

(α, β) -metrics with relatively isotropic mean Landsberg curvature

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Abstract. (α, β) -metrics form an important class of computable Finsler metrics. In this paper, we obtain firstly a formula of mean Cartan torsion for (α, β) -metrics and characterize Riemann metrics among (α, β) -metrics. Further, we obtain a sufficient and necessary condition for an (α, β) -metric to be of relatively isotropic mean Landsberg curvature.

1. Introduction

In Finsler geometry, there are several very important non-Riemannian quantities. The Cartan torsion $\mathbf{C}$ is a primary quantity. There is another quantity which is determined by the Busemann–Hausdorff volume form, that is the so-called distortion τ . The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^k} dx^k$. $\mathbf{C}$, τ and $\mathbf{I}$ are the basic geometric quantities which characterize Riemannian metrics among Finsler metrics. Differentiating $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}$. The horizontal derivative of τ along geodesics is the so-called S -curvature $\mathbf{S} := \tau_{|k} y^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k} y^k$. The Riemann curvature measures the shape of the space while the non-Riemannian quantities describe the change of the “color” on the space. Hence

Mathematics Subject Classification: 53B40, 53C60.

Key words and phrases: Finsler metric, (α, β) -metric, mean Cartan torsion, mean Landsberg curvature, weakly Landsberg metric.

This paper is supported by the National Natural Science Foundation of China (10671214) and by Natural Science Foundation Project of CQ CSTC.

Finsler spaces are “colorful” geometric spaces. It is found that the flag curvature is closely related to these non-Riemannian quantities [3], [9], [10].

By the definition, $\mathbf{J}/\mathbf{I}$ can be regarded as the relative growth rate of the mean Cartan torsion along geodesic. We call a Finsler metric F is of *relatively isotropic mean Landsberg curvature* if F satisfies $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on the Finsler manifold. In particular, when $c = 0$, Finsler metrics with $\mathbf{J} = 0$ are called *weakly Landsberg metrics*. Many known Finsler metrics satisfy $\mathbf{J} + cF\mathbf{I} = 0$ (cf. [3], [4], [9]). In [11], Z. SHEN proves that a projectively flat Randers metric of constant flag curvature on an n -dimensional manifold is either locally Minkowskian or after a scaling, isometric to a Finsler metric on the unit ball $\mathbb{B}^n$ in the following form

$$F_a = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2} \pm \frac{\langle a, y \rangle}{1 + \langle a, x \rangle}, \quad y \in T_x \mathbb{R}^n, \quad (1)$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. The Randers metric in (1) satisfies $\mathbf{J} \pm \frac{1}{2}F_a\mathbf{I} = 0$. In [4], the first author and Z. SHEN classify Randers metrics of isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ satisfying $\mathbf{J} + c(x)F\mathbf{I} = 0$ for some $c(x)$. Further, CHENG–MO–SHEN characterize flag curvature of Finsler metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature [3].

In the past several years, we witness a rapid development in Finsler geometry. Various curvatures have been studied and their geometric meanings are better understood. This is partially due to the study of a special class of Finsler metrics. The special Finsler metrics we are going to discuss are expressed in terms of a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$. They are called (α, β) -metrics. The simplest (α, β) -metrics are the Randers metrics $F = \alpha + \beta$. (α, β) -metrics form an important class of Finsler metrics with many applications in physics and biology (cf. [1]). Most important, (α, β) -metrics are “computable” and they are of many interesting curvature properties.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n -dimensional manifold M . Using α and β one can define a function F on TM as follows

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \quad (2)$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sup_{y \in T_x M} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

We assume that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_0), \quad (3)$$

so that $F = \alpha\phi(s)$, where $s = \beta/\alpha$, is a positive definite Finsler metric if and only if $b(x) := \|\beta_x\|_\alpha < b_0$ for all $x \in M$.

Recently, B. LI and Z. SHEN characterize weakly Landsberg metrics in (α, β) -metrics and show that there exist weakly Landsberg metrics which are not Landsberg metrics in dimension greater than two [8].

