

Distribution of additive and q -additive functions under some conditions II.

By I. KÁTAI (Budapest) and M. V. SUBBARAO (Edmonton)

Abstract. Distribution of additive function over the set of integers having fixed number of prime divisors, and the distribution of q -additive functions over the set of integers for which the value of the sum of divisors function is fixed are investigated.

§1. Introduction

1.1. Notation. $\mathbb{N}, \mathbb{R}, \mathbb{C}$ as usual denote the set of natural, real and complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathcal{P}$ be the set of the primes, p with or without suffixes always denote prime numbers. The letters $c, c_1, c_2, \dots$ denote constants not necessary the same at every occurrence. Let $\Phi(y)$ be the Gaussian distribution function, $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-u^2/2} du$.

1.2. q -additive and q -multiplicative functions. Let $q \geq 2$ be an integer, the q -ary expansion of $n \in \mathbb{N}_0$ is defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1.1}$$

where the digits $\varepsilon_j(n)$ are taken from $\mathbb{A}_q := \{0, 1, \dots, q-1\}$. Let $\mathcal{A}_q$ be the set of q -additive functions, and $\overline{\mathcal{M}}_q$ be the set of q -multiplicative functions of modulus

Mathematics Subject Classification: 11N60, 11K60.

Key words and phrases: additive functions, q -additive functions, distribution function. Supported by OTKA T46993.

M. V. Subbarao passed away on 15th of February, 2006.

1: $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ belongs to $\mathcal{A}_q$ if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0). \quad (1.2)$$

We say that $g : \mathbb{N}_0 \rightarrow \mathbb{C}$ belongs to $\overline{\mathcal{M}}_q$, if $g(0) = 1$,

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0) \quad (1.3)$$

and $|g(n)| = 1$ ($n \in \mathbb{N}_0$).

Let $\alpha(n)$, $\beta_h(n)$ be defined as

$$\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n); \quad \beta_h(n) = \sum_{\varepsilon_j(n)=h} 1 \quad (h = 1, \dots, q-1). \quad (1.4)$$

It is clear that $\alpha, \beta_h \in \mathcal{A}_q$. H. DELANGE [1] proved that for every $g \in \overline{\mathcal{M}}_q$ the limit

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} g(n) = M(g) \quad (1.5)$$

exists and $M(g) \neq 0$, if

$$m_j := \frac{1}{q} \sum_{c \in \mathbb{A}_q} g(cq^j) \neq 0 \quad (j = 0, 1, 2, \dots) \quad (1.6)$$

and

$$\sum (1 - m_j) \quad (1.7)$$

is convergent. If these conditions hold, then

$$M(g) = \prod_{j=0}^{\infty} m_j. \quad (1.8)$$

Hence he deduced that for $f \in \mathcal{A}_q$ the values $f(n)$ possess a limit distribution if and only if both of the series

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f(bq^j), \quad (1.9)$$

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f^2(bq^j) \quad (1.10)$$

are convergent.

Let $f \in \mathcal{A}_q$. Assume that it has the limit distribution

$$F(y) := \lim_{x \rightarrow \infty} \frac{1}{x} \#\{n < x \mid f(n) < y\}. \quad (1.11)$$

Delange proved that $F(y) = P(\xi < y)$, where ξ is the sum of the independent random variables $\xi_0, \xi_1, \dots$, where $P(\xi_j = f(aq^j)) = 1/q$ ($a \in \mathbb{A}_q$). Thus the characteristic function $\varphi(\tau)$ of $F(y)$ can be written as

$$\varphi(\tau) = \prod_{j=0}^{\infty} \left\{ \frac{1}{q} \sum_{a=0}^{q-1} e^{i\tau f(aq^j)} \right\}. \quad (1.12)$$

Let $r_1, r_2, \dots, r_{q-1}$ be nonnegative integers, $\underline{r} = (r_1, \dots, r_{q-1})$ and $S_N(\underline{r}) = \{n < q^N \mid \beta_j(n) = r_j, j = 1, \dots, q-1\}$. Let $r_0 := N - (r_1 + \dots + r_{q-1})$. $S_N(\underline{r})$ is empty if $r_0 < 0$. Let $M(N \mid \underline{r}) := \#S_N(\underline{r})$.

In [5] we proved the following Theorems A, B, C.

Theorem A. *Let $f \in \mathcal{A}_q$, and the series (1.9), (1.10) be convergent. Let $\underline{r}^{(N)} = (r_1^{(N)}, \dots, r_{q-1}^{(N)})$ be such a sequence for which*

$$\left| \frac{qr_j^{(N)}}{N} - 1 \right| < \delta_N \quad (j = 1, \dots, q-1) \quad (1.13)$$

where $\delta_N \rightarrow 0$ ($N \rightarrow \infty$).

Then

$$\lim_{N \rightarrow \infty} \frac{1}{M(N \mid \underline{r})} \#\{n \in S_N(\underline{r}^{(N)}) \mid f(n) < y\} = F(y), \quad (1.14)$$

where $F(y) = P(\xi < y)$.

Theorem B. *Let $g \in \overline{\mathcal{M}}_q$ be such a function for which (1.6) holds and (1.7) is convergent. Let $\underline{r}^{(N)}$ be a sequence satisfying (1.13). Then*

$$\lim_{N \rightarrow \infty} \frac{1}{M(N \mid \underline{r})} \sum_{n \in S_N(\underline{r}^{(N)})} g(n) = M(g). \quad (1.15)$$

Theorem C. *Let $q = 2$, $f \in \mathcal{A}_2$, $f(2^j) = O(1)$ ($j \in \mathbb{N}$),*

$$\eta_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2^j), \quad B_N^2 := \frac{1}{4} \sum_{j=0}^{N-1} (f(2^j) - \eta_N)^2.$$

Assume that $B_N \rightarrow \infty$.

Let $\rho_N \rightarrow 0$, and $k = k^{(N)}$ be such a sequence of integers for which

$$\left| \frac{k}{N} - 1/2 \right| < \rho_N \quad (1.16)$$

holds.

Then

$$\lim_{N \rightarrow \infty} \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N \mid \frac{f(n) - k\eta_N}{B_N} < y, \alpha(n) = k \right\} = \Phi(y), \quad (1.17)$$

the convergence is uniform in y .

In [6] we continued our work and proved the following Theorems D, E.

Let

$$\eta_{N,k} := \frac{k}{N}, \quad \mathcal{E}_{N,k} = \{n < 2^N \mid \alpha(n) = k\}. \quad (1.18)$$

Theorem D. Let $g \in \overline{\mathcal{M}}_2$ be such a function for which

$$\sum_{j=0}^{\infty} (1 - g(2^j)) \quad (1.19)$$

is convergent. Let

$$M_{\xi} := \prod_{j=0}^{\infty} ((1 - \xi) + g(2^j)\xi) \quad (0 < \xi < 1). \quad (1.20)$$

Let $\delta > 0$ be a constant. Then

$$\lim_{N \rightarrow \infty} \max_{\delta \leq \frac{k}{N} \leq 1-\delta} \left| \frac{1}{\binom{N}{k}} \sum_{\substack{n \in \mathcal{E}_{N,k} \\ n \leq q^N}} g(n) - M_{\eta_{N,k}} \right| = 0. \quad (1.21)$$

Theorem E. Let $f \in \mathcal{A}_2$, such that $\sum f(2^j)$, $\sum f^2(2^j)$ are convergent. Let $\xi_0, \xi_1, \dots$ be independent random variables, $P(\xi_{\nu} = 0) = 1 - \eta$, $P(\xi_{\nu} = f(2^{\nu})) = \eta$, $\Theta = \sum_{j=0}^{\infty} \xi_j$,

$$F_{\eta}(y) := P(\Theta < y). \quad (1.22)$$

Then

$$\lim_{N \rightarrow \infty} \max_{\delta \leq \frac{k}{N} \leq 1-\delta} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \{n \in \mathcal{E}_{N,k}, f(n) < y\} - F_{\frac{k}{N}}(y) \right| = 0. \quad (1.23)$$

Here $\delta > 0$ is an arbitrary small constant.

In [6] we mentioned that we would be able to prove

Theorem F. *Let $f \in \mathcal{A}_2$, $f(2^j) = O(1)$. Let $h_N \in \mathcal{A}_2$ be defined by $h_N(2^j) := f(2^j) - \frac{1}{N}A_N$, $A_N = \sum_{j=0}^{N-1} f(2^j)$, $\sigma_N^2(\eta) := (1-\eta)\eta \sum_{j=0}^{N-1} h_N^2(2^j)$.*

Assume that $\lim_{N \rightarrow \infty} \sigma_N(1/2) = \infty$.

