbb{N}$ let

$$S_n(x) = 1^n + 2^n + \ldots + x^n,$$

$$P_n(x) = x \cdot (x+1) \cdot \ldots \cdot (x+n-1).$$

The formerly solved Diophantine equations which can be reduced to elliptic equations concerning $P_n(x)$, $S_n(x)$ and $\binom{x}{n}$, are the followings:

$P_2(k) = P_3(l)$ (Mordell [19]),

$\binom{k}{2} = \binom{l}{3}$ (Avanesov [1]),

$P_2(k) = P_6(l)$ (MacLeod and Barrodale [17]),

$S_2(k) = \binom{l}{2}$ (Avanesov [2] and Uchiyama[36]),

$P_3(k) = P_4(l),\ S_2(k) = \binom{l}{4}$ (Boyd and Kisilevsky [6]),

$\binom{k}{2} = P_3(l)$ (Tzanakis and de Weger [35]),

$\binom{k}{2} = P_4(l)$ (Pintér [20], see also [14], p. 225.),

$\binom{k}{4} = \binom{l}{2}$ (Pintér [21] and de Weger [39]),

$\binom{k}{3} = \binom{l}{4}$ (de Weger [40]),

$\binom{k}{4} = P_2(l), P_3(l)$ (Pintér and de Weger [22]),

$\binom{k}{m} = P_n(l)$, where $(m,n) = (3,6;3,6)$ (Stroeker and de Weger [30]),

$\binom{k}{m} = \binom{l}{n}$, where $(m,n) = (2;3,4,6,8), (3;4,6), (4;6,8)$ (Stroeker and de Weger [31]),

$S_5(k) = \binom{l}{2}$, $S_5(k) = \binom{l}{4}$, $S_m(k) = P_n(l)$, where $(m,n) = (2,5;2,4)$,

$\binom{k}{m} = P_n(l)$, where $(m,n) = (2,4;6), (3,6;2,4)$, $P_4(k) = P_6(l)$ (Hajdu and Pintér [15]).

Here and later on $(k,l) = (a_1,\dots,a_n;b_1,\dots,b_m)$ means that (k,l) can be any of the pairs (a_i,b_j), $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$.

We mention that $S_n(x)$ is a polynomial of degree $n+1$, and $P_n(x)$ is a polynomial of degree n. For the sake of completeness we give all integer solutions of the investigated polynomial equations (although the negative solutions do not have combinatorial meanings). Our results are summarized in the next theorem. We distribute the equations considered into three tables, according to the methods used in their solutions.

Theorem 1. *All integral solutions of the equations in the first columns of Tables 1–3 are exactly the ones appearing in the second columns of the tables, respectively.*

Equation	Solutions
$S_3(k) = P_2(l)$	$(k,l) = (-1,0;-1,0)$
$S_3(k) = P_4(l)$	$(k,l) = (-1,0;-3,-2,-1,0)$
$S_3(k) = P_8(l)$	$(k,l) = (-1,0;-7,-6,-5,-4,-3,-2,-1,0)$
$S_5(k) = \binom{l}{3}$	$(k,l) = (-1,0;0,1,2), (-2,1;3)$
$S_7(k) = \binom{l}{2}$	$(k,l) = (-1,0;0,1), (-2,1;-1,2)$
$P_2(k) = P_4(l)$	$(k,l) = (-1,0;-3,-2,-1,0)$
$P_2(k) = P_8(l)$	$(k,l) = (-1,0;-7,-6,-5,-4,-3,-2,-1,0)$
$P_3(k) = P_6(l)$	$(k,l) = (-2,-1,0;-5,-4,-3,-2,-1,0), (8;-6,1)$
$P_4(k) = P_8(l)$	$(k,l) = (-3,-2,-1,0;-7,-6,-5,-4,-3,-2,-1,0)$

Table 1. Equations which can be solved by Runge's method

Equation	Solutions
$S_3(k) = \binom{l}{3}$	$(k,l) = (-1,0;0,1,2), (-2,1;3)$
$S_3(k) = \binom{l}{6}$	$(k,l) = (-1,0;0,1,2,3,4,5), (-2,1;-1,6)$
$S_3(k) = P_3(l)$	$(k,l) = (-1,0;-2,-1,0)$
$S_3(k) = P_6(l)$	$(k,l) = (-1,0;-5,-4,-3,-2,-2,-1,0)$
$S_5(k) = \binom{l}{2}$	$(k,l) = (-1,0;0,1), (-2,1;-1,2), (-4,3;-23,24),$ $(-9,8;-351,352)$
$S_5(k) = \binom{l}{4}$	$(k,l) = (-1,0;0,1,2,3), (-2,1;-1,4)$
$S_5(k) = P_2(l)$	$(k,l) = (-1,0;-1,0)$
$S_5(k) = P_4(l)$	$(k,l) = (-1,0;-3,-2,-1,0)$

