

On the equality of generalized quasi-arithmetic means

By ZITA MAKÓ (Debrecen) and ZSOLT PÁLES (Debrecen)

Dedicated to the 70th birthday of Professor Zoltán Daróczy

Abstract. Given a continuous strictly monotone function $\varphi : I \rightarrow \mathbb{R}$ and a probability measure μ on the Borel subsets of $[0, 1]$, the two variable mean $M_{\varphi, \mu} : I^2 \rightarrow I$ is defined by

$$M_{\varphi, \mu}(x, y) := \varphi^{-1} \left(\int_0^1 \varphi(tx + (1-t)y) d\mu(t) \right) \quad (x, y \in I).$$

This class of means includes quasi-arithmetic as well as Lagrangian means. The aim of this paper is to study their equality problem, i.e., to characterize those pairs (φ, μ) and (ψ, ν) such that

$$M_{\varphi, \mu}(x, y) = M_{\psi, \nu}(x, y) \quad (x, y \in I)$$

holds. Under at most fourth-order differentiability assumptions for the unknown functions φ and ψ , a complete description of the solution set of the above functional equation is obtained.

1. Introduction

Throughout this paper I will stand for a nonempty open real interval and $\mathcal{CM}(I)$ will denote the class of real valued continuous strictly monotone functions defined on I .

Mathematics Subject Classification: 39B12, 39B22.

Key words and phrases: Generalized quasi-arithmetic means, equality problem.

This research has been supported by the Hungarian Scientific Research Fund (OKTA) Grant K-62316, Nk-6840.

Given $\varphi \in \mathcal{CM}(I)$, the *two variable quasi-arithmetic mean* generated by φ is the function $M_\varphi : I^2 \rightarrow I$ defined by

$$M_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) \quad (x, y \in I).$$

The systematic treatment of these means was first given by HARDY, LITTLEWOOD and PÓLYA [36]. The most basic problem, the characterization of the equality of these means, is solved by the following theorem.

Theorem A ([36]). *Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means M_φ and M_ψ are equal to each other if and only if there exist two real constants $a \neq 0$ and b such that $\psi = a\varphi + b$.*

Another class of means whose definition is related to the Lagrange mean value theorem were introduced by BERRONE and MORO [6], [5]: Given $\varphi \in \mathcal{CM}(I)$, the *two variable Lagrangian mean* generated by φ is the function $L_\varphi : I^2 \rightarrow I$ defined by

$$L_\varphi(x, y) := \begin{cases} \varphi^{-1}\left(\frac{1}{y-x} \int_x^y \varphi(t) dt\right) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \quad (x, y \in I).$$

The equality of these means is characterized by the following result of the paper.

Theorem B ([6]). *Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means L_φ and L_ψ are equal to each other if and only if there exist two real constants $a \neq 0$ and b such that $\psi = a\varphi + b$.*

Both classes of means have a rich literature, see, e.g., the monographs of BORWEIN–BORWEIN [8], MITRINOVIĆ–PEČARIĆ–FINK [51], [52], NICULESCU–PERSSON [55]. The characterization of quasi-arithmetic means was solved independently by KOLMOGOROV [41], NAGUMO [54], DE FINETTI [31] for the case when the number of variables is non-fixed. For the two-variable case, ACZÉL [1]–[4], proved a characterization theorem involving the notion of bisymmetry. This result was extended to the n -variable case by MAKSA–MÜNNICH–MOKKEN [53]. Another characterization is due to MATKOWSKI [49].

A recently rediscovered and blossoming subject is the investigation of the so-called invariance equation and the Gauss-iteration related to quasi-arithmetic means: GAUSS [33], BŁASIŃSKA–LESK–GŁAZOWSKA–MATKOWSKI [7], BURAI [9], [10], DARÓCZY [11]–[15], DARÓCZY–HAJDU [16], DARÓCZY–HAJDU–NG [17], DARÓCZY–LAJKÓ–LOVAS–MAKSA–PÁLES [18], DARÓCZY–MAKSA [19], DARÓCZY–MAKSA–PÁLES [20], [22], DARÓCZY–NG [23], DARÓCZY–PÁLES [25], [27], [26],

[28], [30], [24], [29], DOMSTA–MATKOWSKI [32], GŁAZOWSKA–JARCZYK–MATKOWSKI [34], HAJDU [35], HARUKI–RASSIAS [37], JARCZYK–MATKOWSKI [40], JARCZYK [38], MATKOWSKI [47], [48], [50],

The equality problem of means in various classes of two-variable means has been solved. We refer here to LOSONCZI's works [42]–[46] where the equality of two-variable means is characterized. A key idea in these papers, under high order differentiability assumptions, is to calculate and then compare the partial derivatives of the means at points of the form (x, x) . A paper where also the regularity properties are proved (not just assumed) is due to DARÓCZY–MAKSA–PÁLES [21], where means that are simultaneously quasi-arithmetic and arithmetic means weighted by a weight function are determined without assuming any regularity properties of the unknown functions. A similar problem, the mixed equality problem of quasi-arithmetic and Lagrangian means has been recently considered by J. JARCZYK [39], where the solutions satisfying an additional convexity condition were determined. This equality problem was completely solved by the following result of the second author:

Theorem C ([57]). *Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the means M_φ and L_ψ are equal to each other if and only if one of the following cases holds:*

- (i) *either there exist real constants a, b, c, d with $ac \neq 0$ such that*

$$\varphi(x) = ax + b, \quad \text{and} \quad \psi(x) = cx + d \quad (x \in I);$$

- (ii) *or there exist real constants a, b, c, d with $ac \neq 0$, and $q \notin I$ such that*

$$\varphi(x) = a \ln |x - q| + b, \quad \text{and} \quad \psi(x) = \frac{c}{(x - q)^2} + d \quad (x \in I);$$

- (iii) *or there exist real constants a, b, c, d with $ac \neq 0$, and $q \notin I$ such that*

$$\varphi(x) = a\sqrt{|x - q|} + b, \quad \text{and} \quad \psi(x) = \frac{c}{\sqrt{|x - q|}} + d \quad (x \in I);$$

- (iv) *or there exist real constants a, b, c, d, p, q with $ac \neq 0$ and $p > 0$ such that*

$$\varphi(x) = a \operatorname{arsinh}(p(x - q)) + b \quad \text{and} \quad \psi(x) = \frac{c(x - q)}{\sqrt{1 + p^2(x - q)^2}} + d \quad (x \in I);$$

- (v) *or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $p > 0$, and $I \cap [q - 1/p, q + 1/p] = \emptyset$ such that*

$$\varphi(x) = a \operatorname{arcosh}(p(x - q)) + b \quad \text{and} \quad \psi(x) = \frac{c(x - q)}{\sqrt{p^2(x - q)^2 - 1}} + d \quad (x \in I);$$

(vi) or there exist real constants a, b, c, d, p, q with $ac \neq 0$, $p > 0$, and $I \subseteq [q - 1/p, q + 1/p]$ such that

$$\varphi(x) = a \arcsin(p(x - q)) + b \quad \text{and} \quad \psi(x) = \frac{c(x - q)}{\sqrt{1 - p^2(x - q)^2}} + d \quad (x \in I).$$

In this paper, we consider the following common generalization of quasi-arithmetic and Lagrangian means: Given a continuous strictly monotone function $\varphi : I \rightarrow \mathbb{R}$ and a probability measure μ on the Borel subsets of $[0, 1]$, the two variable mean $M_{\varphi, \mu} : I^2 \rightarrow I$ is defined by

$$M_{\varphi, \mu}(x, y) := \varphi^{-1} \left(\int_0^1 \varphi(tx + (1 - t)y) d\mu(t) \right) \quad (x, y \in I).$$

If $\mu = \frac{\delta_0 + \delta_1}{2}$, then $M_{\varphi, \mu} = M_\varphi$. If $\mu = \text{Lebesgue measure on } [0, 1]$, then $M_{\varphi, \mu} = L_\varphi$.

The aim of this paper is to study the *equality problem of generalized quasi-arithmetic means*, i.e., to characterize those pairs (φ, μ) and (ψ, ν) such that

$$M_{\varphi, \mu}(x, y) = M_{\psi, \nu}(x, y) \quad (x, y \in I) \quad (1)$$

holds. Due to the complexity of the problem, we will not solve it in its natural generality. We shall need at most fourth-order differentiability properties of the unknown functions φ and ψ .

2. Notations and basic assumptions

Given a Borel probability measure μ on the interval $[0, 1]$, we define the k th moment and the k th centralized moment of μ by

$$\hat{\mu}_k := \int_0^1 t^k d\mu(t) \quad \text{and} \quad \mu_k := \int_0^1 (t - \hat{\mu}_1)^k d\mu(t) \quad (k \in \mathbb{N} \cup \{0\}).$$

Clearly, $\hat{\mu}_0 = \mu_0 = 1$ and $\mu_1 = 0$. In view of the binomial theorem, we easily obtain

$$\begin{aligned} \mu_k &= \int_0^1 (t - \hat{\mu}_1)^k d\mu(t) = \int_0^1 \sum_{i=0}^k (-1)^i \binom{k}{i} t^i \hat{\mu}_1^{k-i} d\mu(t) \\ &= \sum_{i=0}^k (-1)^i \binom{k}{i} \hat{\mu}_i \hat{\mu}_1^{k-i} \quad (k \in \mathbb{N}) \end{aligned} \quad (2)$$

and

$$\begin{aligned}\widehat{\mu}_k &= \int_0^1 ((t - \widehat{\mu}_1) + \widehat{\mu}_1)^k d\mu(t) = \int_0^1 \sum_{i=0}^k \binom{k}{i} (t - \widehat{\mu}_1)^i \widehat{\mu}_1^{k-i} d\mu(t) \\ &= \sum_{i=0}^k \binom{k}{i} \mu_i \widehat{\mu}_1^{k-i} \quad (k \in \mathbb{N}).\end{aligned}\quad (3)$$

The statement of the following lemma is obvious.

Lemma 1. *Let μ be a Borel probability measure on $[0, 1]$ and $k \in \mathbb{N}$. Then $\mu_{2k} \geq 0$ and equality can hold if and only if μ is the Dirac measure $\delta_{\widehat{\mu}_1}$.*

(In the sequel, δ_τ will denote the Dirac measure concentrated at the point $\tau \in [0, 1]$.)

On the other hand, the odd-order centralized moments can be zero. One can prove that $\mu_{2k-1} = 0$ holds for all $k \in \mathbb{N}$ if and only if μ is symmetric with respect to its first moment $\widehat{\mu}_1$, i.e., if $\mu(A) = \mu((2\widehat{\mu}_1 - A) \cap [0, 1])$ for all Borel sets $A \subseteq [0, 1]$.

To formulate the main results of this paper, we consider the cases when the first n moments of the measures μ and ν involved in (1) are identical. For $n \in \mathbb{N} \cup \{0, \infty\}$, we say that the *n th-order moment condition $\mathcal{M}_n$ holds* if μ, ν are Borel probability measures on $[0, 1]$, furthermore,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all } 1 \leq k \leq n. \quad (4)$$

Thus the $\mathcal{M}_\infty$ condition means that all the moments of μ and ν are equal, whence, by well-known results of measure and approximation theory, the equality of the two measure μ and ν follows. On the other hand, the condition $\mathcal{M}_0$ simply means that μ, ν are probability measures on the Borel subsets of $[0, 1]$. For $n \in \mathbb{N} \cup \{0\}$, we say that the *exact n th-order moment condition $\mathcal{M}_n^*$ holds* if $\mathcal{M}_n$ is valid but $\mathcal{M}_{n+1}$ fails, i.e.,

$$\widehat{\mu}_k = \widehat{\nu}_k \quad \text{for all } 1 \leq k \leq n \quad \text{and} \quad \widehat{\mu}_{n+1} \neq \widehat{\nu}_{n+1}. \quad (5)$$

It is obvious that, for all pairs of measures μ, ν , exactly one of the conditions $\mathcal{M}_0^*, \mathcal{M}_1^*, \mathcal{M}_2^*, \dots, \mathcal{M}_\infty$ can hold, i.e., $\mathcal{M}_0$ is the union of the pairwise exclusive cases $\mathcal{M}_0^*, \mathcal{M}_1^*, \mathcal{M}_2^*, \dots, \mathcal{M}_\infty$.

In view of the formulae (2) and (3), it is immediate to see that, for $n \geq 2$, $\mathcal{M}_n$ holds if and only if $\widehat{\mu}_1 = \widehat{\nu}_1$ and $\mu_k = \nu_k$ for $2 \leq k \leq n$.

In order to describe the various regularity conditions on the two unknown functions φ and ψ , for $k \in \mathbb{N} \cup \{\infty\}$, we say that the *n th-order regularity condition*

$\mathcal{C}_n$ holds if $\varphi, \psi : I \rightarrow \mathbb{R}$ are n -times continuously differentiable functions with non-vanishing first-order derivatives. For convenience, we also say that $\mathcal{C}_0$ holds if $\varphi, \psi : I \rightarrow \mathbb{R}$ are just continuous strictly monotone functions.