Let $0 < \delta < 1/2$ be a constant. Then

$$\lim_{N \rightarrow \infty} \sup_{\frac{k}{N} \in [\delta, 1-\delta]} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n \in \mathcal{E}_{N,k} \mid \frac{f(n) - \frac{k}{N}A_N}{\sigma_N\left(\frac{k}{N}\right)} < y \right\} - \Phi(y) \right| = 0.$$

Here we shall prove that for the fulfilment of (1.23) the convergence of $\sum f(2^j)$, and of $\sum f^2(2^j)$ is necessary. Namely we shall prove the following

Theorem 1. *Let $f \in \mathcal{A}_2$. Assume that there exists a sequence of integers k_N , $\frac{k_N}{N} \rightarrow \xi$ ($N \rightarrow \infty$), $0 < \xi < 1$ such that*

$$\lim_{N \rightarrow \infty} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k_N}} \# \{n \in \mathcal{E}_{N,k_N}, f(n) < y\} - G(y) \right| = 0$$

with a suitable distribution function $G(y)$. Then both of the series (1.9), (1.10) are convergent and $G(y) = F_\xi(y)$, $F_\xi(y)$ is defined in Theorem E.

1.3. Additive functions. We say that $f : \mathbb{N} \rightarrow \mathbb{R}$ is additive if $f(mn) = f(m) + f(n)$ holds for every coprime pairs of integers. We say that $g : \mathbb{N} \rightarrow \mathbb{C}$ is multiplicative, if $g(1) = 1$, and $g(mn) = g(m) \cdot g(n)$, whenever $(m, n) = 1$. Let $\mathcal{A}, \mathcal{M}$ be the sets of additive, and multiplicative functions, let $\overline{\mathcal{M}} = \{g \in \mathcal{M} \mid |g(n)| = 1 \ (n \in \mathbb{N})\}$. For the sake of brevity we shall write $x_1 = \log x$, $x_2 = \log x_1, \dots$.

Let $\Omega(n)$ = number of distinct prime powers of n , $\mathcal{N}_k = \{n \mid \Omega(n) = k\}$,

$$N_k(x) := \#\{n \leq x, n \in \mathcal{N}_k\}, \quad N_k(x \mid D) := \#\{n \leq x \mid (n, D) = 1, n \in \mathcal{N}_k\}.$$

According to a classical theorem of Erdős and Wintner, if $f \in \mathcal{A}$ and the following three series

$$\sum_{|f(p)| < 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| < 1} \frac{f^2(p)}{p}, \quad \sum_{|f(p)| \geq 1} 1/p \quad (1.24)$$

are convergent, then

$$\lim_x \frac{1}{x} \# \{n \leq x \mid f(n) < y\} = F(y) \quad (1.25)$$

exists at every continuity points of F , where F is a distribution function. They proved also that the convergence of the series in (1.24) is necessary for the existence of satisfying (1.25).

In [6] we proved the following two theorems.

Theorem G. Assume that $f \in \mathcal{A}$, the series (1.24) are convergent and $k = k(x)$ satisfies the inequality

$$\left| \frac{k}{x_2} - 1 \right| < \delta_x, \quad (1.26)$$

where $\delta_x \downarrow 0$. Then

$$\lim_{x \rightarrow \infty} \frac{1}{N_k(x)} \# \{n \leq x, n \in \mathcal{N}_k, f(n) < y\} = F(y), \quad (1.27)$$

where $F(y)$ is defined by (1.25).

Theorem H. Let $g \in \overline{\mathcal{M}}$, and assume that

$$\sum_p \frac{1 - g(p)}{p} \quad (1.28)$$

is convergent. Let $k = k(x)$ be such a sequence for which (1.26) is satisfied. Then

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) = (1 + o_x(1))M(g),$$

$$M(g) = \prod_p e_p, \quad e_p = (1 - 1/p) \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right).$$

Here we shall prove

Theorem 2. Let g be as in Theorem H satisfying the conditions formulated there. Let $\delta > 0$ be a fixed constant, $\xi_{k,x} := \frac{k}{x_2}$. Let

$$M_\eta(g) := \prod_p e_p(\eta), \quad e_p(\eta) = \left(1 - \frac{\eta}{p} \right) \left(1 + \frac{g(p)\eta}{p} + \frac{g(p^2)\eta^2}{p^2} + \dots \right).$$

We have

$$\lim_{x \rightarrow \infty} \sup_{\delta \leq \xi_{k,x} \leq 2-\delta} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) - M_{\xi_{k,x}}(g) \right| = 0.$$

Theorem 3. Let $f \in \mathcal{A}$, $f(p^\alpha) = O(1)$ if $p \in \mathcal{P}$, and $\alpha \in \mathbb{N}$. Let $A_x = \sum_{p \leq x} \frac{f(p)}{p}$, $f^*(p^\alpha) = (p^\alpha) - \frac{\alpha}{x_2} A_x$, $B_x^2 = \sum_{p \leq x} \frac{1}{p} (f^*(p))^2$. Assume that f^* is extended to $\mathbb{N}$ so that $f^* \in \mathcal{A}$. Let $B_x \rightarrow \infty$. Let $\xi_{k,x} := \frac{k}{x_2}$, $\delta \in (0, 1/2)$ be a constant. Then

$$\lim_{x \rightarrow \infty} \max_k \max_{\xi_{k,x} \in [\delta, 2-\delta]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \leq x \mid \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y, n \in \mathcal{N}_k \right\} - \Phi(y) \right| = 0.$$

Theorem 4. Assume that the conditions of Theorem 3 hold true. Let δ, A be positive constants, so that $0 < \delta < 1/2$, $A > 2 + \delta$. Then

$$\lim_{x \rightarrow \infty} \max_{\xi_{k,x} \in [2+\delta, A]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \leq x \mid \frac{f^*(n)}{B_x \sqrt{2}} < y \right\} - \Phi(y) \right| = 0.$$

Theorem 5. Let $f \in \mathcal{A}$, and assume that the 3 series in (1.24) are convergent. For some $\eta \in (0, 2)$ and $p \in \mathcal{P}$ let $\xi_p = \xi_p(\eta)$ be the random variable distributed by $P(\xi_p = f(p^\alpha)) = \left(1 - \frac{\eta}{p}\right) \left(\frac{\eta}{p}\right)^\alpha$ ($\alpha = 0, 1, 2, \dots$). Assume that $\xi_p (p \in \mathcal{P})$ are completely independent, $\Theta(\eta) := \sum \xi_p(\eta)$.

Let $F_\eta(y) := P(\Theta(\eta) < y)$. Let furthermore

$$F_{k,x}(y) := \frac{1}{N_k(x)} \# \{n \leq x, n \in \mathcal{N}_k, f(n) < y\}.$$

Let $0 < \delta < 1/2$.

Then

$$\lim_{x \rightarrow \infty} \max_{\xi_{k,x} \in [\delta, 2-\delta]} \sup_{y \in \mathbb{R}} |F_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0.$$

Theorem 6. Let $g \in \overline{\mathcal{M}}$, (1.28) is convergent. Assume furthermore that $g(2^\alpha) = 1$ ($\alpha = 1, 2, \dots$). Let $A > 2 + \delta$ be constants. In the notations of Theorem 4 we have

$$\lim_{x \rightarrow \infty} \sup_{2+\delta \leq \xi_{k,x} \leq A} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g(n) - M_2^*(g) \right| = 0,$$

where

$$M_2^*(g) = \prod_{p>2} e_p(2).$$

Theorem 7. Let $f \in \mathcal{A}$ be as in Theorem 7. Assume furthermore that $f(2^\alpha) = 0$ ($\alpha = 1, 2, \dots$). Then

$$\lim_{x \rightarrow \infty} \max_{2+\delta \leq \xi_{k,x} \leq A} |F_{k,x}(y) - F_2^*(y)| = 0,$$

where

$$F_2^*(y) = P\left(\sum_{p>2} \xi_p(2) < y\right).$$

Here $\xi_{k,x} = \frac{k}{x_2}$.

Remark. In Theorems 6 and 7 we have to assume something on the values $g(2^\alpha)$ and on $f(2^\alpha)$, since for the function $\nu(n)$ defined by $2^{\nu(n)} \parallel n$,

$$\lim_{x \rightarrow \infty} \frac{1}{N_k(x)} \#\{n \leq x, \nu(n) < c, n \in \mathcal{N}_k\} = 0$$

for every fixed c .

In the proof of some of the theorems we use the following analogue of the Turán–Kubilius inequality.

Theorem 8. Let $f \in \mathcal{A}$, $A_x = \sum_{p \leq x} \frac{f(p)}{p}$, $\tilde{B}_x^2(\eta) := \sum_{p^\alpha \leq \sqrt{x}} \frac{f^2(p^\alpha) \eta^{2\alpha}}{p^\alpha}$. Assume that $f(p^\alpha) = 0$ if $p^\alpha > x^{1/4}$ or if $p \in \mathcal{P}$ and $\alpha > \sqrt{x_2}$. Let $\delta > 0$ be a constant, $\xi_{k,x} := \frac{k}{x_2}$. Then

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f(n) - \xi_{k,x} A_x)^2 \leq c \tilde{B}_x^2(\xi_{k,x}), \quad (1.29)$$

if $\xi_{k,x} \in [\delta, 2 - \delta]$. Here c is an absolute constant.