Table 2. Equations which can be reduced to Mordell-type equations

3. Proof of Theorem 1

The considered Diophantine equations can be divided into three groups.

Equations which can be solved by Runge's method. Consider the Diophantine equation $F(u) = G(v)$, where F and G are monic polynomials with integer coefficients, $F(u)-G(v)$ is irreducible in $\mathbb{Q}[u,v]$ and $\gcd(\deg F, \deg G) > 1$. We can use the method of Runge [25] for computing the integer solutions of such equations. Among the combinatorial Diophantine equations considered in the present paper, there are several ones which can be treated by this method. These equations are collected in Table 1. For example, using that $S_5(k) = \frac{1}{12}(2k^6+6k^5+5k^4-k^2) = \frac{1}{12}(2(k^2+k)^3-(k^2+k)^2)$, the equation $S_5(k) = \binom{l}{3}$ can be transformated to the equation $u^3-u^2 = v^3-6v^2+8v$ with the substitution $u = 2k^2+2k$, $v = 2l$, and the method of Runge can be applied. There are several results and efficient algorithms for finding the integer solutions of Runge-type equations, see for example Masser [18], Schinzel and Grytczuk [13], Szalay [32], Tengely [33] and Walsh [37] and the references given there. Tengely implemented his algorithm from [33] in the Magma computational algebra system [9] and made it accessible on the internet site www.math.klte.hu/∼tengely. We computed all integer solutions of the equations in Table 1 with Tengely's program. The total running time of the program was only a few minutes.

Equations which can be reduced to Mordell-type equations. Under a Mordell-type equation we mean a Diophantine equation $F(u) = G(v)$ with $\deg F = 3$, $\deg G = 2$ or conversely. These equations can be simply solved with Magma by the procedure `IntegralPoints`. The algorithm is based upon a theorem obtained independently by Gebel, Pethő and Zimmer [12] and Stroeker and

Equation	Solutions
$S_1(k) = \binom{l}{4}$	$(k,l) = (-21,20;-7,10), (-6,5;-3,6),$ $(-2,1;-1,4), (-1,0;0,1,2,3)$
$S_1(k) = \binom{l}{8}$	$(k,l) = (-1,0;0,1,2,3,4,5,6,7), (-2,1;-1,8),$ $(-10,9;-3,10;), (-78,77;-7,14), (-221,220;-10,17)$
$S_1(k) = P_4(l)$	$(k,l) = (-16,15;-5,2), (-1,0;-3,-2,-1,0))$
$S_1(k) = P_8(l)$	$(k,l) = (-1,0;-7,-6,-5,-4,-3,-2,-1,0)$
$S_2(k) = \binom{l}{3}$	$(k,l) = (-1,0;0,1,2), (-2,-1), (1,3)$
$S_2(k) = \binom{l}{6}$	$(k,l) = (-1,0;0,1,2,3,4,5), (1;-1,6)$
$S_2(k) = P_3(l)$	$(k,l) = (-1,0;-2,-1,0)$
$S_2(k) = P_6(l)$	$(k,l) = (-1,0;-5,-4,-3,-2,-1,0)$
$S_3(k) = \binom{l}{2}$	$(k,l) = (-4,3;-8,9), (-2,1;-1,2), (-1,0;0,1)$
$S_3(k) = \binom{l}{4}$	$(k,l) = (-2,1;-1,4), (-1,0;0,1,2,3)$
$S_3(k) = \binom{l}{8}$	$(k,l) = (-1,0;0,1,2,3,4,5,6,7), (-3,2;-2,9),$ $(-2,1;-1,8)$
$S_5(k) = \binom{l}{6}$	$(k,l) = (-1,0;0,1,2,3,4,5), (-2,1;-1,6)$
$S_5(k) = P_3(l)$	$(k,l) = (-1,0;-2,-1,0)$
$S_5(k) = P_6(l)$	$(k,l) = (-1,0;-5,-4,-3,-2,-1,0)$
$S_7(k) = \binom{l}{4}$	$(k,l) = (-2,1;-1,4), (-1,0;0,1,2,3)$
$S_7(k) = P_2(l)$	$(k,l) = (-1,0;-1,0)$
$S_7(k) = P_4(l)$	$(k,l) = (-1,0;-3,-2,-1,0)$
$\binom{k}{2} = P_8(l)$	$(k,l) = (0,1;-7,-6,-5,-4,-3,-2,-1,0)$
$\binom{k}{4} = P_4(l)$	$(k,l) = (0,1,2,3;-3,-2,-1,0)$
$\binom{k}{4} = P_8(l)$	$(k,l) = (0,1,2,3;-7,-6,-5,-4,-3,-2,-1,0)$
$\binom{k}{8} = P_2(l)$	$(k,l) = (0,1,2,3,4,5,6,7;-1,0)$
$\binom{k}{8} = P_4(l)$	$(k,l) = (0,1,2,3,4,5,6,7;-3,-2,-1,0)$