3. Basic results

In our first result, we compute the first partial derivatives of the mean $M_{\varphi, \mu}$ at a point of the diagonal of $I \times I$ under a weak regularity assumption. We note that, by Lebesgue theorem, φ is differentiable almost everywhere in I , however the derivative of φ can be equal to zero almost everywhere even if φ is strictly increasing.

Lemma 2. *Let μ be a Borel probability measure, let $\varphi : I \rightarrow \mathbb{R}$ be a continuous strictly monotone function and assume that φ is differentiable at a point $p \in I$ and $\varphi'(p) \neq 0$. Then $\partial_1 M_{\varphi, \mu}(p, p) = \widehat{\mu}_1$.*

PROOF. Using the differentiability of φ at p , one can easily see that the function $f : I \rightarrow \mathbb{R}$ defined by

$$f(x) := \int_0^1 \varphi(tx + (1-t)p) d\mu(t) \quad (x \in I)$$

is differentiable at p and $f'(p) = \int_0^1 t\varphi'(p) d\mu(t) = \varphi'(p)\widehat{\mu}_1$. We have that $M_{\varphi, \mu}(x, p) = \varphi^{-1}(f(x))$ and $\varphi'(p) \neq 0$ implies that φ^{-1} is differentiable at $\varphi(p) = f(p)$. Therefore, by the standard chain rule,

$$\partial_1 M_{\varphi, \mu}(p, p) = (\varphi^{-1})'(f(p)) \cdot f'(p) = \frac{1}{\varphi'(p)} \cdot \varphi'(p)\widehat{\mu}_1 = \widehat{\mu}_1. \quad \square$$

As an immediate consequence of the previous result, we obtain the first necessary condition for the equality of the generalized quasi-arithmetic means. This shows that, under weak regularity assumptions, there is no solution of the equality problem if the exact moment condition $\mathcal{M}_0^*$ holds.

Corollary 3. *Assume $\mathcal{C}_0$ and $\mathcal{M}_0$. Suppose that there exists a point $p \in I$ such that φ and ψ are differentiable at p and $\varphi'(p)\psi'(p) \neq 0$. Then, in order that $M_{\varphi, \mu} = M_{\psi, \nu}$ be valid, it is necessary that*

$$\widehat{\mu}_1 = \widehat{\nu}_1, \quad (6)$$

i.e., $\mathcal{M}_1$ be satisfied.

PROOF. Using Lemma 2 twice and the equality of the means $M_{\varphi,\mu}$ and $M_{\psi,\nu}$, we get

$$\widehat{\mu}_1 = \partial_1 M_{\varphi,\mu}(p, p) = \partial_1 M_{\psi,\nu}(p, p) = \widehat{\nu}_1. \quad \square$$

The necessary condition (6) does not involve the derivatives of φ and ψ explicitly. It remains an open problem to derive the necessity of (6) assuming only the continuity and monotonicity of the functions φ and ψ .

In view of Corollary 3, in the rest of the paper, we may assume that the first-order moment condition $\mathcal{M}_1$ holds.

In our next result, assuming $\mathcal{C}_1$, we obtain a characterization of the equality (1) that does not involve the inverses of the unknown functions φ and ψ .

Theorem 4. Assume $\mathcal{C}_1$ and $\mathcal{M}_1$. Then $M_{\varphi,\mu} = M_{\psi,\nu}$ holds if and only if

$$\int_0^1 \int_0^1 (t-s)\varphi'(tx + (1-t)y)\psi'(sx + (1-s)y)d\mu(t)d\nu(s) = 0 \quad (x, y \in I). \quad (7)$$

PROOF. *Necessity.* In view of the continuous differentiability of $\varphi, \psi : I \rightarrow \mathbb{R}$ and that φ' and ψ' do not vanish anywhere, the means $M_{\varphi,\mu}$ and $M_{\psi,\nu}$ are continuously partially differentiable with respect to their variables. Thus, (1) yields

$$\partial_1 M_{\varphi,\mu}(x, y) = \partial_1 M_{\psi,\nu}(x, y) \quad \text{and} \quad \partial_2 M_{\varphi,\mu}(x, y) = \partial_2 M_{\psi,\nu}(x, y) \quad (x, y \in I).$$

Hence,

$$\partial_1 M_{\varphi,\mu}(x, y)\partial_2 M_{\psi,\nu}(x, y) = \partial_1 M_{\psi,\nu}(x, y)\partial_2 M_{\varphi,\mu}(x, y) \quad (x, y \in I). \quad (8)$$

By an elementary calculation, (8) can be rewritten as

$$\begin{aligned} & \frac{\int_0^1 t\varphi'(tx + (1-t)y)d\mu(t)}{\varphi'(M_{\varphi,\mu}(x, y))} \cdot \frac{\int_0^1 (1-s)\psi'(sx + (1-s)y)d\nu(s)}{\psi'(M_{\psi,\nu}(x, y))} \\ &= \frac{\int_0^1 (1-t)\varphi'(tx + (1-t)y)d\mu(t)}{\varphi'(M_{\varphi,\mu}(x, y))} \cdot \frac{\int_0^1 s\psi'(sx + (1-s)y)d\nu(s)}{\psi'(M_{\psi,\nu}(x, y))} \quad (x, y \in I), \end{aligned}$$

which simplifies to

$$\begin{aligned} & \int_0^1 t\varphi'(tx + (1-t)y)d\mu(t) \int_0^1 (1-s)\psi'(sx + (1-s)y)d\nu(s) \\ &= \int_0^1 (1-t)\varphi'(tx + (1-t)y)d\mu(t) \int_0^1 s\psi'(sx + (1-s)y)d\nu(s) \quad (x, y \in I). \end{aligned} \quad (9)$$

On can easily see that (9) is equivalent to (7).

Sufficiency. We have that (7) is equivalent to (9), which easily yields (8). Therefore, it suffices to prove that (8) implies (1). For the sake of simplicity, denote

$$F(x, y) := M_{\varphi, \mu}(x, y), \quad G(x, y) := M_{\psi, \nu}(x, y) \quad (x, y \in I).$$

Due to the mean value property, we have

$$F(x, x) = x = G(x, x) \quad (x \in I).$$

Thus it remains to prove $F(x, y) = G(x, y)$ for $x \neq y$. Without loss of generality, we can assume that $x < y$. Set $z := F(x, y)$. Then $x < z < y$. By the continuity and strict monotonicity of φ , we have that the mapping $s \mapsto F(t, s)$ is continuous and strictly increasing on I for all fixed $t \in I$. Thus, for $t \in [x, z]$,

$$F(t, z) \leq F(z, z) = z = F(x, y) \leq F(t, y).$$

Therefore, for all $t \in [x, z]$, there exists a unique element $s \in [z, y]$ such that $F(t, s) = z$. Denote this element s by $f(t)$. Then f is a function mapping $[x, z]$ into $[z, y]$ and satisfying the identity

$$F(t, f(t)) = z \quad (t \in [x, z]) \tag{10}$$

and the boundary value conditions

$$f(x) = y \quad \text{and} \quad f(z) = z. \tag{11}$$

Due to the implicit function theorem, f is continuously differentiable on $[x, z]$. Differentiating (10) with respect to the variable t , it follows that

$$f'(t) = -\frac{\partial_1 F(t, f(t))}{\partial_2 F(t, f(t))} \quad (t \in [x, z]).$$

On the other hand, by (8), we have

$$\frac{\partial_1 F(t, f(t))}{\partial_2 F(t, f(t))} = \frac{\partial_1 G(t, f(t))}{\partial_2 G(t, f(t))} \quad (t \in [x, z]),$$

whence it follows that

$$\partial_1 G(t, f(t)) + f'(t) \partial_2 G(t, f(t)) = 0 \quad (t \in [x, z]).$$

Therefore, the mapping $t \mapsto G(t, f(t))$ is constant on $[x, z]$. Thus, by (11) and the definition of z ,

$$G(x, y) = G(x, f(x)) = G(z, f(z)) = G(z, z) = z = F(x, y).$$

This proves the equality of $F(x, y)$ and $G(x, y)$, i.e., the equality of $M_{\varphi, \mu}(x, y)$ and $M_{\psi, \nu}(x, y)$, too. $\square$

Substituting $x = y$ into (7) we get the condition

$$(\widehat{\mu}_1 \widehat{\nu}_0 - \widehat{\mu}_0 \widehat{\nu}_1) \varphi' \psi' = 0,$$

which simplifies to (6) because φ' and ψ' do not vanish anywhere. The result of Corollary 3 states the same condition under a weaker regularity assumption.

Assuming $\mathcal{C}_{n+1}$, we now deduce further conditions that are necessary for the equality (1).

Theorem 5. *Assume $\mathcal{C}_{n+1}$ for some $n \in \mathbb{N}$ and $\mathcal{M}_1$. Then, in order that $M_{\varphi, \mu} = M_{\psi, \nu}$ be valid, it is necessary that*

$$\sum_{i=0}^n \binom{n}{i} (\mu_{i+1} \nu_{n-i} - \mu_i \nu_{n+1-i}) \frac{\varphi^{(i+1)}}{\varphi'} \cdot \frac{\psi^{(n+1-i)}}{\psi'} = 0. \quad (12)$$

Conversely, if φ, ψ are analytic functions and (12) holds for all $n \in \mathbb{N}$, then $M_{\varphi, \mu} = M_{\psi, \nu}$ is satisfied.

PROOF. Denote by m the joint value of $\widehat{\mu}_1$ and $\widehat{\nu}_1$. Substituting $x := u + (1-m)v$ and $y := u - mv$ into (7), in view of Theorem 4, we can see that (7) holds for all $x, y \in I$ if and only if

$$F_u(v) := \int_0^1 \int_0^1 (t-s) \varphi'(u + (t-m)v) \psi'(u + (s-m)v) d\mu(t) d\nu(s) = 0 \quad (13)$$

$$(u \in I, v \in I_u),$$

where $I_u := \{v \in \mathbb{R} \mid (1-m)v, -mv \in I - u\}$ (which is a neighborhood of the origin). If $\mathcal{C}_{n+1}$ holds then, for all fixed $u \in I$, the function F_u is n -times continuously differentiable on I_u . Differentiating F_u n -times by applying the Leibniz rule, we obtain

$$F_u^{(n)}(v) = \int_0^1 \int_0^1 \sum_{i=0}^n \binom{n}{i} \varphi^{(i+1)}(u + (t-m)v) \\ \times \psi^{(n+1-i)}(u + (s-m)v) (t-s)(t-m)^i (s-m)^{n-i} d\mu(t) d\nu(s).$$

Now substituting $v := 0$, we get

$$\begin{aligned}
F_u^{(n)}(0) &= \int_0^1 \int_0^1 \sum_{i=0}^n \binom{n}{i} \varphi^{(i+1)}(u) \psi^{(n+1-i)}(u) (t-s)(t-m)^i (s-m)^{n-i} d\mu(t) d\nu(s) \\
&= \sum_{i=0}^n \binom{n}{i} \int_0^1 \int_0^1 (t-s)(t-m)^i (s-m)^{n-i} d\mu(t) d\nu(s) \varphi^{(i+1)}(u) \psi^{(n+1-i)}(u) \\
&= \sum_{i=0}^n \binom{n}{i} \int_0^1 \int_0^1 ((t-m)^{i+1} (s-m)^{n-i} \\
&\quad - (t-m)^i (s-m)^{n-i+1}) d\mu(t) d\nu(s) \varphi^{(i+1)}(u) \psi^{(n+1-i)}(u) \\
&= \sum_{i=0}^n \binom{n}{i} (\mu_{i+1} \nu_{n-i} - \mu_i \nu_{n-i+1}) \varphi^{(i+1)}(u) \psi^{(n+1-i)}(u).
\end{aligned}$$

If (13) holds, then $F_u^{(n)}(0) = 0$, whence the above formula for $F_u^{(n)}(0)$ divided by $\varphi'(u)\psi'(u)$ yields (12).

Conversely, assume that φ and ψ are analytic and (12) holds for all $n \in \mathbb{N}$. Then, for all fixed $u \in I$, the function F_u is analytic on the open interval I_u . On the other hand, (12) shows that $F_u^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. The equality $F_u(0) = 0$ is a consequence of $\widehat{\mu}_1 = \widehat{\nu}_1$. Therefore, due to its analyticity, the function F_u is identically zero over I_u . Thus (13) holds, whence the equality of the means $M_{\varphi,\mu}$ and $M_{\psi,\nu}$ follows. $\square$

In the particular case $n = 1$, the above theorem yields the following result.

Corollary 6. *Assume $\mathcal{C}_2$ and $\mathcal{M}_1$. Then, in order that $M_{\varphi,\mu} = M_{\psi,\nu}$ be valid, it is necessary that*

$$|\psi'|^{\nu_2} = \alpha |\varphi'|^{\mu_2} \quad (14)$$

for some constant $\alpha > 0$.