In this paper, we study (α, β) -metrics with relatively isotropic mean Landsberg curvature and prove the following

Theorem 1.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric on an n -dimensional manifold M ($n \geq 3$), where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. Then F is of relatively isotropic mean Landsberg curvature, i.e. there exists a scalar function $c = c(x)$ on M such that $\mathbf{J} + c(x)F\mathbf{I} = 0$, if and only if β satisfies*

$$s_{ij} = 0, \quad (4)$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + \sigma b_i b_j, \quad (5)$$

where $k = k(x)$ and $\sigma = \sigma(x)$ are scalar functions on M and $\phi = \phi(s)$ satisfies the following ODE:

$$\{\Psi_1 k + s\sigma\Psi_3\} + c(x)\Phi(\phi - s\phi') = 0, \quad (6)$$

where Φ, Ψ_1, Ψ_3 are defined as follows

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q', \quad (7)$$

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'', \quad (8)$$

$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{1/2} \left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{3/2}} \right]', \quad (9)$$

$$\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta} \quad (10)$$

and

$$\Psi_3 := \frac{s}{b^2 - s^2}\Psi_1 + \frac{b^2}{b^2 - s^2}\Psi_2. \quad (11)$$

By Theorem 1.1, we can see that $\mathbf{J} = 0$ if and only if β satisfies

$$s_{ij} = 0, \quad r_{ij} = k(b^2 a_{ij} - b_i b_j) + \sigma b_i b_j$$

and $\phi = \phi(s)$ satisfies

$$\Psi_1 k + s \sigma \Psi_3 = 0.$$

This is just the result of Proposition 3.1 in [8].

Example 1.2. Let $\phi(s) = 1 + s$. Then $F = \alpha\phi(\beta/\alpha) = \alpha + \beta$ is a Randers metric on the manifold. By a direct computation, we can prove that F is of relatively isotropic mean Landsberg curvature, $\mathbf{J} + c(x)F\mathbf{I} = 0$, if and only if β satisfies (4) and (5) with $k = 2c/b^2$ and $\sigma = 2c(1 - b^2)/b^2$, that is, β is closed and $r_{ij} = 2c(a_{ij} - b_i b_j)$. This result is first given by the first author and Z. SHEN in [4].

2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n -dimensional manifold M . Let

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$$

and $(g^{ij}) := (g_{ij})^{-1}$. For a non-zero vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$, F induces an inner product on $T_x M$

$$g_y(u, v) = g_{ij} u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}$, $v = v^j \frac{\partial}{\partial x^j} \in T_x M$. $g = \{g_y\}$ is called the *fundamental tensor* of F .

Let

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $\mathbf{C} := C_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$ on $TM \setminus \{0\}$. We call $\mathbf{C}$ the *Cartan torsion*. The *mean Cartan torsion* $\mathbf{I} = I_i dx^i$ is defined by

$$I_i := g^{jk} C_{ijk}.$$

Further, we have ([3], [7], [9])

$$I_i = g^{jk} C_{ijk} = \frac{\partial}{\partial y^i} \left[\ln \sqrt{\det(g_{jk})} \right]. \quad (12)$$

For a Finsler metric F , the geodesics are characterized locally by a system of 2nd ODEs:

$$\frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx}{dt} \right) = 0,$$

where

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^m y^l} y^m - [F^2]_{x^l} \right\}. \quad (13)$$

G^i are called the *geodesic coefficients* of F . The *Landsberg curvature* $\mathbf{L} = L_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$ is a horizontal tensor on $TM \setminus \{0\}$ defined by (cf. [9], [10])

$$L_{ijk} := -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j y^k}. \quad (14)$$

F is called a *Landsberg metric* if $\mathbf{L} = 0$. The *mean Landsberg curvature* $\mathbf{J} = J_i dx^i$ is defined by

$$J_i := g^{jk} L_{ijk}. \quad (15)$$

We call F a *weakly Landsberg metric* if $\mathbf{J} = 0$. We say that F is of *relatively isotropic mean Landsberg curvature* if $J_i + c(x) F I_i = 0$ for a scalar function $c(x)$ on M .

Now we consider an (α, β) -metric on an n -dimensional manifold M , $F = \alpha \phi(s)$, $s = \beta/\alpha$. By a linear algebra technique, one obtains (cf. [2], [7], [9])

$$\det(g_{ij}) = \phi^{n+1} (\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi''] \det(a_{ij}), \quad (16)$$

where $b(x) := \|\beta_x\|_\alpha$.