Table 3. Equations which can be reduced to genus 1 equations

Tzanakis [27]. As the algorithm for equations of genus 1 is the extension of this method, we give some details only later. We collected the equations which can be reduced to Mordell-type equations in Table 2. For example, the equation $S_3(k) = \binom{l}{3}$ can be written as $3(k(k+1))^2 = 2l(l-1)(l-2)$, which reduces to the Mordell-type equation $3u^2 = 2v^3 - 6v^2 + 4v$ by the substitution $u = k(k+1)$ and $v = l$. We determined all the integer solutions of these equations with Magma, and listed them in Table 2.

Equations which can be reduced to genus 1 equations. Table 3 contains equations that can be transformated into genus 1 equations with simple integral transformations. As finding the integer solutions of an equation of genus 1 is not at all

automatic, we give some details at this point. The algorithm is based upon a theorem obtained independently by Gebel, Pethő and Zimmer [12] and Stroeker and Tzanakis [27], later extended by Tzanakis [34], Stroeker and de Weger [30] and Stroeker and Tzanakis [29]. The method we use is the so-called *Ellog* method, developed by Stroeker and Tzanakis [29]. In the remaining part of this section, we use the terminology of [29] without any further reference.

3.1. The *Ellog* method. Let $f \in \mathbb{Z}[u, v]$ be irreducible over $\mathbb{Z}$, and consider the Diophantine equation

$$f(u, v) = 0 \tag{1}$$

and the corresponding curve

$$C = \{(u, v) \in \mathbb{Q}^2 \mid f(u, v) = 0\}.$$

If C is of genus 1 and non-empty, then (1) can be transformed into a short Weierstrass equation

$$y^2 = x^3 + Ax + B =: q(x) \tag{2}$$

with a birational transformation. Here $A, B \in \mathbb{Z}$, and the discriminant of $q(x)$, i.e. $4A^3 + 27B^2$ is non-zero. Put

$$E = \{(x, y) \in \mathbb{Q}^2 \mid y^2 = x^3 + Ax + B\}.$$

Let $P \in E$ be a point which is the image of an integral point of C. Then P has a unique representation of the form

$$P = m_1 P_1 + \ldots + m_r P_r + P_{r+1}, \tag{3}$$

where r is the rank of the E, $P_1, \ldots, P_r$ denote a Mordell–Weil basis of E, P_{r+1} is a torsion point and $m_i \in \mathbb{Z}$ $(i = 1, \ldots, r)$. Let

$$\mathcal{L}(P) := \mathcal{L} = m_1 u_1 + \ldots + m_r u_r + u_{r+1} - u_0 + m_0 \omega, \tag{4}$$

where u_i denote the elliptic logarithm of the points P_i $(i = 1, \ldots, r + 1)$, u_0 is the elliptic logarithm of a certain point Q_0 having algebraic coordinates and m_0 is a rational integer. Note that knowing C and the birational transformation, Q_0 can be easily computed. Put $M = \max_{1 \le i \le r}\{|m_i|\}$, and note that $m_0 \le rM + 1$. Combining certain estimates involving e.g. heights of points of E, we get