PROOF. In the case $n = 1$, condition (12) of Theorem 5 results

$$(\mu_1 \nu_1 - \mu_0 \nu_2) \frac{\psi''}{\psi'} + (\mu_2 \nu_0 - \mu_1 \nu_1) \frac{\varphi''}{\varphi'} = 0.$$

Using $\mu_0 = \nu_0 = 1$ and $\mu_1 = \nu_1 = 0$, the above equation can be rewritten as

$$-\nu_2 \frac{\psi''}{\psi'} + \mu_2 \frac{\varphi''}{\varphi'} = 0.$$

After integration, it follows that

$$-\nu_2 \ln |\psi'| + \mu_2 \ln |\varphi'| = \ln \left(|\psi'|^{-\nu_2} \cdot |\varphi'|^{\mu_2} \right)$$

is a constant function, which yields (14). $\square$

Though we assumed $\mathcal{C}_2$ in Corollary 6, the necessary condition (14) involves only the first-order derivatives of φ and ψ . It remains an open problem to derive the necessity of (14) under first-order continuous differentiability.

4. The case when $\mathcal{M}_\infty$ holds

In this section we solve the equality problem (1) if the two measures μ and ν coincide.

Theorem 7. *Assume $\mathcal{C}_0$ and $\mathcal{M}_\infty$. Then $M_{\varphi,\mu} = M_{\psi,\nu}$ holds if and only if*

- (i) *either $\mu = \nu = \delta_\tau$ for some $\tau \in [0, 1]$ and φ, ψ are arbitrary,*
- (ii) *or $\mu = \nu$ is not a Dirac measure and there exist constants $a \neq 0$ and b such that*

$$\psi = a\varphi + b. \quad (15)$$

PROOF. If $\mu = \nu = \delta_\tau$, then one can easily check that both sides of (1) are equal to $\tau x + (1 - \tau)y$, hence (1) is satisfied for any functions φ and ψ .

It is also elementary to see that condition (ii) is sufficient for the equality of the means $M_{\varphi,\mu}$ and $M_{\psi,\mu}$.

To show the necessity of (ii), assume that $M_{\varphi,\mu} = M_{\psi,\nu}$ and $\mu = \nu$ is not a Dirac measure. Define now the function $f : \varphi(I) \rightarrow \mathbb{R}$ by $f := \psi \circ \varphi^{-1}$. To prove that (15) holds for some constants $a \neq 0$ and b , it suffices to show that f is affine (i.e., convex and concave at the same time). Indeed, if f is affine then $f(t) = at + b$ for some constants a and b . Substituting $t = \varphi(x)$, (15) follows. (Note that, by the strict monotonicity of f , a cannot be zero.)

If f is not affine then either it is non-convex or non-concave over $J := \varphi(I)$. Without loss of generality, we can assume that f is non-convex and φ, ψ are strictly increasing functions. Applying the characterization of non-convexity obtained in [58], it follows that there exist a point $q \in J$ such that f is strictly concave at q , i.e., there exists a positive number δ and a constant a such that, for $t \in]q - \delta, q[$ and $s \in]q, q + \delta[$,

$$f(t) < f(q) + a(t - q) \quad \text{and} \quad f(s) \leq f(q) + a(s - q).$$

Substituting $t := \varphi(u)$, $s := \varphi(v)$, and denoting $p := \varphi^{-1}(q)$, it follows that there exists $\eta > 0$ such that, for $u \in]p - \eta, p[$ and $v \in]p, p + \eta[$,

$$\psi(u) < \psi(p) + a(\varphi(u) - \varphi(p)) \quad \text{and} \quad \psi(v) \leq \psi(p) + a(\varphi(v) - \varphi(p)). \quad (16)$$

Introduce the function $\tilde{\varphi}$ by $\tilde{\varphi}(u) := \psi(p) + a(\varphi(u) - \varphi(p))$. Then $\tilde{\varphi}$ is an affine transform of φ , hence we have the identity $M_{\varphi,\mu} = M_{\tilde{\varphi},\mu}$. On the other hand, by (16), for $u \in]p - \eta, p[$ and $v \in]p, p + \eta[$,

$$\psi(u) < \tilde{\varphi}(u), \quad \psi(v) \leq \tilde{\varphi}(v), \quad \text{and} \quad \psi(p) = \tilde{\varphi}(p). \quad (17)$$

By our assumption, μ is not a Dirac measure, hence $M_{\psi,\mu}$ is strictly increasing in both variables. Using also its continuity, we can easily find $x \in]p - \eta, p[$ and $y \in]p, p + \eta[$ such that $M_{\psi,\mu}(x, y) = p$. Define $\tau \in [0, 1]$ by the equality $\tau x + (1 - \tau)y = p$. Using that μ is not the Dirac measure δ_τ , we show that $\mu(] \tau, 1]) > 0$. Indeed, if $\mu(] \tau, 1]) = 0$, then $\mu([0, \tau[) > 0$. If $t \in [0, \tau[$ then $tx + (1 - t)y > \tau x + (1 - \tau)y = p$, hence, by the strict monotonicity of ψ ,

$$\begin{aligned} \psi(p) &= \psi(M_{\psi,\mu}(x, y)) = \int_0^1 \psi(tx + (1 - t)y) d\mu(t) = \int_{[0, \tau]} \psi(tx + (1 - t)y) d\mu(t) \\ &> \int_{[0, \tau]} \psi(\tau x + (1 - \tau)y) d\mu(t) = \int_{[0, \tau]} \psi(p) d\mu(t) = \mu([0, \tau]) \psi(p) = \psi(p), \end{aligned}$$

which is a contradiction. Thus $\mu(] \tau, 1]) > 0$ must be valid. On the other hand, if $t \in] \tau, 1]$ then $p - \eta < x \leq tx + (1 - t)y < \tau x + (1 - \tau)y = p$. Hence, by the first inequality in (17), we have

$$\psi(tx + (1 - t)y) < \tilde{\varphi}(tx + (1 - t)y) \quad (t \in] \tau, 1])$$

and, using the second inequality in (17), we also get

$$\psi(tx + (1 - t)y) \leq \tilde{\varphi}(tx + (1 - t)y) \quad (t \in [0, \tau]).$$

Using these inequalities, $\mu(] \tau, 1]) > 0$, and $M_{\tilde{\varphi},\mu} = M_{\varphi,\mu} = M_{\psi,\mu}$ we finally obtain

$$\begin{aligned} \psi(p) &= \psi(M_{\psi,\mu}(x, y)) = \int_0^1 \psi(tx + (1 - t)y) d\mu(t) \\ &< \int_0^1 \tilde{\varphi}(tx + (1 - t)y) d\mu(t) = \tilde{\varphi}(M_{\tilde{\varphi},\mu}(x, y)) = \tilde{\varphi}(M_{\psi,\mu}(x, y)) = \tilde{\varphi}(p), \end{aligned}$$

which contradicts the last equality in (17). This contradiction proves that f is affine. $\square$

5. The case when $\mathcal{M}_n^*$ holds for some $2 \leq n < \infty$

In this section we characterize the equality problem (1) assuming that at least the first two moments of the measures μ and ν are the same but the measures are not identical. The investigation of this case requires twice continuous differentiability of the unknown functions φ and ψ .

Theorem 8. *Assume $\mathcal{C}_2$ and $\mathcal{M}_n^*$ for some $2 \leq n < \infty$. Then $M_{\varphi,\mu} = M_{\psi,\nu}$ holds if and only if there exist constants $a \neq 0$ and b such that*

$$\psi = a\varphi + b \quad (18)$$

and φ is a polynomial with $\deg \varphi \leq n$.

PROOF. Since $n \geq 2$, condition $\mathcal{M}_n^*$ implies that

$$\mu_2 = \hat{\mu}_2 - \hat{\mu}_1^2 = \hat{\nu}_2 - \hat{\nu}_1^2 = \nu_2 =: \beta.$$

If β were zero, then, by Lemma 1, μ and ν are equal to some Dirac measures δ_τ and δ_σ ($\tau, \sigma \in [0, 1]$), respectively. By Corollary 3, we have $\hat{\mu}_1 = \hat{\nu}_1$ which yields that $\tau = \sigma$. Hence $\mu = \nu$ follows, which is impossible in the case when $\mathcal{M}_n^*$ holds for some $2 \leq n < \infty$. Consequently, β cannot be zero.

By Corollary 6, we have (14), which can be rewritten as $|\psi'|^\beta = \alpha|\varphi'|^\beta$. Hence, $\psi' = a\varphi'$ for some nonzero constant a which proves (18).

Using (18), we have the identity $M_{\psi,\nu} = M_{\varphi,\nu}$, therefore (1) is equivalent to the following equation

$$M_{\varphi,\mu}(x, y) = M_{\varphi,\nu}(x, y) \quad (x, y \in I). \quad (19)$$

Applying the function φ to both sides, we get

$$\int_0^1 \varphi(tx + (1-t)y) d(\mu - \nu)(t) = 0 \quad (x, y \in I). \quad (20)$$

Using a recent result of PÁLES [56], it follows that a function φ satisfying the linear functional equation (20) must be a polynomial, therefore it is infinitely many times continuously differentiable on I . Differentiating (20) $(n+1)$ -times with respect to x and then substituting $y := x$, we obtain

$$\int_0^1 t^{n+1} \varphi^{(n+1)}(x) d(\mu - \nu)(t) = 0 \quad (x \in I),$$

which yields $(\hat{\mu}_{n+1} - \hat{\nu}_{n+1})\varphi^{(n+1)} = 0$. By assumption $\mathcal{M}_n^*$, $\hat{\mu}_{n+1} - \hat{\nu}_{n+1}$ cannot be zero, hence $\varphi^{(n+1)} = 0$. Therefore, φ must be a polynomial with $\deg \varphi \leq n$.

Now assume that φ is a polynomial with $\deg \varphi \leq n$. Then, for fixed $x, y \in I$, the function $f(t) := \varphi(tx + (1-t)y)$ is again a polynomial of degree not bigger than n . Thus, by $\mathcal{M}_n^*$, (20) and hence (19) follows. Now using (18), we can see that (1) holds. $\square$

6. The case when $\mathcal{M}_1^*$ holds

In the investigation of this case we consider two subcases.

Subcase 1: $\mu_2\nu_2 = 0$.

Theorem 9. Assume $\mathcal{C}_2$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 = 0$. Then $M_{\varphi,\mu} = M_{\psi,\nu}$ holds if and only if

- (i) either μ and ψ are arbitrary, $\nu = \delta_{\widehat{\mu}_1}$, and there exist constants $a \neq 0$ and b such that

$$\varphi(x) = ax + b \quad (x \in I). \quad (21)$$

- (ii) or ν and φ are arbitrary, $\mu = \delta_{\widehat{\nu}_1}$, and there exist constants $c \neq 0$ and d such that

$$\psi(x) = cx + d \quad (x \in I), \quad (22)$$

PROOF. If $\mu_2 = \nu_2 = 0$, then $\mu_2 = \nu_2$, which contradicts $\mathcal{M}_1^*$. Thus, only one of the values μ_2 and ν_2 can be equal to zero.

In the first case, μ is equal to a Dirac measure δ_τ for some $\tau \in [0, 1]$. By $\widehat{\mu}_1 = \widehat{\nu}_1$, it follows that $\tau = \widehat{\nu}_1$. Now (14) can be rewritten as $|\psi'|^{\nu_2} = \alpha$, which results that ψ' is a constant function. Hence (22) follows for some constants $a \neq 0$ and b .

Conversely, one can easily check that if condition (ii) holds, then (1) is indeed satisfied.

The case $\nu_2 = 0$ is analogous. □

Subcase 2: $\mu_2\nu_2 \neq 0$.

In our first result, applying Theorem 5, we derive further necessary conditions for the equality (1).

Theorem 10. Assume $\mathcal{C}_2$ and $\mathcal{M}_1$ with $\mu_2\nu_2 \neq 0$ and assume that equality $M_{\varphi,\mu} = M_{\psi,\nu}$ holds. Then

$$\nu_2 \frac{\psi''}{\psi'} = \mu_2 \frac{\varphi''}{\varphi'} =: \Phi. \quad (23)$$

If $\mathcal{C}_3$ is valid then the function $\Phi : I \rightarrow \mathbb{R}$ introduced in (23) satisfies the differential equation

$$\left(\frac{\mu_3}{\mu_2} - \frac{\nu_3}{\nu_2} \right) \Phi' + \left(\frac{\mu_3}{\mu_2^2} - \frac{\nu_3}{\nu_2^2} \right) \Phi^2 = 0. \quad (24)$$

If $\mathcal{C}_4$ is also valid, then Φ satisfies the differential equation

$$\left(\frac{\mu_4}{\mu_2} - \frac{\nu_4}{\nu_2} \right) \Phi'' + \left(\frac{3\mu_4}{\mu_2^2} - \frac{3\nu_4}{\nu_2^2} \right) \Phi \Phi' + \left(\frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3} \right) \Phi^3 = 0. \quad (25)$$

If $\mathcal{M}_1^*$ holds then the three coefficients in this equation do not vanish simultaneously and φ and ψ are analytic functions.