Theorem 9. Let f be as in Theorem 8. Assume that $f(2^\alpha) = 0$ ($\alpha = 1, 2, \dots$). Let δ and $A > 2 + \delta$ be constants. Then, for $(2 + \delta)x_2 \leq k \leq Ax_2$,

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f(n) - 2A_x)^2 \leq c \tilde{B}_x^2(2), \quad (1.30)$$

where c is a constant that may depend on δ and A .

Remark. In Theorem 9

$$\tilde{B}_x^2(2) = \sum_{\substack{p > 2 \\ p^\alpha \leq \sqrt{x}}} \frac{f^2(p^\alpha)}{p^\alpha}.$$

§2. Some lemmas and proof of Theorem 1

Let $f \in \mathcal{A}_2$, and

$$Q_{k,N}(D) := \sup_{y \in \mathbb{R}} \#\{n \in \mathcal{E}_{N,k}, f(n) \in [y, y + D]\}.$$

Lemma 1. *Let $D > 0$ be fixed. If $\limsup_j |f(2^j)| = \infty$, then*

$$\max_{\delta \leq k/N \leq 1-\delta} \frac{Q_{k,N}(D)}{\binom{N}{k}} \rightarrow 0 \quad (N \rightarrow \infty).$$

PROOF. By changing the sign of f , if needed, we may assume that $\limsup f(2^j) = \infty$.

Let $l_1 < l_2 < \dots$ be such a sequence of integers for which: $2D \leq f(2^{l_1})$, $f(2^{l_{h+1}}) \geq 2f(2^{l_h})$.

Let N be a large integer, T be defined such that $l_T \leq N-1 < l_{T+1}$. Let

$$U = \{l_1, l_2, \dots, l_T\}, \quad V = \{0, 1, \dots, N-1\} \setminus U.$$

Let

$$\alpha_1(n) = \sum_{s \in V} \varepsilon_s(n), \quad \alpha_2(n) = \sum_{t \in U} \varepsilon_t(n),$$

$$\mathcal{E}_h := \{n \in \mathcal{E}_{k,N}, \alpha_2(n) = h\}, \quad h = 0, 1, \dots, T.$$

Then

$$\mathcal{E}_{N,k} = \bigcup_{h=0}^T \mathcal{E}_h.$$

Assume that $h \geq 1$. Then

$$\mathcal{E}_h = \bigcup_{a_1, a_2, \dots, a_h} \mathcal{E}_h^{(a_1, \dots, a_h)},$$

where $a_1, a_2, \dots, a_h$ run over all strictly monotonic sequences of length h from the set U ,

$$\mathcal{E}_h^{(a_1, \dots, a_h)} := \{n \in \mathcal{E}_{k,N}; \varepsilon_{a_\nu}(n) = 1$$

$$\text{if } \nu = 1, \dots, h; \varepsilon_b(n) = 0 \text{ if } b \in U \setminus \{a_1, \dots, a_h\}\}.$$

If $n \in \mathcal{E}_h^{(a_1, \dots, a_h)}$, then $n = m + \rho_h$, where

$$\rho_h = \sum_{\nu=1}^h 2^{a_\nu}, \quad m = \sum_{\substack{j=0^{N-1} \\ j \in V}} \delta_j \cdot 2^j, \quad (\delta_j \in \{0, 1\}).$$

It is clear that $|f(\rho_h^{(1)}) - f(\rho_h^{(2)})| > D$ if $\rho_h^{(1)} \neq \rho_h^{(2)}$.

Let y and h be fixed. Then, for a fixed m , no more than one ρ_h may exist for which $f(\rho_h + m) \in [y, y + D]$.

Thus

$$\#\{n \in \mathcal{E}_h \mid f(n) \in [y, y + D]\} \leq \binom{N-T}{k-h}.$$

This inequality holds for $h = 0$ as well.

We have

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = \frac{(N-T)!k!(N-k)!}{N!(k-h)!(N-T-(k-h))!}.$$

It is clear that, if $\{l_\nu\}$ satisfies the conditions stated above, then these conditions hold for every infinite subsequence of it. Therefore we may assume that $T^2/N \rightarrow 0$ as $N \rightarrow \infty$, whence we can deduce that

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = (1 + o_N(1)) \frac{k^h \cdot (N-k)^{T-h}}{N^T},$$

and so

$$\begin{aligned} \frac{Q_{k,N}(D)}{\binom{N}{k}} &\leq (1 + o_N(1)) \sum_{h=0}^T \left(\frac{k}{N}\right)^h \left(1 - \frac{k}{N}\right)^{T-h} \\ &\leq cT \max \left\{ \left(1 - \frac{k}{N}\right)^T, \left(\frac{k}{N}\right)^T \right\} \leq cT(1-\delta)^T \rightarrow 0 \quad \text{as } T \rightarrow \infty. \end{aligned}$$

The proof of Lemma 1 is complete. $\square$

Lemma 2. Let $f \in \mathcal{A}_2$, $f(2^j) = O(1)$, $h_N \in \mathcal{A}_2$,

$$h_N(2^j) := f(2^j) - \frac{1}{N}A_N, \quad A_N = \sum_{j=0}^{N-1} f(2^j), \quad B_N^2 = \sum_{j=0}^{N-1} h_N^2(2^j).$$

Assume that $\limsup_{N \rightarrow \infty} B_N^2 = \infty$. Then

$$\lim_{N \rightarrow \infty} \max_{\frac{k}{N} \in [\delta, 2-\delta]} \frac{Q_{k,N}(D)}{\binom{N}{k}} = 0.$$

PROOF. The assertion is clear from Theorem F. $\square$

PROOF OF THEOREM 1. Assume that the conditions hold. Then $Q_{k_N,N}(D) > c \binom{N}{k_N}$ with $c > 0$, if $\frac{k_N}{N} \in (\delta, 1-\delta)$. Thus $f(2^j) = O(1)$, and B_N^2 is bounded. One can prove simply that

$$\frac{1}{\binom{N}{k_N}} \sum_{\substack{n < 2^N \\ n \in \mathcal{E}_{N,k_N}}} h_N^2(n) = \frac{k_N}{N} \cdot \frac{(N-k_N)}{(N-1)} B_N^2, \quad (2.1)$$

whence

$$\frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |h_N(n)| > \Delta\} < \frac{c(\delta)}{\Delta^2}, \quad (2.2)$$

where $c(\delta)$ is a constant, and Δ is an arbitrary positive number. If f has a limit distribution on $\mathcal{E}_{N,k_N}$, then

$$\limsup_{N \rightarrow \infty} \frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |f(n)| > \Delta\} \leq \varepsilon(\Delta), \quad (2.3)$$

where $\varepsilon(\Delta) \rightarrow 0$ as $\Delta \rightarrow \infty$.

From (2.2), (2.3) we obtain that $|h_N(n) - f(n)| \leq 2\Delta$ holds for at least $(1 - 2\varepsilon(\Delta) - \frac{c(\delta)}{\Delta^2})\binom{N}{k_N}$ integers $n \in \mathcal{E}_{N,k_N}$, whence we obtain that $A_N = O(1)$. Thus $\sum f^2(2^j) < \infty$ holds.

Let $M < N$, $A_{M,N} = A_N - A_M$.

Let $0 < \eta < 1$, $\xi_i(\eta)$ be independent random variables,

$$P(\xi_i(\eta)) = -\eta f(2^i) = 1 - \eta, \quad P(\xi_i(\eta) = (1 - \eta)f(2^i)) = \eta,$$

$$\Theta_M(\eta) := \xi_0(\eta) + \xi_1(\eta) + \dots + \xi_{M-1}(\eta).$$

Since $\sum f^2(2^i) < \infty$, therefore $P(\Theta_M(\eta) < z)$ converges weakly to a distribution function as $M \rightarrow \infty$.

Let

$$G_{M,\eta}(y) = P(\Theta_M(\eta) < y) \rightarrow G_\eta(y) = P(\Theta_\infty(\eta) < y).$$

Let $\tau \in \mathbb{R}$, $g(n) = e^{i\tau f(n)}$, $g_M(n) = \prod_{j=0}^{M-1} g(\varepsilon_j(n)2^j)$,

$$h(n) = \tau f(n), \quad h_M^*(n) = \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j),$$

$$u_M(n) := \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j).$$

Repeating the simple computation used in [5], we can deduce that

$$\begin{aligned} \frac{1}{\binom{N}{k_N}} \sum_{n \in \mathcal{E}_{N,k_N}} (h_M^*(n) - \eta \tau A_{M,N})^2 &\leq c_1(\delta) \sum_{j=M}^{N-1} h^2(2^j) \\ &+ \frac{c_2(\delta)}{N} \sum_{i,j=M}^{N-1} |h(2^i)| \cdot |h(2^j)| \leq c_3(\delta) \sum_{j=M}^{N-1} h^2(2^j), \end{aligned}$$

with suitable constants $c_j(\delta)$, $j = 1, 2, 3$.