$$|\mathcal{L}| < c_1 \cdot \exp(c_2 - c_3 M^2) \tag{5}$$

with some constants c_1, c_2, c_3 depending only on certain parameters of the curves C and E. To get a lower estimation for $|\mathcal{L}|$, one needs a deep result of David [11] providing lower bounds for linear forms in elliptic logarithms. In the following lemma we formulate Tzanakis' variant of this result from [34]. First note that it is always possible to choose a pair of fundamental periods ω_1, ω_2 of the curve E in a way that $\tau := \omega_2/\omega_1$ satisfies

$$|\tau| \geq 1, \quad \Im\tau > 0, \ -\frac{1}{2} < \Re\tau \leq \frac{1}{2} \text{ with } 0 \leq \Re\tau \text{ if } |\tau| = 1.$$

Let ω be the fundamental real period of E, let D denote the degree of the number field generated by the coordinates of Q_0, and let $k = r + 1$ if u_0 is linearly independent of $u_1, \ldots, u_r$ over $\mathbb{Q}$, else set $k = r$. Finally, choose real numbers A_i ($i = 0, \ldots, r+1$) such that $A_0 \geq \max\left(h_E, \frac{3\pi|\omega|^2}{D|\omega_1|^2\Im\tau}\right)$, $A_i \geq \max\left(h_E, \frac{3\pi u_i^2}{D|\omega_1|^2\Im\tau}, \hat{h}(P_i)\right)$ ($i = 1, \ldots, r$), $A_{r+1} \geq \max\left(h_E, \frac{3\pi u_0^2}{D|\omega_1|^2\Im\tau}, \hat{h}(Q_0)\right)$. Here h_E is the height of E and $\hat{h}$ is the Néron–Tate height function.

Lemma 1 (Tzanakis [34])**.** *By the above notation we have*

$$|\mathcal{L}| > \exp\big(-c_4(\log N + c_5)(\log\log N + c_6)^{k+2}\big), \tag{6}$$

where

$$c_4 = 2.9 \cdot 10^{6k+12} D^{2k+4} 4^{2(k+1)^2} (k+2)^{2k^2+13k+23.3} \prod_{i=0}^{k} A_i,$$

$$c_5 = \log De, \quad c_6 = \log De + h_E,$$

and $N = \max\{|m_0|, M\}$.

Combining the upper bound (5) and the lower bound (6) for the linear form, using $N \leq rM + 1$ we obtain an upper estimate for M. This initial bound according to a heuristic argument of Stroeker and Tzanakis [28] is approximately around $10^{(5r^2+15r+28)/2}$, so it is too large to determine all integer solutions of the original equation. We use de Weger's method [38] based upon the LLL-algorithm to reduce this bound. Using the inverse of the birational transformation, after the reduction we can compute all integer solutions of equation (1). Put $\rho_i = u_i/\omega$ ($i = 1, \ldots, r$) and $\rho_0 = u_0/\omega$. In general, ρ_0 is linearly independent of $\rho_1, \ldots, \rho_r$ over $\mathbb{Q}$. In the opposite case, a simpler version of the reduction can be used. Consider the $(r+1)$-dimensional lattice Γ generated by the coloumns of

the matrix

$$A = \begin{pmatrix} 1 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ [K_0\rho_1] & \cdots & [K_0\rho_r] & K_0 \end{pmatrix},$$

where K_0 is a conveniently chosen integer, to be specified later. Compute the LLL-reduced basis of the lattice, and denote by b_1 the shortest vector of this basis. Write

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{r+1} \end{pmatrix} = B^{-1} \cdot \mathbf{x} \quad \text{with} \quad \mathbf{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -[K_0\rho_0] \end{pmatrix} \in \mathbb{R}^{r+1},$$

where B denotes the matrix whose columns are the vectors of the reduced basis. By a lemma of DE WEGER [38]

$$d(\mathbf{x}, \Gamma) \geq 2^{r/2} \|x_{i_0}\| |b_1|$$

holds, where $\|\cdot\|$ denotes the distance from the nearest integer, and $i_0 \in \{1, \dots, r+1\}$ is chosen so that $\|x_{i_0}\|$ is minimal among $\|x_1\|, \dots, \|x_{r+1}\|$. Then we have the following result.