PROOF. If $\mathcal{C}_2$ is valid then, from (14), we get that (23) holds. By this definition of the function Φ , we have that

$$\frac{\varphi''}{\varphi'} = \frac{\Phi}{\mu_2} \quad \text{and} \quad \frac{\psi''}{\psi'} = \frac{\Phi}{\nu_2}. \quad (26)$$

To show (24), assume $\mathcal{C}_3$. Differentiating the equalities in (26), it follows that

$$\frac{\varphi'''}{\varphi'} = \frac{\Phi'}{\mu_2} + \frac{\Phi^2}{\mu_2^2} \quad \text{and} \quad \frac{\psi'''}{\psi'} = \frac{\Phi'}{\nu_2} + \frac{\Phi^2}{\nu_2^2}. \quad (27)$$

In the particular case $n = 2$, condition (12) of Theorem 5 yields

$$(\mu_1\nu_2 - \mu_0\nu_3)\frac{\psi'''}{\psi'} + 2(\mu_2\nu_1 - \mu_1\nu_2)\frac{\varphi''}{\varphi'} \cdot \frac{\psi''}{\psi'} + (\mu_3\nu_0 - \mu_2\nu_1)\frac{\varphi'''}{\varphi'} = 0. \quad (28)$$

Using $\mu_1 = \nu_1 = 0$ and the identities (26), (27), equation (28) can be rewritten as

$$-\nu_3\left(\frac{\Phi'}{\nu_2} + \frac{\Phi^2}{\nu_2^2}\right) + \mu_3\left(\frac{\Phi'}{\mu_2} + \frac{\Phi^2}{\mu_2^2}\right) = 0.$$

which results the differential equation (24).

If the regularity assumption $\mathcal{C}_4$ holds, then differentiating (27) again, one obtains

$$\frac{\varphi''''}{\varphi'} = \frac{\Phi''}{\mu_2} + 3\frac{\Phi\Phi'}{\mu_2^2} + \frac{\Phi^3}{\mu_2^3} \quad \text{and} \quad \frac{\psi''''}{\psi'} = \frac{\Phi''}{\nu_2} + 3\frac{\Phi\Phi'}{\nu_2^2} + \frac{\Phi^3}{\nu_2^3}. \quad (29)$$

On the other hand, in the particular case $n = 3$, condition (12) of Theorem 5 yields

$$\begin{aligned} (\mu_1\nu_3 - \mu_0\nu_4)\frac{\psi''''}{\psi'} + 3(\mu_2\nu_2 - \mu_1\nu_3)\frac{\varphi''}{\varphi'} \cdot \frac{\psi'''}{\psi'} \\ + 3(\mu_3\nu_1 - \mu_2\nu_2)\frac{\varphi'''}{\varphi'} \cdot \frac{\psi''}{\psi'} + (\mu_4\nu_0 - \mu_3\nu_1)\frac{\varphi''''}{\varphi'} = 0. \end{aligned} \quad (30)$$

Using $\mu_1 = \nu_1 = 0$, applying (26), (27), and (29), equation (30) can be rewritten in the following form:

$$\begin{aligned}
& -\nu_4 \left(\frac{\Phi''}{\nu_2} + 3 \frac{\Phi\Phi'}{\nu_2^2} + \frac{\Phi^3}{\nu_2^3} \right) + 3\mu_2\nu_2 \frac{\Phi}{\mu_2} \left(\frac{\Phi'}{\nu_2} + \frac{\Phi^2}{\nu_2^2} \right) \\
& - 3\mu_2\nu_2 \left(\frac{\Phi'}{\mu_2} + \frac{\Phi^2}{\mu_2^2} \right) \frac{\Phi}{\nu_2} + \mu_4 \left(\frac{\Phi''}{\mu_2} + 3 \frac{\Phi\Phi'}{\mu_2^2} + \frac{\Phi^3}{\mu_2^3} \right) = 0,
\end{aligned}$$

which results (25), an at most second-order differential equation for Φ . Introduce the notations

$$\eta := \frac{\mu_4}{\mu_2} - \frac{\nu_4}{\nu_2}, \quad \gamma := \frac{3\mu_4}{\mu_2^2} - \frac{3\nu_4}{\nu_2^2}, \quad \delta := \frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}. \quad (31)$$

First we show that the constants η , γ , and δ , cannot be simultaneously zero if $\mathcal{M}_1^*$ holds. The equations $\eta = 0$ and $\gamma = 0$ form a system of linear equations for the unknowns μ_4, ν_4 . The determinant of this system is nonzero because $\mu_2 - \nu_2 \neq 0$ by $\mathcal{M}_1^*$. Thus $\mu_4 = \nu_4 = 0$. Then the equation $\delta = 0$ yields $\mu_2 = \nu_2$, which again contradicts $\mathcal{M}_1^*$. Therefore, the coefficients in (25) do not vanish simultaneously.

To show that φ and ψ are analytic, in view of (26), it suffices to show that Φ is analytic.

If $\eta \neq 0$, then (25) is an explicit second-order differential equation for Φ . Applying the results on the analyticity of the solutions of such equations, it follows that Φ is analytic.

If $\eta = 0$, then (25) could be rewritten as

$$\Phi(\gamma\Phi' + \delta\Phi^2) = 0. \quad (32)$$

We show that this equation is satisfied if and only if

$$\gamma\Phi' + \delta\Phi^2 = 0. \quad (33)$$

Denote

$$J := \{t \in I : \gamma\Phi'(t) + \delta\Phi^2(t) \neq 0\}.$$

Then J is an open subset of I . By (32), Φ has to be zero on J . By the openness of J , it follows that Φ' is also zero on J . Hence J must be empty which means that (33) holds. If $\gamma \neq 0$, then (33) is a first-order explicit differential equation for Φ . Thus, it follows that Φ is analytic. If $\gamma = 0$, then δ cannot be zero, therefore $\Phi = 0$, which again yields the analyticity of Φ . $\square$

In our second result, we obtain a necessary and sufficient condition for the equality problem (1) under the additional assumption that Φ satisfies a first-order polynomial differential equation.

Theorem 11. Assume $\mathcal{C}_3$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$. Suppose that (23) holds and that there exists integer numbers $0 \leq 2n \leq k$ and a constant vector $(c_0, \dots, c_n) \neq (0, \dots, 0)$ such that the function $\Phi : I \rightarrow \mathbb{R}$ introduced in (23) satisfies the following first-order polynomial differential equation

$$\sum_{i=0}^n c_i \Phi^{k-2i} (\Phi')^i = 0. \quad (34)$$

Then $M_{\varphi, \mu} = M_{\psi, \nu}$ holds if and only if

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that

$$\varphi(x) = ax + b, \quad \text{and} \quad \psi(x) = cx + d \quad (x \in I); \quad (35)$$

(ii) or there exist real constants a, b, c, d, p, q with $ac(p-q) \neq 0, pq > 0$ such that

$$\varphi(x) = ae^{px} + b \quad \text{and} \quad \psi(x) = ce^{qx} + d \quad (x \in I) \quad (36)$$

and, for $n \in \mathbb{N}$,

$$\sum_{i=0}^n \binom{n}{i} p^i q^{n-i} (\mu_{i+1}\nu_{n-i} - \mu_i\nu_{n+1-i}) = 0; \quad (37)$$

(iii) or there exist real constants a, b, c, d, p, q with $ac(p-q) \neq 0, (p-1)(q-1) > 0$, and $x_0 \notin I$ such that, for $x \in I$,

$$\begin{aligned} \varphi(x) &= \begin{cases} a|x - x_0|^p + b, & \text{if } p \neq 0 \\ a \ln |x - x_0| + b, & \text{if } p = 0 \end{cases} \quad \text{and} \\ \psi(x) &= \begin{cases} c|x - x_0|^q + d, & \text{if } q \neq 0 \\ c \ln |x - x_0| + d, & \text{if } q = 0 \end{cases} \end{aligned} \quad (38)$$

and, for $n \in \mathbb{N}$,

$$\sum_{i=0}^n \binom{p-1}{i} \binom{q-1}{n-i} (\mu_{i+1}\nu_{n-i} - \mu_i\nu_{n+1-i}) = 0. \quad (39)$$

PROOF. To solve (34), we distinguish three cases.

Case 1: $\Phi = 0$ (which is trivially a solution of (34)). Then $\varphi'' = 0$, whence $\varphi' = a$, and by (14), also $\psi' = c$ for some nonzero constants a and c . Therefore, in this case, statement (i) of the theorem must be valid.

Conversely, if (i) holds, then, for all $x, y \in I$,

$$M_{\varphi, \mu}(x, y) = \hat{\mu}_1 x + (1 - \hat{\mu}_1)y \quad \text{and} \quad M_{\psi, \nu}(x, y) = \hat{\nu}_1 x + (1 - \hat{\nu}_1)y,$$

hence the equality of the means follows from $\hat{\mu}_1 = \hat{\nu}_1$.

In the rest of the proof we may assume that Φ is not identically zero. Denote by J a maximal subinterval of I where Φ does not vanish. Clearly, J is open and nonempty and (34) can be rewritten as

$$\sum_{i=0}^n c_i \left(\frac{\Phi'(x)}{\Phi^2(x)} \right)^i = 0 \quad (x \in J). \quad (40)$$

Therefore, the values of the function $\frac{\Phi'}{\Phi^2}$ on J are equal to the roots of the polynomial $P(x) := \sum_{i=0}^n c_i x^i$. Due to the continuity, we get that

$$\frac{\Phi'(x)}{\Phi^2(x)} = c \quad (x \in J), \quad (41)$$

where the constant c is one of the roots of the polynomial P . Now we can consider the cases $c = 0$ and $c \neq 0$.

Case 2: $c = 0$. Then, (41) says that $\Phi' = 0$ on J . Thus, there exists a nonzero constant p such that $\Phi = \mu_2 p$ on J . If J were a proper subinterval of I , then one of the endpoints of J , say α , would be contained in I . By the continuity, we have $\Phi(\alpha) = \mu_2 p \neq 0$, which results that J is not maximal. The contradiction so obtained shows that $J = I$.

Using the definition of Φ , we get that $\varphi'' = p\varphi'$. Integrating this equality, we can find a constant b such that $\varphi' = p(\varphi - b)$. This is a first-order linear differential equation for φ , whose general solution is of the form $\varphi(x) = ae^{px} + b$ for some constant a . Of course, ap cannot be zero, otherwise φ is not strictly monotone. Using (14), it follows that ψ is also of the form stated in (36) of (ii), where $q = (\mu_2/\nu_2)p$. Clearly $pq = (\mu_2/\nu_2)p^2 > 0$. The condition $\mathcal{M}_1^*$, i.e., $\mu_2 \neq \nu_2$ implies that $q \neq p$. The functions φ and ψ are obviously analytic, hence, Theorem 5 can be applied. Using

$$\frac{\varphi^{(j)}(x)}{\varphi'(x)} = p^{j-1}, \quad \frac{\psi^{(j)}(x)}{\psi'(x)} = q^{j-1}, \quad (x \in I, j \in \mathbb{N}),$$

one can see that (12) is equivalent to (37), therefore, by Theorem 5, the means $M_{\varphi, \mu}$ and $M_{\psi, \nu}$ are identical if and only if (37) holds for all $n \in \mathbb{N}$.

Case 3: $c \neq 0$. Then, with the notation $p := 1 + 1/(\mu_2 c) \neq 1$, (41) can be rewritten as

$$\frac{\Phi'(x)}{\Phi^2(x)} = \frac{1}{\mu_2(p-1)} \quad (x \in J).$$

Integrating this equality, it follows, for some x_0 , that

$$\frac{1}{\Phi(x)} = \frac{x - x_0}{\mu_2(p-1)} \quad (x \in J). \quad (42)$$

Hence x_0 cannot be in J . If J were a proper subinterval of I , then one of the endpoints of J , say α , would be an element of I . By taking the limit $x \rightarrow \alpha$ in the above equation, it follows that Φ has a finite nonzero limit at α . By continuity, this yields that $\Phi(\alpha) = \frac{\mu_2(p-1)}{\alpha - x_0} \neq 0$, showing that J is not maximal. The contradiction so obtained proves that $J = I$. Applying (42) and the definition (23) of the function Φ , we get

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\Phi(x)}{\mu_2} = \frac{p-1}{x - x_0} \quad (x \in J).$$

Integrating this equation, it results that

$$\varphi'(x) = \begin{cases} ap|x - x_0|^{p-1}, & \text{if } p \neq 0 \\ a|x - x_0|^{-1}, & \text{if } p = 0 \end{cases}$$

for some constant a . After integration this yields that φ is of the form (38). Using (14), we get that ψ is also of the form (38) with $q := 1 + (\mu_2/\nu_2)(p-1)$. Obviously, $(p-1)(q-1) = (\mu_2/\nu_2)(p-1)^2 > 0$. We also have $ac \neq 0$ otherwise φ or ψ is not strictly monotone. The condition $p \neq q$ follows from $\mu_2 \neq \nu_2$.

Now assume that $x_0 \leq \inf I$ (the case $x_0 \geq \sup I$ is analogous). In view of (38), the functions φ and ψ are analytic and we have

$$\begin{aligned} \frac{\varphi^{(j)}(x)}{\varphi'(x)} &= (j-1)! \binom{p-1}{j-1} (x - x_0)^{1-j}, \\ \frac{\psi^{(j)}(x)}{\psi'(x)} &= (j-1)! \binom{q-1}{j-1} (x - x_0)^{1-j}, \quad (x \in I, j \in \mathbb{N}). \end{aligned}$$

Using these formulae, we can see that (12) is valid if and only if (39) holds. Therefore, by Theorem 5, the equality of the means $M_{\varphi, \mu}$ and $M_{\psi, \nu}$ is equivalent to the validity of condition (39) for all $n \in \mathbb{N}$. $\square$

Subcase 2.A: $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq (0, 0)$.