We have

$$g(n) = g_M(n)e^{ih_M^*(n)} = g_M(n)e^{i\tau\eta A_{M,N}} + g_M(n)\left(e^{ih_M^*(n)} - e^{i\eta\tau A_{M,N}}\right),$$

whence $|g(n) - g_M(n)e^{i\eta\tau A_{M,N}}| \leq |h_M^*(n) - \eta\tau A_{M,N}|$, and in the notations

$$M_{N, \frac{k}{N}}(\tau) := \frac{1}{\binom{N}{k}} \sum_{\substack{n < 2^N \\ n \in \mathcal{E}_{N,k}}} g(n),$$

$$\varphi_{M,\eta}(\tau) = \prod_{l=0}^{M-1} (\eta e^{i\tau(1-\eta)f(2^l)} + (1-\eta)e^{-i\tau\eta f(2^l)}),$$

we obtain that

$$\left| M_{N, k_N/N}(\tau) - e^{i\frac{k_N}{N}\tau A_{M,N}} \cdot \frac{1}{\binom{N}{k_N}} \sum_{n < 2^N} g_M(n) \right| \leq c_4(\delta)|\tau| \sqrt{\sum_{j \geq M} f^2(2^j)}.$$

Arguing as in [5], we can deduce that

$$\begin{aligned} \frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} g_M(n) &= (1 + o_N(1)) \prod_{j=0}^{M-1} \left(\left(1 - \frac{k}{N}\right) + \frac{k}{N} \cdot g(2^j) \right) \\ &= (1 + o_N(1)) e^{i\tau \frac{k}{N} A_M} \varphi_{M, \frac{k}{N}}(\tau), \end{aligned}$$

thus

$$\left| M_{N, k_N/N}(\tau) - e^{i\frac{k_N}{N}\tau A_N} \varphi_{M, \frac{k_N}{N}}(\tau) \right| \leq o_N(1) + c_5(\delta)\varepsilon_M|\tau|,$$

where

$$\varepsilon_M^2 = \sum_{j=M}^{\infty} f^2(2^j), \quad \varepsilon_M \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Let $\psi_\eta(\tau) = \lim_{N \rightarrow \infty} M_{N, \frac{k_N}{N}}(\tau)$. From the condition we know that ψ_η exists. Furthermore $\lim_{N \rightarrow \infty} \varphi_{M, \frac{k_N}{N}}(\tau) = \varphi_{M,\eta}(\tau)$ obviously holds (due to $\frac{k_N}{N} \rightarrow \eta$). Finally, we shall prove that $\lim A_N$ exists.

Assume indirectly that $\alpha = \liminf A_N$, $\beta = \limsup A_N$, $\alpha \neq \beta$, $N_\nu \nearrow \infty$, $R_\mu \rightarrow \infty$, $A_{N_\nu} \rightarrow \alpha$ ($\nu \rightarrow \infty$), $R_\mu \rightarrow \beta$ ($\mu \rightarrow \infty$). Then

$$\begin{aligned} &\left| M_{N_\nu, \frac{k_{N_\nu}}{N_\nu}}(\tau) - M_{R_\mu, \frac{k_{R_\mu}}{R_\mu}}(\tau) - e^{i\frac{k_{N_\nu}}{N_\nu}\tau A_{N_\nu}} \varphi_{M, \frac{k_{N_\nu}}{N_\nu}}(\tau) - e^{i\frac{k_{R_\mu}}{R_\mu}\tau A_{R_\mu}} \varphi_{M, \frac{k_{R_\mu}}{R_\mu}}(\tau) \right| \\ &\leq o_{\min(N_\nu, R_\mu)}(1) + c_6(\delta)\varepsilon_M|\tau|. \end{aligned}$$

It is clear that $\varphi_{M,\lambda}$ is continuous uniformly in $\lambda \in [\delta, 1 - \delta]$, and $\lim_{M \rightarrow \infty} \varphi_{M,\lambda}(\tau)$ is continuous as well. Hence we obtain that $|e^{i\alpha\tau} - e^{i\beta\tau}| = 0$. This holds only if $\alpha = \beta$.
The proof is completed. $\square$

§3. Some useful lemmas

The following two lemmas can be found in [7], pages 59 and 60.

Lemma 3 (Wintner, Fréchet–Shohat). *Let $F_n(z)$ ($n = 1, 2, \dots$) be a sequence of distribution functions. For each non-negative integer k let*

$$\alpha_k = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

exist.

Then there is a subsequence $F_{n_j}(z)$, ($n_1 < n_2 < \dots$), which converges weakly to a limiting distribution $F(z)$ for which

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, 2, \dots).$$

Moreover, if the set of moments α_k determine $F(z)$ uniquely, then as $n \rightarrow \infty$ the distributions $F_n(z)$ converge weakly to $F(z)$.

Lemma 4. *In the notations of Lemma 3 let the series*

$$\phi(t) = \sum_{l=0}^{\infty} \alpha_l \frac{(it)^l}{l!}$$

converge absolutely in a disc of complex t -values $|t| < \tau$, $\tau > 0$.

Then the α_k determine the distribution function $F(u)$ uniquely. Moreover, the characteristic function $\phi(t)$ of this distribution had the above representation in the disc $|t| < \tau$, and can be analytically continued into the strip $|\operatorname{Im}(t)| < \tau$.

Remark. The proof of Lemma 3 can be found in [3], while the proof of Lemma 4 is given in [7], (Vol. I, page 60).

Remark. The characteristic function $\varphi(t) = e^{-t^2/2}$ of the standard normal distribution can be written as

$$\varphi(t) = \sum_{l=0}^{\infty} \frac{\mu_{2l}(it)^{2l}}{2^l l!}, \quad \mu_{2l} = \frac{(2l)!}{2^l \cdot l!}$$

($l = 0, 1, 2, \dots$). The expansion is absolute convergent on the whole complex plane.

Lemma 5 (Newton–Girard formulas). *Let $\mathcal{B}$ be a finite set of primes, $M = \#\mathcal{B}$, $\psi : \mathcal{B} \rightarrow \mathbb{R}$,*

$$E_l = (-1)^l \sum_{\substack{p_1 < \dots < p_l \\ p_\nu \in \mathcal{B}}} \psi(p_1) \dots \psi(p_l), \quad s_h = \sum_{p \in \mathcal{B}} \psi^h(p).$$

Then

$$E_1 + s_1 = 0$$

$$2E_2 + E_1 s_1 + s_2 = 0$$

$$\vdots$$

$$rE_r + E_{r-1}s_1 + \dots + E_1 s_{r-1} + s_r = 0 \quad (r = 1, 2, \dots, M).$$

We shall use some of the results from the book of TENENBAUM [4] (Part II., Chapter II. 6).

Let

$$\nu(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} (1 - 1/p)^z$$

be defined in $|z| < 2$. Since $\nu(z)$ is analytic in the open set $|z| < 2$, therefore

$$\nu(z) = \sum_{m=0}^{\infty} \frac{\nu^{(m)}(0)}{m!} z^m, \quad \left| \frac{\nu^{(m)}(0)}{m!} \right| \leq \frac{c}{(2 - \delta/2)^m}$$

with any $\delta > 0$ and a suitable constant $c = c(\delta)$.

Let

$$b_m := \frac{\nu^{(m)}(0)}{m!}$$

$$Q_{0,k}(y) = \sum_{l=0}^{k-1} \frac{1}{l!} b_{k-1-l} y^l.$$

For some polynomial $P(x) \in \mathbb{R}[x]$, $P(x) = \sum u_l x^l$, let $\|P\|(x) = \sum |u_l| x^l$.

We have

$$Q_{0,k}(y + \lambda) - Q_{0,k}(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} Q_{0,k}^{(\mu)}(y) \cdot \lambda^\mu,$$

and so

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} \sum_{k-1 \geq l \geq \mu} \frac{1}{(l-\mu)!} |b_{k-1-l}| y^{l-\mu}$$

$$= \sum_{t=0}^{k-2} \frac{1}{t!} \left(\sum_{l=t+1}^{k-1} \frac{1}{(l-t)!} |b_{k-1-l}| \right) y^t = \sum_{t=0}^{k-2} d_t \cdot y^t.$$

It is clear that $d_t \leq \frac{c}{t!}$ with a suitable constant c .