In order to study the geometric properties of (α, β) -metrics, one needs a formula for the geodesic coefficients of an (α, β) -metric. Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2} (b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2} (b_{i|j} - b_{j|i}), \\ s^i_j &:= a^{ih} s_{hj}, & r_j &:= b^m r_{mj}, & s_j &:= b_i s^i_j = b^m s_{mj}, \end{aligned}$$

where “|” denotes the covariant derivative with respect to the Levi-Civita connection of α . We will denote $r_{00} := r_{ij} y^i y^j$, $s^i_0 := s^i_j y^j$, etc. Let G^i and $\bar{G}^i$ denote the geodesic coefficients of F and α respectively in the same coordinate system. By a direct computation, one gets the following formula [7], [9]:

$$G^i = \bar{G}^i + \alpha Q s^i_0 + \Theta \left\{ -2\alpha Q s_0 + r_{00} \right\} \left\{ \frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i \right\}, \quad (17)$$

where

$$\Theta := \frac{Q - sQ'}{2\Delta}.$$

The Landsberg curvature of (α, β) -metrics is given by Z. SHEN [12] as follows

$$L_{jkl} = -\frac{\rho}{6\alpha^3} \{h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3(E_j h_{kl} + E_k h_{jl} + E_l h_{jk})\}, \quad (18)$$

where

$$\rho = \phi(\phi - s\phi'),$$

$$h_j = b_j - \alpha^{-1} s y_j,$$

$$h_{jk} = a_{j\ k} - \alpha^{-2} y_j y_k,$$

$$C_j = \alpha(X_4 r_{00} + Y_4 \alpha s_0) h_j - 3Q'' D_j,$$

$$E_j = \alpha(X_6 r_{00} + Y_6 \alpha s_0) h_j - (Q - sQ') D_j,$$

$$D_j = \frac{\alpha^2}{\Delta} (\Delta s_{j0} + r_{j0} - Q \alpha s_j) - \frac{1}{\Delta} (r_{00} - Q \alpha s_0) y_j,$$

where $y_j := a_{jk} y^k$ and

$$X_4 = \frac{1}{2\Delta^2} \{ -2\Delta Q'' + 3[(Q - sQ') + (b^2 - s^2)Q'']Q'' \},$$

$$X_6 = \frac{1}{2\Delta^2} \{ (Q - sQ')^2 + [2(s + b^2 Q) - (b^2 - s^2)(Q - sQ')]Q'' \},$$

$$Y_4 = -2QX_4 + \frac{3Q'Q''}{\Delta},$$

$$Y_6 = -2QX_6 + \frac{(Q - sQ')Q'}{\Delta}.$$

Then the mean Landsberg curvature of (α, β) -metrics is given by B. LI and Z. SHEN [8] as follows

$$\begin{aligned} J_j = & -\frac{1}{2\alpha^4 \Delta} \left\{ \frac{2\alpha^3}{b^2 - s^2} \left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (s_0 + r_0) h_j \right. \\ & + \frac{\alpha^2}{b^2 - s^2} \left[\Psi_1 + s \frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_j \\ & + \alpha \left[-\alpha^2 Q' s_0 h_j + \alpha Q (\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0} \right. \\ & \left. \left. + \alpha^2 (r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0) y_j \right] \frac{\Phi}{\Delta} \right\}. \end{aligned} \quad (19)$$

For our aim, we need the following formula for the mean Cartan torsion of (α, β) -metrics.

Lemma 2.1. *For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$, the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F} \frac{\Phi}{\Delta} (\phi - s\phi') h_i. \quad (20)$$

PROOF. By use of (12) and (16), after a direct computation, one can obtain

$$I_i = \frac{1}{2\alpha} \left\{ (n+1) \frac{\phi'}{\phi} - (n-2) \frac{s\phi''}{\phi - s\phi'} + \frac{(b^2 - s^2)\phi''' - 3s\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''} \right\} h_i. \quad (21)$$

Further, by use of a Maple programm, one can get (20). $\square$

By Deicke's theorem, a Finsler metric is Riemannian metric if and only if $\mathbf{I} = 0$ [2]. By (3) and the assumption $\phi(s) > 0$, we have $\phi(s) - s\phi'(s) > 0$, $|s| \leq b < b_0$ (cf. [7]). Thus, from Lemma 2.1, we have the following