Theorem 12. Assume $\mathcal{C}_3$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq (0, 0)$. Then $M_{\varphi, \mu} = M_{\psi, \nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 11 is satisfied.

PROOF. By Theorem 10, we have that (24) holds. We show that (24) is not a trivial equation, i.e., one of the coefficients different from zero. Indeed, if both coefficients were zero, then we would get a homogeneous system of linear equations for the unknowns μ_3 and ν_3 . Since the determinant of this linear system is $(\mu_2 - \nu_2)/(\mu_2\nu_2)^2 \neq 0$ hence $\mu_3 = \nu_3 = 0$, which contradicts the assumption $(\mu_3, \nu_3) \neq (0, 0)$ of the theorem.

Thus (24) is a nontrivial first-order polynomial differential equation for Φ . The statement now follows from Theorem 10. $\square$

If $\mu_3 = \nu_3 = 0$, then the necessary condition (24) of Theorem 10 does not result any information. Thus, we may apply differential equation (25). Unfortunately, this equation can be solved explicitly if $\mu_2\nu_4 = \nu_2\mu_4$. In the remaining cases, we shall use again the necessary condition (12) of Theorem 5 in the cases $n = 4$ and $n = 5$.

Subcase 2.B: $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0, 0)$, and $\mu_2\nu_4 = \nu_2\mu_4$.

Theorem 13. Assume $\mathcal{C}_4$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0, 0)$, and $\mu_2\nu_4 = \nu_2\mu_4$. Then $M_{\varphi, \mu} = M_{\psi, \nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 11 is satisfied.

PROOF. As we have shown in Theorem 10, the functions φ and ψ are analytic on I and Φ defined by (23) satisfies

$$\eta\Phi'' + \gamma\Phi\Phi' + \delta\Phi^3 = 0, \quad (43)$$

where the constants η, γ, δ are defined by (31).

Now, by $\mu_2\nu_4 = \nu_2\mu_4$, we have that $\eta = 0$ then, (43) is an equation of the form (34). Thus, by Theorem 10, one of the alternatives (i), (ii), or (iii) must be valid. $\square$

Subcase 2.C: $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0, 0)$, and $\mu_2\nu_4 \neq \nu_2\mu_4$.

Theorem 14. Assume $\mathcal{C}_4$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0, 0)$, $\mu_2\nu_4 \neq \nu_2\mu_4$, $(\mu_5, \nu_5) \neq (0, 0)$, and

$$(\mu_5 - \nu_5)^2 + (\mu_4 - 3\mu_2\nu_2)^2 + (\nu_4 - 3\mu_2\nu_2)^2 \neq 0. \quad (44)$$

Then $M_{\varphi, \mu} = M_{\psi, \nu}$ holds if and only if one of the alternatives (i), (ii), or (iii) of Theorem 11 is satisfied.

PROOF. As we have shown in Theorem 10, the functions φ and ψ are analytic on I and Φ defined by (23) satisfies (43), where the constants η, γ, δ are defined by (31).

By condition $\mu_2\nu_4 \neq \nu_2\mu_4$, we have that $\eta \neq 0$, therefore (43) is a second-order differential equation that cannot be solved explicitly. However, using this equation, the second and third (and also higher-order) derivatives of Φ can be expressed as a polynomial of Φ and Φ' . With the notations $\alpha := -\gamma/\eta$ and $\beta := -\delta/\eta$, easily follows from (43) that

$$\Phi'' = \alpha\Phi\Phi' + \beta\Phi^3 \quad \text{and} \quad \Phi''' = \alpha(\Phi')^2 + (\alpha^2 + 3\beta)\Phi^2\Phi' + \alpha\beta\Phi^4. \quad (45)$$

In the particular case $n = 4$, condition (12) of Theorem 5 yields

$$\begin{aligned} & (\mu_1\nu_4 - \mu_0\nu_5) \frac{\psi''''}{\psi'} + 4(\mu_2\nu_3 - \mu_1\nu_4) \frac{\varphi''}{\varphi'} \cdot \frac{\psi''''}{\psi'} + 6(\mu_3\nu_2 - \mu_2\nu_3) \frac{\varphi'''}{\varphi'} \cdot \frac{\psi'''}{\psi'} \\ & + 4(\mu_4\nu_1 - \mu_3\nu_2) \frac{\varphi''''}{\varphi'} \cdot \frac{\psi''}{\psi'} + (\mu_5\nu_0 - \mu_4\nu_1) \frac{\varphi''''}{\varphi'} = 0. \end{aligned} \quad (46)$$

Differentiating (29), we get that

$$\begin{aligned} \frac{\varphi''''}{\varphi'} &= \frac{\Phi'''}{\mu_2} + 4 \frac{\Phi\Phi''}{\mu_2^2} + 3 \frac{(\Phi')^2}{\mu_2^2} + 6 \frac{\Phi^2\Phi'}{\mu_2^3} + \frac{\Phi^4}{\mu_2^4}, \\ \frac{\psi''''}{\psi'} &= \frac{\Phi'''}{\nu_2} + 4 \frac{\Phi\Phi''}{\nu_2^2} + 3 \frac{(\Phi')^2}{\nu_2^2} + 6 \frac{\Phi^2\Phi'}{\nu_2^3} + \frac{\Phi^4}{\nu_2^4}. \end{aligned} \quad (47)$$

Now using $\mu_1 = \nu_1 = \mu_3 = \nu_3 = 0$, and (47), equation (46) simplifies to the following (at most) third-order differential equation for Φ :

$$\begin{aligned} & \left(\frac{\mu_5}{\mu_2} - \frac{\nu_5}{\nu_2} \right) \Phi''' + 4 \left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2} \right) \Phi\Phi'' + 3 \left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2} \right) (\Phi')^2 \\ & + 6 \left(\frac{\mu_5}{\mu_2^3} - \frac{\nu_5}{\nu_2^3} \right) \Phi^2\Phi' + \left(\frac{\mu_5}{\mu_2^4} - \frac{\nu_5}{\nu_2^4} \right) \Phi^4 = 0. \end{aligned}$$

Substituting the formulae from (45) into the above equation, we get

$$\begin{aligned} & \left(\frac{\mu_5}{\mu_2} - \frac{\nu_5}{\nu_2} \right) (\alpha(\Phi')^2 + (\alpha^2 + 3\beta)\Phi^2\Phi' + \alpha\beta\Phi^4) + 4 \left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2} \right) (\alpha\Phi^2\Phi' + \beta\Phi^4) \\ & + 3 \left(\frac{\mu_5}{\mu_2^2} - \frac{\nu_5}{\nu_2^2} \right) (\Phi')^2 + 6 \left(\frac{\mu_5}{\mu_2^3} - \frac{\nu_5}{\nu_2^3} \right) \Phi^2\Phi' + \left(\frac{\mu_5}{\mu_2^4} - \frac{\nu_5}{\nu_2^4} \right) \Phi^4 = 0. \end{aligned}$$

Finally, we obtain the following (at most) first-order differential equation for Φ :

$$\begin{aligned} & \left(\frac{3 + \alpha\mu_2}{\mu_2^2} \mu_5 - \frac{3 + \alpha\nu_2}{\nu_2^2} \nu_5 \right) (\Phi')^2 \\ & + \left(\frac{6 + 4\alpha\mu_2 + (\alpha^2 + 3\beta)\mu_2^2}{\mu_2^3} \mu_5 - \frac{6 + 4\alpha\nu_2 + (\alpha^2 + 3\beta)\nu_2^2}{\nu_2^3} \nu_5 \right) \Phi^2 \Phi' \\ & + \left(\frac{1 + 4\beta\mu_2^2 + \alpha\beta\mu_2^3}{\mu_2^4} \mu_5 - \frac{1 + 4\beta\nu_2^2 + \alpha\beta\nu_2^3}{\nu_2^4} \nu_5 \right) \Phi^4 = 0. \end{aligned} \quad (48)$$

In the next step we show that the three constant coefficients in this equation cannot be simultaneously zero. Indeed, if all these coefficients are zero then, using that $(\mu_5, \nu_5) \neq (0, 0)$, we can see that the following two vectors in $\mathbb{R}^3$ are linearly dependent:

$$\begin{aligned} u &= (u_1, u_2, u_3) := \left(\frac{3 + \alpha\mu_2}{\mu_2^2}, \frac{6 + 4\alpha\mu_2 + (\alpha^2 + 3\beta)\mu_2^2}{\mu_2^3}, \frac{1 + 4\beta\mu_2^2 + \alpha\beta\mu_2^3}{\mu_2^4} \right) \\ v &= (v_1, v_2, v_3) := \left(\frac{3 + \alpha\nu_2}{\nu_2^2}, \frac{6 + 4\alpha\nu_2 + (\alpha^2 + 3\beta)\nu_2^2}{\nu_2^3}, \frac{1 + 4\beta\nu_2^2 + \alpha\beta\nu_2^3}{\nu_2^4} \right). \end{aligned} \quad (49)$$

Therefore, their vectorial product is zero, i.e., $u_i v_j = u_j v_i$ for all $1 \leq i < j \leq 3$. The equations corresponding to the cases $(i, j) = (1, 2)$ and $(i, j) = (1, 3)$ are

$$\mu_2(3 + \alpha\mu_2)(6 + 4\alpha\nu_2 + (\alpha^2 + 3\beta)\nu_2^2) = \nu_2(3 + \alpha\nu_2)(6 + 4\alpha\mu_2 + (\alpha^2 + 3\beta)\mu_2^2)$$

and

$$\mu_2^2(3 + \alpha\mu_2)(1 + 4\beta\nu_2^2 + \alpha\beta\nu_2^3) = \nu_2^2(3 + \alpha\nu_2)(1 + 4\beta\mu_2^2 + \alpha\beta\mu_2^3).$$

After some calculations, simplifying also by the factor $\mu_2 - \nu_2 \neq 0$, we arrive at

$$6\alpha(\mu_2 + \nu_2) + (\alpha^2 - 9\beta)\mu_2\nu_2 + 18 = 0 \quad (50)$$

and

$$\alpha(\mu_2^2 + \nu_2^2 + \mu_2\nu_2) + \alpha\beta\mu_2^2\nu_2^2 + 3(\mu_2 + \nu_2) = 0. \quad (51)$$

Multiplying (50) by $(\mu_2 + \nu_2)$, (51) by 6, and subtracting the equations so obtained, finally dividing by $\mu_2\nu_2 \neq 0$, we get

$$(\alpha^2 - 9\beta)(\mu_2 + \nu_2) - 6\alpha\beta\mu_2\nu_2 + 6\alpha = 0. \quad (52)$$

Using the formulae

$$\alpha = -\frac{\gamma}{\eta} = -\frac{3(\mu_4\nu_2^2 - \nu_4\mu_2^2)}{\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2)},$$

$$\beta = -\frac{\delta}{\eta} = -\frac{\mu_4\nu_2^3 - \nu_4\mu_2^3 + 3\mu_2^2\nu_2^2(\mu_2 - \nu_2)}{\mu_2^2\nu_2^2(\mu_4\nu_2 - \nu_4\mu_2)}, \quad (53)$$

equations (50) and (52) can be rewritten in the form

$$\frac{9(\mu_2 - \nu_2)(3\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2) + \mu_4\nu_4(\mu_2 - \nu_2))}{(\mu_4\nu_2 - \nu_4\mu_2)^2} = 0,$$

$$\frac{9(\mu_2 - \nu_2)^2(3\mu_2\nu_2(\mu_4\nu_2 + \nu_4\mu_2) - \mu_4\nu_4(\mu_2 + \nu_2))}{\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2)^2} = 0,$$

respectively. Using $\mu_2 - \nu_2 \neq 0$ and $\mu_4\nu_2 - \nu_4\mu_2 \neq 0$, we get

$$3\mu_2\nu_2(\mu_4\nu_2 - \nu_4\mu_2) + \mu_4\nu_4(\mu_2 - \nu_2) = 0,$$

$$3\mu_2\nu_2(\mu_4\nu_2 + \nu_4\mu_2) - \mu_4\nu_4(\mu_2 + \nu_2) = 0.$$

Adding up, and subtracting these two equations, we obtain

$$6\mu_4\mu_2\nu_2^2 - 2\mu_4\nu_4\nu_2 = 0, \quad 6\nu_4\mu_2^2\nu_2 - 2\mu_4\nu_4\mu_2 = 0,$$

whence it follows that

$$\mu_4 = \nu_4 = 3\mu_2\nu_2. \quad (54)$$

In this case, (53) simplifies to

$$\alpha = -3\left(\frac{1}{\mu_2} + \frac{1}{\nu_2}\right), \quad \beta = -\left(\frac{1}{\mu_2^2} + \frac{1}{\nu_2^2}\right).$$

Therefore, for the vectors u and v defined in (49), we get

$$u = v = \left(\frac{-3}{\mu_2\nu_2}, \frac{6(\mu_2 + \nu_2)}{\mu_2^2\nu_2^2}, \frac{3(\mu_2^2 + \nu_2^2) - \mu_2^2\nu_2^2}{\mu_2^3\nu_2^3}\right).$$

Thus, the differential equation (48) reduces to the following form

$$\frac{-3}{\mu_2\nu_2}(\mu_5 - \nu_5)(\Phi')^2 + \frac{6(\mu_2 + \nu_2)}{\mu_2^2\nu_2^2}(\mu_5 - \nu_5)\Phi'\Phi^2$$

$$+ \frac{3(\mu_2^2 + \nu_2^2) - \mu_2^2\nu_2^2}{\mu_2^3\nu_2^3}(\mu_5 - \nu_5)\Phi^4 = 0.$$

The coefficients of this equation can simultaneously vanish if and only if $\mu_5 = \nu_5$. However, this equality, together with (54) contradicts the condition (44) of the theorem. The contradiction so obtained shows that the coefficients of (48) cannot be identically zero under the assumptions of the theorem. Thus, (48) is a non-trivial first-order polynomial differential equation of the form (34). Therefore, it follows that φ and ψ satisfy one of the alternatives of Theorem 11. $\square$

If either $\mu_5 = \nu_5 = 0$ or $\mu_5 = \nu_5$ and $\mu_4 = \nu_4 = 3\mu_2\nu_2$, then (48) is useless, thus we need to apply the necessary condition (12) of Theorem 5 in the case $n = 5$.