We formulate the above assertion as

Lemma 6. *We have*

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{t=0}^{k-2} d_t y^t, \quad d_t < \frac{c}{t!}$$

with a suitable constant c .

Let

$$N_k^*(x) = \frac{x}{x_1} Q_{0,k}(x_2).$$

Lemma 7. *Let δ satisfy $0 < \delta < 1$. Then, for $x \geq 3$, $1 \leq k \leq (2 - \delta)x_2$*

$$N_k(x) = N_k^*(x) + O_\delta \left(\frac{x_2}{k} N_k^*(x) \cdot \frac{1}{x_1} \right).$$

(See TENENBAUM [4] Theorem 5 in p. 205.)

Let $1 \leq D \leq x^{\varepsilon_x}$, where $0 < \varepsilon_x < 0, 1$. Let $\eta_D := \frac{\log D}{x_1}$, $\Theta_D := \log(1 - \eta_D)$,

$$\psi_{k,D}(y) := \frac{1}{1 - \eta_D} \left\{ 1 + \Theta_D \cdot \frac{Q'_{0,k}(y)}{Q_{0,k}(y)} + \dots + \Theta_D^{k-1} \frac{Q_{0,k}^{(k-1)}(y)}{Q_{0,k}(y)} \right\}. \quad (3.1)$$

After easy computation we have

$$N_k^* \left(\frac{x}{D} \right) = \frac{N_k^*(x)}{D} \psi_{k,D}(x_2). \quad (3.2)$$

§4. Proof of Theorem 3

Assume that the conditions of Theorem 3 are satisfied.

Let h be completely additive, $J_x = [K_x, x^{\varepsilon_x}]$,

$$h(p) = \begin{cases} \frac{f^*(p)}{B_x} & \text{if } p \in J_x \\ 0 & \text{if } p \notin J_x, \end{cases}$$

where $\varepsilon_x \downarrow 0$, $K_x \uparrow \infty$ so slowly that

$$\lim_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k, n \leq x, \left| h(n) - \frac{f^*(n)}{B_x} \right| > \varepsilon \right\} = 0 \quad (4.1)$$

for each $\varepsilon > 0$.

$$\lim_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \{n \in \mathcal{N}_k \mid n \leq x, \exists p > K_x, p^2 \mid n\} = 0. \quad (4.2)$$

Let $p < K_x$, count those $n \in \mathcal{N}_k$ for which $p^\alpha \mid n$. The size of those n is no more than

$$N_{k-\alpha} \left(\frac{x}{p^\alpha} \right) < \frac{cx}{p^\alpha x_1} \frac{x_2^{k-\alpha-1}}{(k-1-\alpha)!} \leq \frac{c_1(2-\delta/2)^{\alpha+1}}{p^\alpha} N_k(x), \quad (4.3)$$

assuming e.g. that $p^\alpha \leq x_1$. Hence we obtain that

$$\sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k, \left| \sum_{\substack{p^\alpha \parallel n \\ p < K_x}} \frac{f^*(p^\alpha)}{B_x} \right| > \varepsilon \right\} \rightarrow 0 \quad (x \rightarrow \infty)$$

if $K_x \uparrow \infty$ sufficiently slowly. Since the number of prime divisors p in $(x^{\varepsilon_x}, x]$ of n is less than $\frac{1}{\varepsilon_x}$ therefore (4.1) clearly holds.

(4.2) can be proved easily. We use (4.3) if $K_x \leq p \leq x_1$ with $\alpha = 2$, and for $p > x_1$ we use the obvious

$$\# \{n \in \mathcal{N}_k, n \leq x \mid \exists p^2 \mid n, p > x_1\} \leq \sum_{p > x_1} \frac{x}{p^2} \leq \frac{x}{x_1}$$

inequality.

Thus (4.2) is true.

We have

$$\frac{1}{B_x^2} \sum_{p < K_x} \frac{f^{*2}(p)}{p} \ll \frac{\log \log K_x}{B_x^2}, \quad \frac{1}{B_x^2} \sum_{x^{\varepsilon_x} < p < x} \frac{f^{*2}(p)}{p} \ll \frac{\log 1/\varepsilon_x}{B_x^2},$$

and so

$$\sum \frac{h^2(p)}{p} = 1 + H_x, \quad |H_x| \ll \frac{\log \log K_x + \log 1/\varepsilon_x}{B_x^2}. \quad (4.4)$$

Assuming that K_x and $1/\varepsilon_x$ are increasing sufficiently slowly, we can and will assume that $H_x \rightarrow 0$ ($x \rightarrow \infty$).

To prove the theorem it is enough to show that for every $r = 1, 2, \dots$,

$$\sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} \frac{h^r(n)}{\beta_k^r} - \mu_r \right| \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

and then apply the Frechet-Shohat theorem.

Let us consider the sum

$$U_{k,r}(x) := \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} h^r(n). \quad (4.5)$$

Since h is completely additive, therefore

$$U_{k,r}(x) = \sum_{s=1}^r \sum_{l_1+\dots+l_s=r} \frac{c(r; l_1, \dots, l_s)}{N_k(x)} \sum_{p_1, p_2, \dots, p_s}^* h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s} \left(\frac{x}{p_1 \dots p_s} \right) \quad (4.6)$$

where star indicates that we sum over all those s tuples $p_1, \dots, p_s$ of primes for which $p_i \neq p_j$, if $i \neq j$. Here $c(r; l_1, \dots, l_s) = \frac{r!}{l_1! \dots l_s!}$.

Let

$$V_{k,r}(x \mid l_1, \dots, l_s) = \frac{1}{N_{k-s}^*(x)} \sum_{p_1, \dots, p_s}^* h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s}^* \left(\frac{x}{p_1 \dots p_s} \right), \quad (4.7)$$

$$\tilde{U}_{k,r}(x) = \sum_{s=1}^r \frac{c(r; l_1, \dots, l_s) \cdot N_{k-s}^*(x)}{N_k(x)} V_{k,r}(x \mid l_1, \dots, l_s). \quad (4.8)$$

From Lemma 7 we can deduce simply that $U_{k,r}(x) - \tilde{U}_{k,r}(x) \rightarrow 0$ ($x \rightarrow \infty$) uniformly as $\frac{k}{x} \in [\delta, 2-\delta]$. We estimate $V_{k,r}(x \mid l_1, \dots, l_s)$ by using (3.1), (3.2) with $D = p_1 \dots p_s$. We can write $\psi_{k,D}(y)$ as a convergent power series of η_D .

We try to estimate

$$E(l_1, t_1; l_2, t_2; \dots; l_s, t_s) := \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1)(\log p_1)^{t_1}}{p_1 x_1^{t_1}} \dots \frac{h^{l_s}(p_s) \cdot (\log p_s)^{t_s}}{p_s x_1^{t_s}}. \quad (4.9)$$

Let

$$\kappa(l, t) := \frac{1}{x_1^t} \sum_{p \in J_x} \frac{h^l(p)(\log p)^t}{p} \quad (l = 1, 2, \dots; t = 0, 1, \dots). \quad (4.10)$$

From (4.4) we have

$$\kappa(2, 0) = 1 + H_x, \quad |H_x| < \frac{c \log \log K_x + \log 1/\varepsilon_x}{B_x^2}. \quad (4.11)$$

We have

$$\begin{aligned} \kappa(1, 0) &= \frac{1}{B_x} \sum_{p \in J_x} \frac{1}{p} \left(f(p) - \frac{A_x}{x_2} \right) = \frac{1}{B_x} \sum_{p < x_2} \frac{1}{p} \left(f(p) - \frac{A_x}{x_2} \right) \\ &\quad - \frac{1}{B_x} \sum_{p < B_x} \frac{1}{p} \left(f(p) - \frac{A_x}{x_2} \right) - \frac{1}{B_x} \sum_{x^{\varepsilon_x} \leq p < x} \frac{1}{p} \left(f(p) - \frac{A_x}{x_2} \right) \\ &= \sum_1 - \sum_x - \sum_3. \end{aligned}$$

Since $x_2 - \sum_{p < x_2} 1/p = O(1)$, therefore

$$\sum_1 = \frac{1}{B_x} \left(A_x - \frac{A_x}{x_2} \sum_{p < x_2} 1/p \right) = \frac{A_x}{x_2 B_x} \left(x_2 - \sum_{p < x_2} 1/p \right) = O\left(\frac{1}{B_x}\right).$$

Furthermore

$$\sum_1 = O\left(\frac{\log \log K_x}{B_x}\right), \quad \sum_2 = O\left(\frac{\log 1/\varepsilon_x}{B_x}\right).$$

Consequently

$$|\kappa(1, 0)| \leq \frac{c(\log \log K_x + \log 1/\varepsilon_x)}{B_x}$$

with a suitable constant c .

It is known that

$$\sum_{p < y} \frac{(\log p)^s}{p} < c \frac{(\log y)^s}{s}$$

for $s \geq 1$.