Lemma 2 (DE WEGER [38])**.** *Let* $K_1 = \frac{c_1}{\omega} \exp c_2$, $K_2 = c_3$. *Then by the above notation,*

$$\|x_{i_0}\| |b_1| > 2^{r/2} \sqrt{(r^2+r)K_3^2 + 2rK_3 + 1}$$

implies that

$$M^2 \leq K_2^{-1} \left(\log K_0 K_1 - \log \sqrt{2^{-r} \|x_{i_0}\|^2 |b_1|^2 - rK_3^2} - rK_3 - 1 \right).$$

To use this result, we choose K_0 somewhat larger than $\left(2^{r/2} K_3 \sqrt{r^2+r}\right)^{r+1}$. Then by Lemma 2 (if the condition is satisfied) we get a new bound for M of the size $(K_2^{-1} \log K_3)^{1/2}$. We iterate this process (always with the new values of K_0 and K_3), until the new bound cannot be improved. Using this reduced bound we can determine the integer points of the curve C by the help of the inverse of the birational transformation, and hence all integer solutions of equation (1). In the next subsection we illustrate the *Ellog* method by an example.

3.2. Application of the *Ellog* method. The algorithm discussed in the previous subsection can always be used in cases when equation (1) has the form $F(u) = G(v)$, where $F, G \in \mathbb{Z}[x]$ with $\deg F = 4$, $\deg G = 2$ (quartic case) or $\deg F = \deg G = 3$ (cubic case). Among the equations in Table 3 the followings reduce to quartic ones:

$$S_1(k) = \binom{l}{4}, \qquad S_1(k) = \binom{l}{8}, \qquad S_1(k) = P_4(l), \qquad S_1(k) = P_8(l),$$

$$S_3(k) = \binom{l}{2}, \qquad S_3(k) = \binom{l}{4}, \qquad S_3(k) = \binom{l}{8}, \qquad S_7(k) = \binom{l}{4},$$

$$S_7(k) = P_2(l), \qquad S_7(k) = P_4(l), \qquad \binom{k}{2} = P_8(l), \qquad \binom{k}{4} = P_4(l)$$

$$\binom{k}{4} = P_8(l), \qquad \binom{k}{8} = P_2(l), \qquad \binom{k}{8} = P_4(l).$$

To transform these equations to the desired shape, we make use of the fact that all of $S_{2i-1}(x)$, $\binom{x}{2i}$ and $P_{2i}(x)$ can be written in the form $F(G(x))$ where $F, G \in \mathbb{Q}[x]$ with $\deg G = 2$, $\deg F = i$. For example, we have $S_7(k) = \frac{1}{24}(3k^8 + 12k^7 + 14k^6 - 7k^4 + 2k^2) = \frac{1}{24}(3(k^2+k)^4 - 4(k^2+k)^3 + 2(k^2+k)^2)$ and $P_4(l) = l(l+1)(l+2)(l+3) = (l^2 + 3l)(l^2 + 3l + 2)$. Hence the equation $S_7(k) = P_4(l)$ can be transformated to the equation $3u^4 - 4u^3 + 2u^2 = 6v^2 + 24v$ with the substitution $u = k^2 + k$, $v = 2(l^2 + 3l)$.

Note that the program package Magma contains a procedure (namely `IntegralQuarticPoints`) which is able to locate all integral points on quartic equations in some cases. (For details see the Magma manual [9].) However, in the previous versions of Magma this procedure apparently contains some error, and we solved all these equations following the *Ellog* method step-by-step. In case of each equation, we obtained exactly the solutions listed in Table 3. Note that in the new version of Magma (V.2.13–9) distributed by the beginning of 2007 the procedure `IntegralQuarticPoints` seems to be correct, and by its help we have also solved the above quartic equations except for $\binom{k}{8} = P_2(l)$, $\binom{k}{8} = P_4(l)$ and $S_1(k) = \binom{l}{8}$. In these cases Magma is only able to guarantee that `IntegralQuarticPoints` gives all integral points in a subgroup of the curve of finite index. In the other cases we have obtained the same solutions as in Table 3.