Theorem 15. Assume $\mathcal{C}_4$ and $\mathcal{M}_1^*$ with $\mu_2\nu_2 \neq 0$, $(\mu_3, \nu_3) = (0, 0)$, $\mu_2\nu_4 \neq \nu_2\mu_4$,

$$(\mu_6, \nu_6, 0) \neq \left(\frac{5\mu_2\mu_4^2}{6\mu_2^2 - \mu_4}, \frac{5\nu_2\nu_4^2}{6\nu_2^2 - \nu_4}, 3\mu_2\nu_2(\nu_2\mu_4 - \mu_2\nu_4) - (\mu_2 - \nu_2)\mu_4\nu_4 \right) \quad (55)$$

and

$$(\mu_6, \nu_6, 0) \neq \left(\frac{F}{E}, \frac{G}{E}, D \right), \quad (56)$$

where

$$\begin{aligned} D &:= 45\mu_2^7\nu_2^7(\mu_2 - \nu_2)^3(\mu_2\nu_4 - \mu_4\nu_2)^4((\mu_2 - \nu_2)\mu_4\nu_4 + 3\mu_2\nu_2(\mu_2\nu_4 - \mu_4\nu_2)) \\ &\quad (\nu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(7\mu_2 - 8\nu_2)\mu_4^2\nu_4 \\ &\quad + \mu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(8\mu_2 - 7\nu_2)\mu_4\nu_4^2 \\ &\quad - 6\mu_2^2\nu_2^2(\mu_2 + \nu_2)(6\mu_2^2 - 5\mu_2\nu_2 + 6\nu_2^2)\mu_4\nu_4 \\ &\quad - 6\mu_2\nu_2^3(\mu_2 - 2\nu_2)(7\mu_2^2 + \mu_2\nu_2 - \nu_2^2)\mu_4^2 \\ &\quad - 6\mu_2^3\nu_2(2\mu_2 - \nu_2)(\mu_2^2 - \mu_2\nu_2 - 7\nu_2^2)\nu_4^2 + 45\mu_2^3\nu_2^4(\mu_2 - \nu_2)(7\mu_2 - 2\nu_2)\mu_4 \\ &\quad + 45\mu_2^4\nu_2^3(\mu_2 - \nu_2)(2\mu_2 - 7\nu_2)\nu_4), \\ E &:= 4\nu_2^2(\mu_2 - \nu_2)(\mu_2 - 2\nu_2)^2\mu_4^3\nu_4 + 4\mu_2^2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)^2\mu_4\nu_4^3 \\ &\quad + \mu_2\nu_2(\mu_2 - \nu_2)(7\nu_2^2 - 22\mu_2\nu_2 + 7\mu_2^2)\mu_4^2\nu_4^2 - 6\mu_2\nu_2^4(2\mu_2 + \nu_2)(\mu_2 - 2\nu_2)\mu_4^3 \\ &\quad - 6\mu_2^4\nu_2(\mu_2 + 2\nu_2)(2\mu_2 - \nu_2)\nu_4^3 + 3\mu_2^2\nu_2^3(7\mu_2^2 - 24\mu_2\nu_2 - \nu_2^2)\mu_4^2\nu_4 \\ &\quad + 3\mu_2^3\nu_2^2(\mu_2^2 + 24\mu_2\nu_2 - 7\nu_2^2)\mu_4\nu_4^2 + 90\mu_2^3\nu_2^3(\mu_2 - \nu_2)(\mu_2\nu_4 - \nu_2\mu_4)^2, \\ F &:= 3\mu_2\nu_2(\mu_2\nu_4 - \mu_4\nu_2)(\mu_2^2\nu_2\nu_4 - \mu_2\mu_4\nu_2^2 + (3\nu_2 - 3\mu_2)\mu_4\nu_4) \\ &\quad (6\mu_2^3\nu_2\nu_4 - 6\mu_2^2\mu_4\nu_2^2 + \mu_2(7\mu_2 - 2\nu_2)\mu_4\nu_4 + 5\nu_2(\mu_2 - 2\nu_2)\mu_4^2), \\ G &:= 3\mu_2\nu_2(\mu_2\nu_4 - \mu_4\nu_2)(\mu_2^2\nu_2\nu_4 - \mu_2\mu_4\nu_2^2 + (3\nu_2 - 3\mu_2)\mu_4\nu_4) \\ &\quad (6\mu_2^2\nu_2^2\nu_4 - 6\mu_2\mu_4\nu_2^3 + \nu_2(2\mu_2 - 7\nu_2)\mu_4\nu_4 + 5\mu_2(2\mu_2 - \nu_2)\nu_4^2). \end{aligned} \quad (57)$$

Then $M_{\varphi, \mu} = M_{\psi, \nu}$ holds if and only if either one of the alternatives (i), (ii), (iii) of Theorem 11 is satisfied.

PROOF. Following the argument of the proof of the previous theorem, we get that the functions φ and ψ are analytic on I and Φ defined by (23) satisfies

(43), where the constants η, γ, δ are defined by (31). We have $\eta \neq 0$, and, with the notations $\alpha := -\gamma/\eta$ and $\beta := -\delta/\eta$, (43) yields (45) and

$$\Phi'''' = (4\alpha^2 + 6\beta)\Phi(\Phi')^2 + (\alpha^3 + 9\alpha\beta)\Phi^3\Phi' + (\alpha^2\beta + 3\beta^2)\Phi^5. \quad (58)$$

Differentiating (47), we get that

$$\begin{aligned} \frac{\varphi''''}{\varphi'} &= \frac{\Phi''''}{\mu_2} + 5\frac{\Phi\Phi'''}{\mu_2^2} + 10\frac{\Phi'\Phi''}{\mu_2^2} + 10\frac{\Phi^2\Phi''}{\mu_2^3} + 15\frac{\Phi(\Phi')^2}{\mu_2^3} + 10\frac{\Phi^3\Phi'}{\mu_2^4} + \frac{\Phi^5}{\mu_2^5}, \\ \frac{\psi''''}{\psi'} &= \frac{\Phi''''}{\nu_2} + 5\frac{\Phi\Phi'''}{\nu_2^2} + 10\frac{\Phi'\Phi''}{\nu_2^2} + 10\frac{\Phi^2\Phi''}{\nu_2^3} + 15\frac{\Phi(\Phi')^2}{\nu_2^3} + 10\frac{\Phi^3\Phi'}{\nu_2^4} + \frac{\Phi^5}{\nu_2^5}. \end{aligned} \quad (59)$$

In the particular case $n = 5$, condition (12) of Theorem 5 yields

$$\begin{aligned} &(\mu_1\nu_5 - \mu_0\nu_6)\frac{\psi''''}{\psi'} + 5(\mu_2\nu_4 - \mu_1\nu_5)\frac{\varphi''}{\varphi'} \cdot \frac{\psi''''}{\psi'} + 10(\mu_3\nu_3 - \mu_2\nu_4)\frac{\varphi'''}{\varphi'} \cdot \frac{\psi''''}{\psi'} \\ &+ 10(\mu_4\nu_2 - \mu_3\nu_3)\frac{\varphi''''}{\varphi'} \cdot \frac{\psi'''}{\psi'} + 5(\mu_5\nu_1 - \mu_4\nu_2)\frac{\varphi''''}{\varphi'} \cdot \frac{\psi''}{\psi'} + (\mu_6\nu_0 - \mu_5\nu_1)\frac{\varphi''''}{\varphi'} = 0. \end{aligned}$$

Now using $\mu_1 = \nu_1 = \mu_3 = \nu_3 = 0$, and the identities (26), (27), (29), (45), (47), (58), and (59), we obtain

$$\begin{aligned} &-\nu_6\left(\frac{\Phi''''}{\nu_2} + 5\frac{\Phi\Phi'''}{\nu_2^2} + 10\frac{\Phi'\Phi''}{\nu_2^2} + 10\frac{\Phi^2\Phi''}{\nu_2^3} + 15\frac{\Phi(\Phi')^2}{\nu_2^3} + 10\frac{\Phi^3\Phi'}{\nu_2^4} + \frac{\Phi^5}{\nu_2^5}\right) \\ &+ 5\mu_2\nu_4\frac{\Phi}{\mu_2} \cdot \left(\frac{\Phi'''}{\nu_2} + 4\frac{\Phi\Phi''}{\nu_2^2} + 3\frac{(\Phi')^2}{\nu_2^2} + 6\frac{\Phi^2\Phi'}{\nu_2^3} + \frac{\Phi^4}{\nu_2^4}\right) \\ &- 10\mu_2\nu_4\left(\frac{\Phi'}{\mu_2} + \frac{\Phi^2}{\mu_2^2}\right) \cdot \left(\frac{\Phi''}{\nu_2} + 3\frac{\Phi\Phi'}{\nu_2^2} + \frac{\Phi^3}{\nu_2^3}\right) \\ &+ 10\mu_4\nu_2\left(\frac{\Phi''}{\mu_2} + 3\frac{\Phi\Phi'}{\mu_2^2} + \frac{\Phi^3}{\mu_2^3}\right) \cdot \left(\frac{\Phi'}{\nu_2} + \frac{\Phi^2}{\nu_2^2}\right) \\ &- 5\mu_4\nu_2\left(\frac{\Phi'''}{\mu_2} + 4\frac{\Phi\Phi''}{\mu_2^2} + 3\frac{(\Phi')^2}{\mu_2^2} + 6\frac{\Phi^2\Phi'}{\mu_2^3} + \frac{\Phi^4}{\mu_2^4}\right) \cdot \frac{\Phi}{\nu_2} \\ &+ \mu_6\left(\frac{\Phi''''}{\mu_2} + 5\frac{\Phi\Phi'''}{\mu_2^2} + 10\frac{\Phi'\Phi''}{\mu_2^2} + 10\frac{\Phi^2\Phi''}{\mu_2^3} + 15\frac{\Phi(\Phi')^2}{\mu_2^3} + 10\frac{\Phi^3\Phi'}{\mu_2^4} + \frac{\Phi^5}{\mu_2^5}\right) = 0, \end{aligned}$$

Using now the formulae (45) and (58), the above equation reduces to the following first-order differential equation for Φ :

$$\begin{aligned} &(A_1\mu_6 + A_2\nu_6 + A_3)\Phi(\Phi')^2 + (B_1\mu_6 + B_2\nu_6 + B_3)\Phi^3\Phi' \\ &+ (C_1\mu_6 + C_2\nu_6 + C_3)\Phi^5 = 0, \end{aligned} \quad (60)$$

where

$$\begin{aligned}
A_1 &:= \mu_2^2 \nu_2^5 ((4\alpha^2 + 6\beta)\mu_2^2 + 15\alpha\mu_2 + 15) \\
A_2 &:= -\mu_2^5 \nu_2^2 ((4\alpha^2 + 6\beta)\nu_2^2 + 15\alpha\nu_2 + 15) \\
A_3 &:= -5\mu_2^3 \nu_2^3 ((\alpha\nu_2 + 3)\mu_2^2 \nu_4 - (\alpha\mu_2 + 3)\mu_4 \nu_2^2) \\
B_1 &:= \mu_2 \nu_2^5 ((\alpha^3 + 9\alpha\beta)\mu_2^3 + (5\alpha^2 + 25\beta)\mu_2^2 + 10\alpha\mu_2 + 10) \\
B_2 &:= -\mu_2^5 \nu_2 ((\alpha^3 + 9\alpha\beta)\nu_2^3 + (5\alpha^2 + 25\beta)\nu_2^2 + 10\alpha\nu_2 + 10) \\
B_3 &:= 5\mu_2^2 \nu_2^2 ((\alpha^2 \nu_2^2 + 4\alpha\nu_2 + \beta\nu_2^2 + 4)\mu_2^3 \nu_4 - (\alpha^2 \mu_2^2 + 4\alpha\mu_2 + \beta\mu_2^2 + 4)\mu_4 \nu_2^3 \\
&\quad + (2\alpha\mu_2^2 + 6\mu_2)\mu_4 \nu_2^2 - (2\alpha\nu_2^2 + 6\nu_2)\mu_2^2 \nu_4) \\
C_1 &:= \nu_2^5 ((\alpha^2 \beta + 3\beta^2)\mu_2^4 + 5\alpha\beta\mu_2^3 + 10\beta\mu_2^2 + 1) \\
C_2 &:= -\mu_2^5 ((\alpha^2 \beta + 3\beta^2)\nu_2^4 + 5\alpha\beta\nu_2^3 + 10\beta\nu_2^2 + 1) \\
C_3 &:= 5\mu_2 \nu_2 ((\alpha\beta\nu_2^3 + 4\beta\nu_2^2 + 1)\mu_2^4 \nu_4 - (\alpha\beta\mu_2^3 + 4\beta\mu_2^2 + 1)\mu_4 \nu_2^4 \\
&\quad + (2\mu_2^3 \beta + 2\mu_2)\mu_4 \nu_2^3 - (2\nu_2^3 \beta + 2\nu_2)\mu_2^3 \nu_4).
\end{aligned}$$