Let Λ_x be defined by

$$\Lambda_x := \frac{c(\log \log K_x + \log 1/\varepsilon_x)}{B_x} + \frac{r}{K_x} \geq |\kappa(1, 0)| + \frac{r}{K_x}. \quad (4.12)$$

It is known that

$$\sum_{p < y} \frac{(\log p)^s}{p} < c \frac{(\log y)^s}{s},$$

for $s \geq 1$.

Hence, by using the Cauchy–Schwarz inequality,

$$\kappa(1, t) \leq \left(\sum_{p \in J_x} \frac{h^2(p)}{p} \right)^{1/2} \left(\frac{1}{x_1^{2t}} \sum_{p \leq x^{\varepsilon_x}} \frac{(\log p)^{2t}}{p} \right) \leq \frac{c\varepsilon_x^t}{\sqrt{t}}. \quad (4.13)$$

For $l \geq 2$, $t \geq 1$

$$|\kappa(l, t)| \leq c\varepsilon_x^t \kappa(l, 0), \quad (4.14)$$

$$|\kappa(l, 0)| \leq c \left(\frac{1}{B_x} \right)^{l-2}. \quad (4.15)$$

Assume first that there exists at least one $(l_j, t_j) = (1, 0)$. Assume that $(l_j, t_j) = (1, 0)$ if $j = 1, \dots, h$ and $(l_j, t_j) \neq (1, 0)$ if $j > h$. We have

$$\begin{aligned} & E(l_1, t_1; \dots; l_s, t_s) \\ &= \sum_{p_{h+1}, \dots, p_s}^* \left(\prod_{\nu=h+1}^s \frac{h^{l_\nu}(p_\nu)}{p_\nu} \cdot \frac{(\log p_\nu)^{t_\nu}}{x_1^{t_\nu}} \right) \left\{ \sum_{p_1, \dots, p_h}^{**} \frac{h(p_1)}{p_1} \dots \frac{h(p_h)}{p_h} \right\} \end{aligned}$$

where $*$ means that $p_{h+1}, \dots, p_s$ are distinct primes, and $**$ means that $p_1, \dots, p_h$ are distinct primes, none of them belongs to the set $\{p_{h+1}, \dots, p_s\}$. First we estimate the inner sum. Let us apply Lemma 5 with $\mathcal{B} = \{p \mid p < x^{\varepsilon_x}\} \setminus \{p_{h+1}, \dots, p_s\}$, $\psi(p) = \frac{h(p)}{p}$. In the notation of Lemma 5 $\sum_{p_1, \dots, p_h}^{**} \frac{h(p_1)}{p_1} \dots \frac{h(p_h)}{p_h} = (-1)^h h! E_h$. Since $|E_1| = \left| \sum_{p \in \mathcal{B}} \frac{h(p)}{p} \right| \leq \Lambda_x$ (see (4.12)), from the Newton–Girard formulas (by using induction on h e.g.) we obtain that $|E_h| \leq c\Lambda_x$, where c is a constant that may depend on r at most.

Thus

$$E(l_1, t_1; \dots; l_s, t_s) \leq c\Lambda_x \kappa(l_{h+1}, t_{h+1}) \dots \kappa(l_s, t_s).$$

By the inequalities (4.13), (4.14), (4.15) we have

$$E(l_1, t_1; \dots; l_s, t_s) \leq c_1 \Lambda_x \varepsilon_x^{t_1 + \dots + t_s} \prod_{l_j \geq 2} \left(\frac{1}{B_x} \right)^{l_j - 2}. \quad (4.16)$$

c_1 is a constant which may depend on r .

Similarly, if $(l_j, t_j) \neq (1, 0)$ holds for every j , then

$$E(l_1, t_1, \dots, l_s, t_s) \leq c_1 \varepsilon_x^{t_1 + \dots + t_s} \prod_{l_j \geq 2} \left(\frac{1}{B_x} \right)^{l_j - 2}. \quad (4.17)$$

We can observe that the right hand side of (4.16), (4.17) tends to zero except the case, when for every j , $(l_j, t_j) = (2, 0)$. This can be happen only if $r = 2R$ is even. Observe that

$$E(2, 0; \dots, 2, 0) = \sum_{p_1, \dots, p_R}^* \frac{h^2(p_1)}{p_1} \dots \frac{h^2(p_R)}{p_R},$$

and hence we can deduce easily that

$$E(2, 0; \dots; 2, 0) = \kappa(2, 0)^R + o_x(1) = 1 + o_x(1). \quad (4.18)$$

Let us go back to (4.7). See furthermore (3.1):

$$V_{k,r}(x \mid l_1, \dots, l_s) = \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} T_{k-s}(\eta_{p_1 \dots p_s}),$$

where

$$\begin{aligned} T_{k-s}(W) &= \frac{1}{1-W} \left\{ 1 + \log(1-W) \cdot S_1 + \log^2(1-W) S_2 + \dots \right. \\ &\quad \left. + \log^{k-s-1}(1-W) \cdot S_{k-s-1} \right\} \\ S_j &:= \frac{Q_{0,k-s-1}^{(j)}(x_2)}{Q_{0,k-s-1}(x_2)}. \end{aligned}$$

Let

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) = \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} \left(\frac{\log p_1 \dots p_s}{x_1} \right)^T.$$

Then

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) = \sum_{t_1 + \dots + t_s = T} \frac{T!}{t_1! \dots t_s!} E(l_1, t_1; \dots, l_s, t_s).$$

In the case $T = 0$ it was already proved that $V_{k,r}^{(0)}(x \mid l_1, \dots, l_s) = o_x(1)$, except the case when $l_1 = l_2 = \dots = l_s = 2$, $s = R$, $r = 2R$, when $V_{k,2R}^{(0)}(x \mid 2, \dots, 2) = 1 + o_x(1)$.

Let now $T \geq 1$. From (4.16), (4.17) we obtain that

$$V_{k,r}^{(T)}(x \mid l_1, \dots, l_s) \leq c \varepsilon_x^T. \quad (4.19)$$

Let $u(w) = p_0 + p_1 w + \dots$ be a power series with nonnegative coefficients, and assume that it converges in the disc $|w| < 1$.

Since

$$\sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} u\left(\frac{\log p_1 \dots p_s}{x_1}\right) = \sum_{T=0}^{\infty} p_T V_{k,r}^{(T)}(x \mid l_1, \dots, l_s),$$

from (4.19) we obtain that the left hand side of (4.20) is less than

$$\leq \sum_{T=0}^{\infty} p_T c \varepsilon_x^T = cu(\varepsilon_x).$$

Since the coefficients of the Taylor expansion of $\frac{w}{1-w}$ and of $(-1)^j \frac{(\log(1-w))^j}{1-w}$ is positive, and they converge for $|w| < 1$, therefore

$$\left| \sum_{p_1, \dots, p_s}^* \frac{h^{l_1}(p_1) \dots h^{l_s}(p_s)}{p_1 \dots p_s} u\left(\frac{\log p_1 \dots p_s}{x_1}\right) \right| \leq cu(\varepsilon_x)$$

holds, if

$$u(w) = \frac{(-1)^j \log^j(1-w)}{1-w}, \quad j = 1, \dots, k-s-1,$$

and if

$$u(w) = \frac{w}{1-w},$$

$l_1, \dots, l_s$ arbitrary, and in the case $u(w) = 1$, $(l_1, \dots, l_s) \neq (2, \dots, 2)$ the left hand side tends to 0.

Consequently, by Lemma 6 $V_{k,r}(x \mid l_1, \dots, l_s) \rightarrow 0$ ($x \rightarrow \infty$) if $(l_1, \dots, l_s) \neq (2, \dots, 2)$, while for $s = R$, $r = 2R$,

$$V_{k,2R}(x \mid 2, \dots, 2) = 1 + o_x(1).$$

We are almost ready. We have to observe only that

$$\frac{N_{k-s}^*(x_2)}{N_k^*(x_2)} = \frac{k(k-1) \dots k-(s-1)}{x_2^s} = (1 + o_x(1)) \xi_{k,x_2}^s.$$

The proof is complete.

§5. Proof of Theorem 4

Theorem 10. *Let $0 < \delta$, $A > 2 + \delta$ be constants. Then for all $k \in [(2 + \delta)x_2, Ax_2]$ we have*

$$N_k(x) = \frac{cx_1}{2^k} \left\{ 1 + O_A(x_1^{-\delta^2/5}) \right\}.$$

See [4].

To prove the theorem we can use the argument of the proof of Theorem 5. Instead of (3.1), (3.2) we can use the formula

$$N_k^*(x) = \frac{cx_1}{2^k}, \quad N_k^*\left(\frac{x}{D}\right) = \frac{1}{D} N_k^*(x) \left(1 - \frac{\log D}{x_1}\right).$$

We omit the details.