Proposition 2.2. *An (α, β)-metric F is a Riemannian metric if and only if $\Phi = 0$.*

In the following we always assume that F is not a Riemannian metric, that is, $\Phi \neq 0$. From (19) and (20), we have the following

$$\begin{aligned} J_j + c(x)FI_j = & -\frac{1}{2\alpha^4\Delta} \left\{ \frac{2\alpha^3}{b^2 - s^2} \left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (s_0 + r_0)h_j \right. \\ & + \frac{\alpha^2}{b^2 - s^2} \left[\Psi_1 + s\frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Qs_0)h_j \\ & + \alpha \left[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0} \right. \\ & + \alpha^2 (r_{j0} - 2\alpha Qs_j) - (r_{00} - 2\alpha Qs_0)y_j \left. \right] \frac{\Phi}{\Delta} \\ & \left. + c(x)\alpha^4 \Phi(\phi - s\phi')h_j \right\}. \end{aligned} \quad (22)$$

3. Necessary conditions

We have known that $\mathbf{J}$ can be expressed in terms of α , β and $\phi(s)$, $\phi'(s)$ and etc, where $s = \beta/\alpha$. But the formula (19) is very complicated. So the equation $\mathbf{J} + c(x)F\mathbf{I} = 0$ is complicated too because one has to deal with the terms $\phi(s)$, $\phi'(s)$, etc. To overcome this difficulty, a useful technique is to take a special local coordinate system at a point x as in [12] such that

$$\alpha_x = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta_x = by^1,$$

where $b = \|\beta_x\|_\alpha$. Then we take another special coordinate: $(s, u^A) \rightarrow (y^i)$ given by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^A = u^A,$$

where $\bar{\alpha} = \sqrt{\sum_{A=2}^n (y^A)^2}$. We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$

Because the expression (22) involves r_{ij} , s_{ij} etc, one needs the following expressions:

$$\begin{aligned} r_1 &= br_{11}, \quad r_A = br_{1A}, \quad s_1 = 0, \quad s_A = bs_{1A}, \\ r_{00} &= \frac{s^2 \bar{\alpha}^2}{b^2 - s^2} r_{11} + 2 \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} \bar{r}_{10} + \bar{r}_{00}, \\ r_{10} &= \frac{s \bar{\alpha}}{\sqrt{b^2 - s^2}} r_{11} + \bar{r}_{10}, \quad s_{10} = \bar{s}_{10}, \end{aligned}$$

where $\bar{r}_{10} = r_{1A} u^A$, $\bar{s}_{10} = s_{1A} u^A$, $\bar{r}_{00} = r_{AB} u^A u^B$. We have $\bar{r}_0 = r_A u^A = b \bar{r}_{10}$, $\bar{s}_0 = s_A u^A = b \bar{s}_{10}$.

By a direct computation and using the formula (22), one can show that $J_1 + c(x)FI_1 = 0$ is equivalent to that

$$\{\Psi_3[s^2 r_{11} \bar{\alpha}^2 + (b^2 - s^2) \bar{r}_{00}] - b^2 \Psi_2 \bar{r}_{00}\} + c(x) s b^2 \Phi(\phi - s\phi') \bar{\alpha}^2 = 0 \quad (23)$$

and

$$(2s\Psi_1 + b^2\Psi_2)\bar{r}_{10} + b^2(\Psi_2 - 2Q\Psi_1)\bar{s}_{10} = 0, \quad (24)$$

$J_A + c(x)FI_A = 0$ ($A = 2, \dots, n$) is equivalent to that

$$\begin{aligned} &\{\Psi_3[s^2 r_{11} \bar{\alpha}^2 + (b^2 - s^2) \bar{r}_{00}] - b^2 \Psi_2 \bar{r}_{00}\} y_A + b^2 \frac{\Phi}{\Delta} [\bar{r}_{00} y_A - (\bar{r}_{A0} + \Delta \bar{s}_{A0}) \bar{\alpha}^2] \\ &\quad + c(x) s b^2 \Phi(\phi - s\phi') \bar{\alpha}^2 y_A = 0 \end{aligned} \quad (25)$$

and

$$\begin{aligned} &s\{2s\Psi_1 + b^2\Psi_2\}\bar{r}_{10} y_A + s b^2\{\Psi_2 - 2Q\Psi_1\}\bar{s}_{10} y_A \\ &\quad + b^2 \frac{\Phi}{\Delta} \{s(\bar{r}_{10} y_A - r_{1A} \bar{\alpha}^2) - (b^2 Q + \Delta s)(\bar{s}_{10} y_A - s_{1A} \bar{\alpha}^2)\} = 0. \end{aligned} \quad (26)$$