If the coefficients in equation (60) vanish simultaneously, then μ_6, ν_6 and $\xi = 1$ is a nontrivial solution of the following system of homogeneous linear equations

$$\begin{aligned}
A_1 \mu_6 + A_2 \nu_6 + A_3 \xi &= 0, & B_1 \mu_6 + B_2 \nu_6 + B_3 \xi &= 0, \\
C_1 \mu_6 + C_2 \nu_6 + C_3 \xi &= 0.
\end{aligned} \tag{61}$$

Therefore, the value D defined in (57), which is the determinant D of this system has to be zero. The constant D was factorized by using the Maple 9 symbolic package. Thus, in order that D be zero, we have two possibilities. The first (simpler) case is when

$$(\mu_2 - \nu_2)\mu_4 \nu_4 + 3\mu_2 \nu_2 (\mu_2 \nu_4 - \mu_4 \nu_2) = 0.$$

Then, again using Maple 9, we get the following values for the solutions μ_6 and ν_6 of the linear system (61):

$$\mu_6 = \frac{5\mu_2 \mu_4^2}{6\mu_2^2 - \mu_4}, \quad \nu_6 = \frac{5\nu_2 \nu_4^2}{6\nu_2^2 - \nu_4}.$$

This, however, contradicts the assumption (55). Thus, in this case the three coefficients of (60) cannot vanish simultaneously.

The second case is when the last factor of D is zero, i.e., when

$$\begin{aligned} & \nu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(7\mu_2 - 8\nu_2)\mu_4^2\nu_4 \\ & + \mu_2(\mu_2 - \nu_2)(2\mu_2 - \nu_2)(\mu_2 - 2\nu_2)(8\mu_2 - 7\nu_2)\mu_4\nu_4^2 \\ & - 6\mu_2^2\nu_2^2(\mu_2 + \nu_2)(6\mu_2^2 - 5\mu_2\nu_2 + 6\nu_2^2)\mu_4\nu_4 \\ & - 6\mu_2\nu_2^3(\mu_2 - 2\nu_2)(7\mu_2^2 + \mu_2\nu_2 - \nu_2^2)\mu_4^2 \\ & - 6\mu_2^3\nu_2(2\mu_2 - \nu_2)(\mu_2^2 - \mu_2\nu_2 - 7\nu_2^2)\nu_4^2 + 45\mu_2^3\nu_2^4(7\mu_2 - 2\nu_2)(\mu_2 - \nu_2)\mu_4 \\ & + 45\mu_2^4\nu_2^3(\mu_2 - \nu_2)(2\mu_2 - 7\nu_2)\nu_4 = 0. \end{aligned}$$

Calculating with the help of the Maple 9 package, we get the following values for the unknowns μ_6 and ν_6 :

$$\mu_6 = \frac{F}{E}, \quad \nu_6 = \frac{G}{E},$$

where E, F , and G are given by (57). In view of condition (56), we get again a contradiction. Thus, in this case, the three coefficients of (60) cannot be simultaneously zero

Therefore, in each case, Φ satisfies a nontrivial first order polynomial differential equation of the form (34). Hence, one of the alternatives of Theorem 11 must be valid. $\square$

7. Applications

In this section we demonstrate some possible applications of our results.

Example 1. Consider the functional equation

$$\varphi^{-1}\left(\frac{\varphi\left(\frac{2x+y}{3}\right) + \varphi\left(\frac{x+2y}{3}\right)}{2}\right) = \psi^{-1}\left(\frac{\psi(x) + 16\psi\left(\frac{x+y}{2}\right) + \psi(y)}{18}\right), \quad (62)$$

where $\varphi, \psi : I \rightarrow \mathbb{R}$ are continuous strictly monotone functions.

Equation (62) is an obvious particular case of the equality problem (1), where the measures μ and ν are given by

$$\mu = \frac{\delta_{1/3} + \delta_{2/3}}{2} \quad \text{and} \quad \nu = \frac{\delta_0 + 16\delta_{1/2} + \delta_1}{18}.$$

Then, $\hat{\mu}_1 = \hat{\nu}_1 = \frac{1}{2}$ and, for $k \in \mathbb{N}$,

$$\mu_k = \frac{(-1)^k + 1}{2 \cdot 6^k} \quad \text{and} \quad \nu_k = \frac{(-1)^k + 1}{18 \cdot 2^k}.$$

Hence

$$\begin{aligned}\mu_1 &= 0, & \mu_2 &= \frac{1}{36}, & \mu_3 &= 0, & \mu_4 &= \frac{1}{1296}, & \dots, \\ \nu_1 &= 0, & \nu_2 &= \frac{1}{36}, & \nu_3 &= 0, & \nu_4 &= \frac{1}{144}, & \dots.\end{aligned}$$

Thus the exact moment condition $\mathcal{M}_3^*$ holds. If $\mathcal{C}_4$ is assumed, then, by Theorem 8, $\varphi, \psi : I \rightarrow \mathbb{R}$ satisfy (62) if and only if there exist constants $a \neq 0$ and b such that

$$\psi = a\varphi + b$$

and φ is an arbitrary strictly monotone polynomial with $\deg \varphi \leq 3$.

It remains an open problem to find the solutions of (62) under the regularity assumption $\mathcal{C}_0$ only.

Example 2. Consider the functional equation

$$\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) = \psi^{-1}\left(\int_0^1 2t\psi(tx + (1-t)y)dt\right), \quad (63)$$

where $\varphi, \psi : I \rightarrow \mathbb{R}$ are continuous strictly monotone functions.

Equation (63) is also a particular case of the equality problem (1), where the measures μ and ν are now given by

$$\mu = \frac{\delta_0 + 2\delta_1}{3} \quad \text{and} \quad d\nu(t) = 2tdt.$$

Then, $\hat{\mu}_1 = \hat{\nu}_1 = \frac{2}{3}$ and, for $k \in \mathbb{N}$, we have

$$\mu_k = \int_0^1 \left(t - \frac{2}{3}\right)^k d\mu(t) = \frac{(-2)^k + 2}{3^{k+1}}$$

and

$$\nu_k = \int_0^1 2t \left(t - \frac{2}{3}\right)^k dt = \frac{6k + 10 - (-2)^{k+3}}{(k+1)(k+2)3^{k+2}}.$$

Hence

$$\begin{aligned}\mu_1 &= 0, & \mu_2 &= \frac{2}{9}, & \mu_3 &= -\frac{2}{27}, & \mu_4 &= \frac{2}{27}, & \dots, \\ \nu_1 &= 0, & \nu_2 &= \frac{1}{18}, & \nu_3 &= -\frac{1}{135}, & \nu_4 &= \frac{1}{135}, & \dots.\end{aligned}$$

Thus the exact moment condition $\mathcal{M}_1^*$ holds. Since $\mu_3 \neq 0 \neq \nu_3$, Theorem 12 can be applied. If $\mathcal{C}_3$ is assumed, then, one of the alternatives (i), (ii), and (iii) of Theorem 11 holds.

If the alternative (i) is valid then there exist real constants a, b, c, d with $ac \neq 0$ such that φ and ψ are given by (35), i.e., they are affine functions. In this case, the means $\mathcal{M}_{\varphi, \mu}(x, y)$ and $\mathcal{M}_{\psi, \nu}(x, y)$ are equal to the weighted arithmetic mean $\frac{2x+y}{3}$.

If (ii) were valid, then there exist real constants a, b, c, d, p, q with $acpq(p-q) \neq 0$ such that (36) and (37) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (37) yields

$$q(\mu_1\nu_1 - \mu_0\nu_2) + p(\mu_2\nu_0 - \mu_1\nu_1) = 0, \quad (64)$$

whence $q = 4p$. If $n = 2$, then (37) implies

$$q^2(\mu_1\nu_2 - \mu_0\nu_3) + pq(\mu_2\nu_1 - \mu_1\nu_2) + p^2(\mu_3\nu_0 - \mu_2\nu_1) = 0, \quad (65)$$

resulting $q^2 = 10p^2$, which contradicts $q = 4p$.

If (iii) is valid then there exist real constants a, b, c, d, p, q with $ac(p-1)(q-1)(p-q) \neq 0$ and $x_0 \notin I$ such that (38) and (39) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (39) yields

$$(q-1)(\mu_1\nu_1 - \mu_0\nu_2) + (p-1)(\mu_2\nu_0 - \mu_1\nu_1) = 0, \quad (66)$$

whence $q = 4p - 3$. If $n = 2$, then (39) implies

$$\begin{aligned} \frac{(q-1)(q-2)}{2}(\mu_1\nu_2 - \mu_0\nu_3) + (p-1)(q-1)(\mu_2\nu_1 - \mu_1\nu_2) \\ + \frac{(p-1)(p-2)}{2}(\mu_3\nu_0 - \mu_2\nu_1) = 0, \end{aligned} \quad (67)$$

which results $p = 0$ and $q = 4p - 3 = -3$. Instead of showing now that (39) holds for all $n \geq 3$, we prove that the functions $\varphi, \psi : I \rightarrow \mathbb{R}$ given by (38) satisfy (63). For simplicity, we assume that $x_0 = 0 \leq \inf I$. Then $\varphi(x) = a \ln x + b$ and $\psi(x) = cx^{-3} + d$.

On one hand, we have

$$\varphi^{-1}\left(\frac{2\varphi(x) + \varphi(y)}{3}\right) = \sqrt[3]{x^2y}.$$

On the other hand,

$$\begin{aligned} \psi^{-1}\left(\int_0^1 2t\psi(tx + (1-t)y)dt\right) &= \left(\int_0^1 \frac{2t}{(tx + (1-t)y)^3}dt\right)^{-\frac{1}{3}} \\ &= \left(\frac{2t(y-x) - y}{(y-x)^2(tx + (1-t)y)^2}\right)\Big|_{t=0}^{t=1}\right)^{-\frac{1}{3}} \end{aligned}$$

$$= \left(\frac{2(y-x)-y}{(y-x)^2 x^2} - \frac{-y}{(y-x)^2 y^2} \right)^{-\frac{1}{3}} = \left(\frac{1}{x^2 y} \right)^{-\frac{1}{3}} = \sqrt[3]{x^2 y},$$

which proves the equality in (63).

Example 3. Consider the functional equation

$$\varphi^{-1} \left(\frac{2\varphi(x) + \varphi(y)}{3} \right) = \psi^{-1} \left(\frac{4\psi(x) + 4\psi\left(\frac{x+y}{2}\right) + \psi(y)}{9} \right), \quad (68)$$

where $\varphi, \psi : I \rightarrow \mathbb{R}$ are continuous strictly monotone functions.

Equation (68) is an obvious particular case of the equality problem (1), where the measures μ and ν are given by

$$\mu = \frac{\delta_0 + 2\delta_1}{3} \quad \text{and} \quad \nu = \frac{\delta_0 + 4\delta_{1/2} + 4\delta_1}{9}.$$

Then, $\hat{\mu}_1 = \hat{\nu}_1 = \frac{2}{3}$ and, for $k \in \mathbb{N}$, we have

$$\mu_k = \int_0^1 \left(t - \frac{2}{3} \right)^k d\mu(t) = \frac{(-2)^k + 2}{3^{k+1}}$$

and

$$\nu_k = \int_0^1 \left(t - \frac{2}{3} \right)^k d\nu(t) = \frac{(-4)^k + 4(-1)^k + 4 \cdot 2^k}{9 \cdot 6^k}.$$

Hence

$$\begin{aligned} \mu_1 &= 0, & \mu_2 &= \frac{2}{9}, & \mu_3 &= -\frac{2}{27}, & \mu_4 &= \frac{2}{27} & \dots, \\ \nu_1 &= 0, & \nu_2 &= \frac{1}{9}, & \nu_3 &= -\frac{1}{54}, & \nu_4 &= \frac{1}{36}, & \dots \end{aligned}$$

Thus the exact moment condition $\mathcal{M}_1^*$ holds. Since $\mu_3 \neq 0 \neq \nu_3$, Theorem 12 can be applied. If $\mathcal{C}_3$ is assumed, then, one of the alternatives (i), (ii), and (iii) of Theorem 12 holds.