§6. Proof of Theorem 8

If (1.29) holds for $f_1, f_2 \in \mathcal{A}$, then it holds for $f = f_1 + f_2$. Let $\gamma < 1/4$ be a small positive constant, $f_1(p^\alpha) = f(p^\alpha)$ if $p^\alpha < x^\gamma$, and $f_1(p^\alpha) = 0$ if $p^\alpha \geq x^\gamma$, and let $f_2(p^\alpha) = f(p^\alpha) - f_1(p^\alpha)$.

We have

$$\begin{aligned} S &:= \sum_{\substack{n \in N_k \\ n \leq x}} f_2^2(n) \leq \sum_{p_1 \neq p_2} |f_2(p_1^{\alpha_1})| \cdot |f_2(p_2^{\alpha_2})| \cdot N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \\ &\quad + \sum_{p^\alpha} |f_2(p^\alpha)| \cdot N_{k-\alpha}\left(\frac{x}{p^\alpha}\right). \end{aligned}$$

From the conditions of the theorem $f_2(p_i^{\alpha_i}) = f(p_i^{\alpha_i}) = 0$ if $p_i^{\alpha_i} > x^{1/4}$, or if $\alpha_i > \sqrt{x_2}$.

Assume that $p_i^{\alpha_i} \leq x^{1/4}$ and $\alpha_i \leq \sqrt{x_2}$. Then

$$N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \leq \frac{cN_k(x)}{p_1^{\alpha_1} p_2^{\alpha_2}} \xi_{k,x}^{\alpha_1+\alpha_2}, \quad (6.1)$$

(c is an absolute constant) and we deduce that

$$\frac{1}{N_k(x)} S \leq c \left(\sum_{x^\gamma < p^\alpha \leq x^{1/4}} \frac{|f_2(p^\alpha)|}{p^\alpha} \xi_{k,x}^\alpha \right)^2 + c \tilde{B}_x^2(\xi_{k,x}).$$

Since

$$\sum \frac{|f_2(p^\alpha)| \xi_{k,x}^\alpha}{p^\alpha} \leq \left(\sum \frac{\xi_{k,x}^\alpha}{p^\alpha} \right)^{1/2} \tilde{B}_x(\xi_{k,x}),$$

where in the right hand side $x^\gamma < p^\alpha < x^{1/4}$, $\alpha \leq \sqrt{x_2}$, thus $p = 2$ cannot occur.

Therefore

$\sum_{p^\alpha, p \geq 2} \frac{\xi_{k,x}^\alpha}{p^\alpha}$ is bounded by an absolute constant, and so

$$\frac{1}{N_k(x)} S \leq c \tilde{B}_x^2(\xi_{k,x}).$$

Let

$$U_x = \sum \frac{f_2(p)}{p} = \sum_{x^\gamma \leq p < x^{1/4}} \frac{f(p)}{p}.$$

Since

$$\frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f_2(n) - U_x)^2 \leq \frac{2}{N_k(x)} S + 2|U_x|^2,$$

and

$$|U_x|^2 \leq \left\{ \sum_{x^\gamma < p < x^{1/4}} \frac{1}{p} \right\} \tilde{B}_x^2\left(\frac{k}{x_2}\right),$$

therefore (1.29) holds for f_2 .

Let now f_3 be defined on prime powers p^β such that $f_3(p^\beta) = f_1(p^\alpha) - f_1(p^{\alpha-1})$ ($\alpha = 1, 2, \dots$). Then, with the classical meaning of summation,

$$f_1(n) = \sum_{p^\beta | n} f_3(p^\beta).$$

Let

$$f_4(n) = \sum_{p|n} f_3(p), \quad f_5(n) = \sum_{\substack{p^\beta | n \\ \beta \geq 2}} f_3(p^\beta).$$

Let us estimate first

$$\begin{aligned} S_1 &:= \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} f_5(n)^2 = \sum_{\substack{p_1^{\alpha_1}, p_2^{\alpha_2} \\ p_1 \neq p_2}} f_3(p_1^{\alpha_1}) f_3(p_2^{\alpha_2}) N_{k-\alpha_1-\alpha_2} \left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}} \right) \\ &+ \sum_{\substack{p \\ \alpha_1, \alpha_2}} f_3(p^{\alpha_1}) f_3(p^{\alpha_2}) N_{k-\max(\alpha_1, \alpha_2)} \left(\frac{x}{p^{\max(\alpha_1, \alpha_2)}} \right) = S_2 + S_3. \end{aligned}$$

Since $f_3(p_i^{\alpha_i}) = 0$, if $p_k^{\alpha_i} > x^\gamma$, or if $\alpha_i = 1$, or $\alpha_i > \sqrt{x_2}$, from (6.1) we deduce that

$$\frac{S_2}{N_k(x)} \leq \left(\sum_{\alpha \geq 2} \frac{|f_3(p^\alpha)|}{p^\alpha} \xi_{k,x} \right)^2 + 2 \sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=2}^{\alpha_1} \sum_p \frac{|f_3(p^{\alpha_1}) f_3(p^{\alpha_2})|}{p^{\alpha_1}} \xi_{k,x}^{\alpha_1}.$$

The first sum on the right hand side is less than $c\tilde{B}_x^2(\xi_{k,x})$. To estimate the second sum we start from

$$\left| \xi_{k,x}^{\alpha_1} f_3(p^{\alpha_1}) f_3(p^{\alpha_2}) \right| \leq 2f_3^2(p_1^\alpha) \xi_{k,x}^{2\alpha_1} + f_3^2(p_2^\alpha)$$

and deduce that it is less than

$$4\tilde{B}_x(\xi_{k,x}) + 4 \sum_{\alpha_2=2}^{\infty} \sum_p \frac{|f_3(p_2^\alpha)|^2}{p^{\alpha_2}} \sum \frac{1}{1-1/p} \leq 4\tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Thus

$$\frac{S_2}{N_k(x)} \leq c_1 \tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Finally we prove that

$$T := \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} (f_4(n) - \xi_{k,x_2} A_x^*)^2 \leq c B_x^2 N_k(x),$$

$$A_x^* = \sum_{p < x^\gamma} \frac{f_4(p)}{p}. \quad (6.2)$$

Let $\rho_x := \sum_{p < x^\gamma} 1/p$.

Let $\tilde{f}_4(p) = f_4(p) - \frac{A_x^*}{\rho_x}$, $\tilde{f}_4(n) = \sum_{p|n} \tilde{f}_4(p)$. Then $\tilde{f}_4(n) = f_4(n) - \frac{k}{\rho_x} A_x^*$ if $n \in \mathcal{N}_k$.

Let

$$\tilde{T} = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} \tilde{f}_4(n)^2. \quad (6.3)$$

We shall prove that

$$\tilde{T} \leq c N_k(x) \sum_{p \leq x} \frac{\tilde{f}_4^2(p)}{p}. \quad (6.4)$$

Hence (6.2) easily follows.

We have

$$\tilde{T} = \sum_{p_1 \neq p_2} \tilde{f}_4(p_1) \tilde{f}_4(p_2) N_{k-2} \left(\frac{x}{p_1 p_2} \right) + \sum_p \tilde{f}_4^2(p) N_{k-1} \left(\frac{x}{p} \right).$$

From Lemma 7 we obtain that

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 + \text{error, where}$$

$$\begin{aligned}\tilde{T}_1 &= \sum_{p_1, p_2} \tilde{f}_4(p_1) \tilde{f}_4(p_2) N_{k-2}^* \left(\frac{x}{p_1 p_2} \right), \\ \tilde{T}_2 &= \sum \tilde{f}_4^2(p) \left(N_{k-1} \left(\frac{x}{p} \right) - N_{k-2} \left(\frac{x}{p_2} \right) \right),\end{aligned}$$

where the error is clearly less than $c N_{k-2}(x) \sum \frac{\tilde{f}_4^2(p)}{p}$.

Let

$$E_l := \sum_{p < x^\gamma} \frac{\tilde{f}_4(p)}{p} \frac{(\log p)^l}{x_1^l}.$$

It is clear that $E_0 = 0$, and

$$|E_l| \leq \left(\sum \frac{\tilde{f}_4^2(p)}{p} \right)^{1/2} \left(\sum_{p < x^\gamma} \frac{(\log p)^{2l}}{p x_1^{2l}} \right)^{1/2} \leq 2 \left(\sum \frac{\tilde{f}_4^2(p)}{p} \right)^{1/2} \frac{\gamma^l}{\sqrt{2l}},$$

if $x > x_0$, and $l \geq 0$. Thus

$$\left| \sum_{p_1, p_2} \frac{\tilde{f}_4(p_1)}{p_1} \frac{\tilde{f}_4(p_2)}{p_2} \frac{(\log p_1 p_2)^\nu}{x_1^\nu} \right| = \left| \sum_{l=0}^{\nu-1} E_l E_{\nu-l} \cdot \binom{\nu}{l} \right| \leq 4(2\gamma)^l. \quad (6.5)$$

We have

$$\frac{\tilde{T}_1}{N_{k-2}(x)} = \sum_{p_1, p_2} \frac{\tilde{f}_4(p_1)}{p_1} \frac{\tilde{f}_4(p_2)}{p_2} \psi_{k-2, p_1 p_2}(x_2),$$

where $\psi_{k-2, p_1 p_2}$ is defined in (3.1). By using Lemma 6 and (6.5), furthermore that $\tilde{T}_2 \ll \xi_{k,x} \sum \frac{\tilde{f}_4^2(p)}{p} N_k(x)$, we get (6.4).