Lemma 3.1. *($n \geq 3$) For an (α, β) -metric F , if $J_j + c(x)FI_j = 0$ at a point x , then we have*

$$s_{AB} = 0, \quad (27)$$

$$r_{AB} = kb^2\delta_{AB}, \quad (28)$$

$$r_{11} = \sigma b^2, \quad (29)$$

where $k = k(x)$ and $\sigma = \sigma(x)$ are scalar functions on M .

PROOF. It follows from (23) and (25) that

$$\bar{r}_{00}y_A - (\bar{r}_{A0} + \Delta\bar{s}_{A0})\bar{\alpha}^2 = 0. \quad (30)$$

Since $n \geq 3$, (30) implies (27) and (28). Letting $\sigma := r_{11}/b^2$, we obtain (29). $\square$

Lemma 3.2. *($n \geq 3$) For an (α, β) -metric F , if $J_i + c(x)FI_i = 0$ at a point x , then*

$$s_{1A} = 0, \quad r_{1A} = 0. \quad (31)$$

PROOF. It follows from (24) and (26) that

$$\{s\bar{r}_{10} - (b^2Q + \Delta s)\bar{s}_{10}\}y_A - \{r_{1A} - (b^2Q + \Delta s)s_{1A}\}\bar{\alpha}^2 = 0. \quad (32)$$

Since $n \geq 3$, we obtain from (32) that

$$sr_{1A} - (b^2Q + \Delta s)s_{1A} = 0. \quad (33)$$

Then we can claim that $s_{1A} = 0$ and $r_{1A} = 0$. See Lemma 4.2 in [8] for more details. $\square$

From Lemma 3.1 and 3.2, one obtains the following

Corollary 3.3. *For an arbitrary (α, β) -metric F on an n -dimensional manifold M ($n \geq 3$), if $J_i + c(x)FI_i = 0$, then β must be closed.*

Now, plugging (28) and (29) into (23) yield

$$\{\Psi_1k + s\sigma\Psi_3\} + c(x)\Phi(\phi - s\phi') = 0. \quad (34)$$

Let us summarize what we have proved.

Proposition 3.4. *Let $F = \alpha\phi(s)$ be an (α, β) -metric on an n -dimensional manifold ($n \geq 3$). Suppose that $J_i + c(x)FI_i = 0$ at a point x . Then β satisfies*

$$s_{ij} = 0, \quad (35)$$

$$r_{ij} = k(b^2a_{ij} - b_ib_j) + \sigma b_ib_j \quad (36)$$

and $\phi = \phi(s)$ satisfies

$$\{\Psi_1k + s\sigma\Psi_3\} + c(x)\Phi(\phi - s\phi') = 0.$$

4. Sufficient conditions

In this section, we are going to prove the sufficient conditions for an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$ to be of relatively isotropic mean Landsberg curvature. Assume that α and β satisfy (4) and (5), we have

$$s_{j0} = 0, \quad s_j = 0, \quad s_0 = 0, \quad (37)$$

$$r_{j0} = k(b^2 y_j - \beta b_j) + \sigma \beta b_j, \quad r_0 = \sigma \beta b^2, \quad (38)$$

$$r_{00} = k(b^2 \alpha^2 - \beta^2) + \sigma \beta^2. \quad (39)$$

Substituting them into (22), we obtain

$$J_j + c(x)FI_j = -\frac{1}{2\Delta} \{ \Psi_1 k + s\sigma\Psi_3 + c(x)\Phi(\phi - s\phi') \} h_j.$$

By our assumption (6) on ϕ , we have

$$J_j + c(x)FI_j = 0.$$

This completes the proof of the sufficient conditions.

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(Received June 18, 2007)