Clearly, if (i) holds, then φ and ψ are affine functions and the two means on the left and right hand sides of (68) are equal to the weighted arithmetic mean $\frac{2x+y}{3}$.

If (ii) holds, then there exist constants a, b, c, d, p, q with $acpq(p-q) \neq 0$ such that (36) and (37) are satisfied for all $n \in \mathbb{N}$. In the case $n = 1$, (37) simplifies to (64), which results $q = 2p$. Instead of showing that (37) holds for all $n \geq 2$, we prove that the functions φ and ψ given by (36) are solutions of (68). Indeed,

$$\varphi^{-1} \left(\frac{2\varphi(x) + \varphi(y)}{3} \right) = \frac{1}{p} \ln \left(\frac{2e^{px} + e^{py}}{3} \right) = \frac{1}{2p} \ln \left(\frac{2e^{px} + e^{py}}{3} \right)^2$$

$$\begin{aligned}
&= \frac{1}{2p} \ln \left(\frac{4e^{2px} + 4e^{2p\frac{x+y}{2}} + e^{2py}}{9} \right) \\
&= \frac{1}{q} \ln \left(\frac{4e^{qx} + 4e^{q\frac{x+y}{2}} + e^{qy}}{9} \right) = \psi^{-1} \left(\frac{4\psi(x) + 4\psi\left(\frac{x+y}{2}\right) + \psi(y)}{9} \right).
\end{aligned}$$

In this case, we can also see that the means on the two sides of (39) are weighted exponential means.

If (iii) were valid then there exist real constants a, b, c, d, p, q with $ac(p-1)(q-1)(p-q) \neq 0$ and $x_0 \notin I$ such that (38) and (39) hold for all $n \in \mathbb{N}$. In the case $n = 1$, (39) simplifies to (66) whence $q = 2p - 1$ follows. If $n = 2$, then (39) yields (67) which results $p = 1$ contradicting the conditions on the parameters. Therefore, there is no solution of (68) in the case (iii).

References

- [1] J. ACZÉL, The notion of mean values, *Norske Vid. Selsk. Forh., Trondhjem* **19**, no. 23 (1947), 83–86.
- [2] J. ACZÉL, On mean values, *Bull. Amer. Math. Soc.* **54** (1948), 392–400.
- [3] J. ACZÉL, On mean values and operations defined for two variables, *Norske Vid. Selsk. Forh., Trondhjem* **20**, no. 10 (1948), 37–40.
- [4] J. ACZÉL, On the theory of means, *Colloq. Math.* **4** (1956), 33–55.
- [5] L. R. BERRONE, The mean value theorem: functional equations and Lagrangian means, *Epsilon* **14**, no. 1(40) (1998), 131–151.
- [6] L. R. BERRONE and J. MORO, Lagrangian means, *Aequationes Math.* **55**, no. 3 (1998), 217–226.
- [7] J. BŁASIŃSKA-LESK, D. GŁAZOWSKA and J. MATKOWSKI, An invariance of the geometric mean with respect to Stolarsky mean-type mappings, *Results Math.* **43**, no. 1–2 (2003), 42–55.
- [8] J. M. BORWEIN and P. B. BORWEIN, Pi and the AGM, *John Wiley & Sons Inc., New York*, 1987, A study in analytic number theory and computational complexity.
- [9] P. BURAI, Extension theorem for a functional equation, *J. Appl. Anal.* **12**, no. 2 (2006), 293–299.
- [10] P. BURAI, A Matkowski–Sutô type equation, *Publ. Math. Debrecen* **70**, no. 1–2 (2007), 233–247.
- [11] Z. DARÓCZY, On a class of means of two variables, Vol. 55, no. 1–2, *Publ. Math. Debrecen*, 1999, 177–197.
- [12] Z. DARÓCZY, Matkowski–Sutô type problem for conjugate arithmetic means, *Rocznik Nauk.-Dydakt. Prace Mat.* **17** (2000), 89–100.
- [13] Z. DARÓCZY, Gaussian iteration of mean values and the existence of $\sqrt{2}$, *Teaching Math. Comp. Sci.* **1**, no. 1 (2003), 35–42.
- [14] Z. DARÓCZY, Quasi-arithmetic elements of a given class of means, *Publ. Math. Debrecen* **65**, no. 3–4 (2004), 317–322.

- [15] Z. DARÓCZY, Functional equations involving means and Gauss compositions of means, *Nonlinear Anal.* **63**, no. 5–7 (2005), e417–e425.
- [16] Z. DARÓCZY and G. HAJDU, On linear combinations of weighted quasi-arithmetic means, *Aequationes Math.* **69**, no. 1–2 (2005), 58–67.
- [17] Z. DARÓCZY, G. HAJDU and C. T. NG, An extension theorem for a Matkowski–Sutô problem, *Colloq. Math.* **95**, no. 2 (2003), 153–161.
- [18] Z. DARÓCZY, K. LAJKÓ, R. L. LOVAS, GY. MAKSA and ZS. PÁLES, Functional equations involving means, *Acta Math. Hungar.* **116**, no. 1–2 (2007), 79–87.
- [19] Z. DARÓCZY and GY. MAKSA, On a problem of Matkowski, *Colloq. Math.* **82**, no. 1 (1999), 117–123.
- [20] Z. DARÓCZY, GY. MAKSA and ZS. PÁLES, Extension theorems for the Matkowski–Sutô problem, *Demonstratio Math.* **33**, no. 3 (2000), 547–556.
- [21] Z. DARÓCZY, GY. MAKSA and ZS. PÁLES, On two-variable means with variable weights, *Aequationes Math.* **67**, no. 1–2 (2004), 154–159.
- [22] Z. DARÓCZY, GY. MAKSA and ZS. PÁLES, Functional equations involving means and their Gauss composition, *Proc. Amer. Math. Soc.* **134**, no. 2 (2006), 521–530.
- [23] Z. DARÓCZY and C. T. NG, A functional equation on complementary means, *Acta Sci. Math. (Szeged)* **66** (2000), 603–611.
- [24] Z. DARÓCZY and ZS. PÁLES, Középértékek Gauss-féle kompozíciója és a Matkowski–Sutô probléma megoldása, *Mat. Lapok*, no. 3–4 (1998–99 (2003)), 1–53.
- [25] Z. DARÓCZY and ZS. PÁLES, On means that are both quasi-arithmetic and conjugate arithmetic, *Acta Math. Hungar.* **90**, no. 4 (2001), 271–282.
- [26] Z. DARÓCZY and ZS. PÁLES, Gauss-composition of means and the solution of the Matkowski–Sutô problem, *Publ. Math. Debrecen* **61**, no. 1–2 (2002), 157–218.
- [27] Z. DARÓCZY and ZS. PÁLES, A Matkowski–Sutô Type Problem for Quasi-Arithmetic Means of Order α , Functional Equations – Results and Advances, *Adv. Math. (Dordr.)*, vol. 3, (Z. Daróczy and Zs. Páles, eds.), *Kluwer Acad. Publ., Dordrecht*, 2002, 189–200.
- [28] Z. DARÓCZY and ZS. PÁLES, The Matkowski–Sutô problem for weighted quasi-arithmetic means, *Acta Math. Hungar.* **100**, no. 3 (2003), 237–243.
- [29] Z. DARÓCZY and ZS. PÁLES, A Matkowski–Sutô-type problem for weighted quasi-arithmetic means, *Ann. Univ. Sci. Budapest. Sect. Comput.* **22** (2003), 69–81.
- [30] Z. DARÓCZY and ZS. PÁLES, On functional equations involving means, *Publ. Math. Debrecen* **62**, no. 3–4 (2003), 363–377.
- [31] B. DE FINETTI, Sul concetto di media, *Giornale dell’ Istituto, Italiano degli Attuari* **2** (1931), 369–396.
- [32] J. DOMSTA and J. MATKOWSKI, Invariance of the arithmetic mean with respect to special mean-type mappings, *Aequationes Math.* **71**, no. 1–2 (2006), 70–85.
- [33] C. F. GAUSS, Bestimmung der Anziehung Eines Elliptischen Ringes, Akademische Verlagsgesellschaft M. B. H., Leipzig, 1927, Nachlass zur Theorie des arithmetisch-geometrischen Mittels und der Modulfunktion.
- [34] D. GLAZOWSKA, W. JARCZYK and J. MATKOWSKI, Arithmetic mean as a linear combination of two quasi-arithmetic means, *Publ. Math. Debrecen* **61**, no. 3–4 (2002), 455–467.
- [35] G. HAJDU, An Extension Theorem for the Matkowski–Sutô Problem for Conjugate Arithmetic Means, Functional Equations – Results and Advances, *Advances in Mathematics*, vol. 3, (Z. Daróczy and Zs. Páles, eds.), *Kluwer Acad. Publ., Dordrecht*, 2002, 201–208.
- [36] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, Inequalities, *Cambridge University Press, Cambridge*, 1934, (first edition), 1952 (second edition).

- [37] H. HARUKI and TH. M. RASSIAS, New characterizations of some mean-values, *J. Math. Anal. Appl.* **202**, no. 1 (1996), 333–348.
- [38] J. JARCZYK, Invariance of weighted quasi-arithmetic means with continuous generators, *Publ. Math. Debrecen* **71**, no. 3–4 (2007), 279–294.
- [39] J. JARCZYK, When Lagrangean and quasi-arithmetic means coincide, *J. Inequal. Pure Appl. Math.* (2008) (*to appear*).
- [40] J. JARCZYK and J. MATKOWSKI, Invariance in the class of weighted quasi-arithmetic means, *Ann. Polon. Math.* **88**, no. 1 (2006), 39–51.
- [41] A. KOLMOGOROV, Sur la notion de la moyenne, *Rend. Accad. dei Lincei* (6) **12** (1930), 388–391.
- [42] L. LOSONCZI, Equality of two variable weighted means: reduction to differential equations, *Aequationes Math.* **58**, no. 3 (1999), 223–241.
- [43] L. LOSONCZI, Equality of Cauchy mean values, *Publ. Math. Debrecen* **57**, no. 1–2 (2000), 217–230.
- [44] L. LOSONCZI, Homogeneous Cauchy Mean Values, Functional Equations – Results and Advances, Advances in Mathematics, vol. 3, (Z. Daróczy and Zs. Páles, eds.), *Kluwer Acad. Publ., Dordrecht*, 2002, 209–218.
- [45] L. LOSONCZI, Equality of two variable Cauchy mean values, *Aequationes Math.* **65**, no. 1–2 (2003), 61–81.
- [46] L. LOSONCZI, Equality of two variable means revisited, *Aequationes Math.* **71**, no. 3 (2006), 228–245.
- [47] J. MATKOWSKI, Invariant and complementary quasi-arithmetic means, *Aequationes Math.* **57**, no. 1 (1999), 87–107.
- [48] J. MATKOWSKI, On iteration semigroups of mean-type mappings and invariant means, *Aequationes Math.* **64**, no. 3 (2002), 297–303.
- [49] J. MATKOWSKI, Convex functions with respect to a mean and a characterization of quasi-arithmetic means, *Real Anal. Exchange* **29**, no. 1 (2003/2004), 229–246.
- [50] J. MATKOWSKI, Lagrangian mean-type mappings for which the arithmetic mean is invariant, *J. Math. Anal. Appl.* **309**, no. 1 (2005), 15–24.
- [51] D. S. MITRINOVIĆ, J. E. PEČARIĆ and A. M. FINK, Inequalities Involving Functions and Their Integrals and Derivatives, Vol. 53, Mathematics and its Applications (East European Series), *Kluwer Academic Publishers Group, Dordrecht*, 1991.
- [52] D. S. MITRINOVIĆ, J. E. PEČARIĆ and A. M. FINK, Classical and New Inequalities in Analysis, Vol. 61, Mathematics and its Applications (East European Series, *Kluwer Academic Publishers Group, Dordrecht*, 1993.
- [53] Á. MÜNNICH, GY. MAKSA and R. J. MOKKEN, Collective judgement: combining individual value judgements, *Math. Social Sci.* **37**, no. 3 (1999), 211–233.
- [54] M. NAGUMO, Über eine Klasse der Mittelwerte, *Jap. Jour. of Math.* **7** (1930), 71–79.
- [55] C. P. NICULESCU and L.-E. PERSSON, Convex Functions and Their Applications, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, *Springer-Verlag, New York*, 2006, A contemporary approach.
- [56] ZS. PÁLES, On functional equations characterizing polynomials, manuscript.
- [57] ZS. PÁLES, On the equality of quasi-arithmetic means and Lagrangian means, manuscript.

- [58] Zs. PÁLES, Nonconvex functions and separation by power means, *Math. Inequal. Appl.* **3**, no. 2 (2000), 169–176.

ZITA MAKÓ
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, PF. 12
HUNGARY

E-mail: zmako@math.klte.hu

ZS. PÁLES
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, PF. 12
HUNGARY

E-mail: pales@math.klte.hu

(Received April 18, 2007, revised October 24, 2007)