Now we turn to the cubic case. From Table 3 the following equations belong to this group:

$$S_2(k) = \binom{l}{3}, \qquad S_2(k) = \binom{l}{6}, \qquad S_2(k) = P_3(l), \qquad S_2(k) = P_6(l),$$

$$S_5(k) = \binom{l}{6}, \qquad S_5(k) = P_3(l), \qquad S_5(k) = P_6(l).$$

In this case no implemented version of the procedure is available, and we follow the *Ellog* method step-by-step for each equation. As an example, we illustrate the algorithm for finding the integer solutions of the equation $S_2(k) = P_6(l)$. Using $S_2(k) = \frac{1}{6}(2k^3 + 3k^2 + k)$ and $P_6(l) = l(l+1)(l+2)(l+3)(l+4)(l+5) = (l^2+5l)(l^2+5l+4)(l^2+5l+6)$, substituting $u = k$, $v = l^2 + 5l$, we get

$$f(u,v) = 2u^3 + 3u^2 + u - 6v^3 - 60v^2 - 144v = 0. \tag{7}$$

Put

$$C = \{(u,v) \in \mathbb{Q}^2 \mid f(u,v) = 0\}.$$

We use Magma to perform the following computations. By a birational transformation, equation (7) can be transformated into the short Weierstrass equation

$$y^2 = x^3 - 1008x + 2985993.$$

Set

$$E = \{(x,y) \in \mathbb{Q}^2 \mid y^2 = x^3 - 1008x + 2985993\}.$$

It turns out that the rank of E is $r = 6$, and the only torsion point of E is $\mathcal{O}$. Further, a basis of the Mordell-Weil group of E is

$$P_1 = (24, 1725), \qquad P_2 = (-36, 1725), \qquad P_3 = (234, 3945),$$
$$P_4 = (354, -6855), \qquad P_5 = (36, -1731), \qquad P_6 = (-144, 381).$$

Let

$$P = m_1P_1 + \ldots + m_6P_6 \quad (m_i \in \mathbb{Z}, i = 1, \ldots, 6)$$

be a point of E, which is the image of an integer point of C. In this case the linear form (4) is of the shape

$$\mathcal{L} = m_0\omega + m_1u_1 + m_2u_2 + m_3u_3 + m_4u_4 + m_5u_5 + m_6u_6 - u_0,$$

where ω is the fundamental real period of E, and u_0 and u_i are the elliptic logarithms of the points Q_0 and P_i $(i = 1, \ldots, 6)$, respectively. We have

$$Q_0 = \left(6\frac{1439 + 6902 \cdot \sqrt[3]{3}}{144 - \sqrt[3]{3}}, 361\frac{\sqrt[3]{3}}{3} + 864\sqrt[3]{9} + 8649\right)$$

and

$$\omega = 0.704584\ldots, \quad u_0 = 0.091196\ldots, \quad u_1 = 0.220969\ldots, \quad u_2 = 0.255688\ldots,$$

$$u_3 = 0.128958\ldots, \quad u_4 = 0.598701\ldots, \quad u_5 = 0.490562\ldots, \quad u_6 = 0.340110\ldots.$$

In this particular case, after some calculations (5) reads as

$$|\mathcal{L}| < 1.106568 \cdot 10^7 \cdot \exp(-0.598086M^2).$$

On the other hand, by Lemma 1 we obtain the lower bound

$$|\mathcal{L}| > \exp(-1.22724 \cdot 10^{354}(\log(N) + 2.09862)(\log(\log(N)) + 28.62165)^9).$$

Using that $N \leq 6M + 1$ and combining the upper and lower estimates for the linear form $\mathcal{L}$, we get the initial bound

$$M < K_3 = 2.753 \cdot 10^{185}.$$

We reduce this bound by the LLL-algorithm, using Lemma 2. The constants K_1 and K_2 are given by

$$K_1 = 1.570524 \cdot 10^7, \quad K_2 = 0.598086.$$

The reduction steps are summarized in the following table:

bound for M	K_0	new bound for M
$2.753 \cdot 10^{185}$	$5.3 \cdot 10^{1322}$	67
67	$6.4 \cdot 10^{47}$	15
15	$5.5 \cdot 10^{16}$	9

After the third iteration we obtain $M \leq 9$, which cannot be improved further. Using the inverse of the birational transformation, we can compute all the integer points of C. These are given by

$$(u, v) = (-1, 0; -6, -4, 0), (-33, -26), (-14, -13), (-11, -11), (2, -5).$$

In view of the original substitution, all integer solutions of the equation $S_2(k) = P_6(l)$ are

$$(k, l) = (-1, 0; -5, -4, -3, -2, -1, 0).$$

The integer solutions of all other cubic and quartic equations can be determined with a similar process and the solutions are exactly those which are summarized in Table 3. Hence the theorem is proved. □

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