Since $\tilde{f}_4^2(p) \leq 2f_4^2(p) + 2\frac{|A_x^*|^2}{\rho_x^2}$, therefore

$$\sum \frac{\tilde{f}_4^2(p)}{p} \leq 2B_x^2 + 2\frac{|A_x^*|^2}{\rho_x}, \quad A_x^{*2} \leq \sum \frac{1}{p} B_x^2,$$

and so

$$\sum \frac{\tilde{f}_4^2(p)}{p} \leq cB_x^2.$$

Finally $f_4(n) - \xi_{k,x_2} A_x^* = \tilde{f}_4(n) + \left(\frac{k}{\rho_x} - \xi_{k,x_2} \right) A_x^*$, and so

$$T \leq 2\tilde{T} + \left| \frac{k}{\rho_x} - \xi_{k,x_2} \right|^2 |A_x^*|^2 N_k(x).$$

Furthermore $|A_x^*|^2 \leq B_X^2 \rho_x$, and so

$$\left| \frac{k}{\rho_x} - \frac{k}{x_2} \right|^2 |A_x^*|^2 \leq \rho_x \left| \frac{k(x_2 - \rho_x)}{\rho_x x_2} \right|^1 B_x^2 = o_x(1) B_x^2.$$

Thus (6.2) holds true.

The proof of the theorem is complete.

§7. Proof of Theorem 9

We can argue similarly as in §6. Since now $N_k(x) = N_k^*(x)(1 + O_A(x_1^{-\delta^2/5}))$ (Lemma 8), $N_k^*\left(\frac{x}{D}\right) = \frac{1}{N} N_k^*(x) - \frac{\log D}{D x_1} N_k^*(x)$, we obtain our theorem easier than that of Theorem 10.

We omit the details.

§8. Proof of Theorem 5

Assume that the conditions of the theorem hold. Let $\mathcal{B}$ be such a sequence of primes for which $\sum_{p \in \mathcal{B}} 1/p < \infty$. Let $\rho(Y) := \sum_{\substack{bp > Y \\ p \in \mathcal{B}}} 1/p$. Then $\rho(Y) \rightarrow 0$ as $Y \rightarrow \infty$.

Count

$$S_Y := \#\{n \leq x \mid n \in \mathcal{N}_k, p \mid n \text{ for some } p > Y, p \in \mathcal{B}\}.$$

Then

$$\begin{aligned} S_Y &\leq \sum_{\substack{Y < p < x^{1-\delta_x} \\ p \in \mathcal{B}}} N_{k-1}\left(\frac{x}{p}\right) + \sum_{\substack{\nu < x^{\delta_x} \\ \nu \in \mathcal{N}_{k-1}}} \pi\left(\frac{x}{\nu}\right) \leq \\ &\leq N_{k-1}(x) \sum_{\substack{Y < p < x^{1-\delta_x} \\ p \in \mathcal{B}}} \frac{1}{p} \frac{\log x}{(\log x - \log p)} + \frac{3x}{x_1} \sum_{\substack{\nu < x^{\delta_x} \\ \nu \in \mathcal{N}_{k-1}}} 1/\nu \\ &\leq N_{k-1}(x) \cdot \frac{1}{\delta_x} \rho(Y) + \frac{3x}{x_1} \cdot \frac{1}{(k-1)!} \left(\sum_{p < x^{\delta_x}} 1/p \right)^{k-1}, \end{aligned}$$

whence

$$\frac{S_Y}{N_k(x)} \leq \frac{k}{x_2} \cdot \frac{1}{\delta_x} \rho(Y) + 3 \left(\frac{x_2 - \log 1/\delta_x}{x_2} \right)^{k-1} \leq \frac{k}{x_2} \frac{1}{\delta_x} \rho(Y) + 3e^{-\frac{(k-1)}{x_2} \log \frac{1}{\delta_x}}.$$

The second sum is small if δ_x is small, the first sum is small if $\frac{\rho(Y)}{\delta_x}$ is small, i.e. if Y is large.

Thus, by choosing $\delta_x = \sqrt{\rho(Y)}$ for example, we obtain that

$$\frac{S_Y}{N_k(x)} = o_Y(1).$$

From the convergence of the three series it is obvious that there is a sequence $\rho_p \downarrow 0$ such that for the set $\mathcal{B}_1 = \{p \mid |f(p)| > \rho_p\}$, $\sum_{p \in \mathcal{B}_1} 1/p < \infty$. Let $\mathcal{B}_1$ be fixed. Let $\mathcal{B}_2 = \{p^\alpha \mid p \in \mathcal{P}, \alpha \geq 2\}$, and let

$$S_Y^* := \#\{n \leq x \mid n \in \mathcal{N}_k, p^\alpha \mid n \text{ for some } p^\alpha \in \mathcal{B}_2, p^\alpha > Y\}.$$

This is clear:

$$S_Y \leq \sum_{\substack{Y < p^\alpha \leq \sqrt{x} \\ p^\alpha \in \mathcal{B}_2}} N_{k-\alpha} \left(\frac{x}{p^\alpha} \right) + \sum_{x \geq p^\alpha \geq \sqrt{x}} \frac{x}{p^\alpha} \leq c N_k(x) \sum_{p^\alpha \geq Y} \left(\frac{k}{x_2} \right)^\alpha \frac{1}{p^\alpha} + cx^{3/4},$$

and so

$$\frac{S_Y^*}{N_k(x)} \leq c \sum_{2^\alpha \geq Y} \left(\frac{k}{x_2 \cdot 2} \right)^\alpha + \frac{1}{Y^{1/10}} \sum_{\substack{\alpha \geq 2 \\ p \geq 3}} \left(\frac{k}{x_2 \cdot p^{9/10}} \right)^\alpha + cx^{3/4}.$$

The first sum on the right hand side is $\ll \frac{Y^{-\delta/2 \log 2}}{1 - \frac{k}{2x_2}}$, the second sum after $\frac{1}{Y^{1/10}}$ is bounded by an absolute constant.

Thus

$$\limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{S_Y^*}{N_k(x)} \leq \varepsilon(Y),$$

where $\varepsilon(Y) \rightarrow 0$ as $Y \rightarrow \infty$.

Let $Y = Y_x$ be tending to infinity slowly. For some $n \leq x$ let $n = A(n) \cdot B(n)$, where $A(n) = \prod_{\substack{p^\alpha \parallel n \\ p < Y}} p^\alpha$, and $B(n) = \frac{n}{A(n)}$. Consider the set of integers $n \in \mathcal{N}_k$ up to x . Let us drop those n for which $p \mid n$ for some $p \in \mathcal{B}_1$, $p > Y$ and those for which $p^\alpha \mid n$ for some $p^\alpha \in \mathcal{B}_2$, $p^\alpha > Y$. The number of the dropped elements is $\ll \varepsilon_1(Y) N_k(x)$, where $\varepsilon_1(Y) \rightarrow 0$ uniformly as $\frac{k}{x_2} \in [\delta, 2-\delta]$. Let $f^* \in \mathcal{A}$ defined on prime powers p^α as follows:

$$f^*(p^\alpha) = \begin{cases} 0 & \text{if } \alpha \geq 2 \\ 0 & \text{if } \alpha = 1, p \leq Y, \text{ if } p \geq \sqrt{x}, \text{ or if } p \in \mathcal{B}_1 \\ f(p) & \text{if } \alpha = 1, p \in (Y, \sqrt{x}]. \end{cases}$$

From Theorem 8 we have

$$\frac{1}{N_k(x)} \sum \left(f^*(n) - \xi_{k,x} \sum \frac{f^*(p)}{p} \right)^2 \leq c \sum_{Y \leq p \leq \sqrt{x}} \frac{f^{*2}(p)}{p}. \quad (8.1)$$

$$\limsup_x \sup_{\xi_{k,x} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, |f^*(n)| \geq \lambda\} \leq \varepsilon_2(Y), \quad \varepsilon_2(Y) \rightarrow 0 \quad (8.2)$$

valid for every $\lambda > 0$.

Let $\mathcal{M}_Y$ be the set of those m , the largest prime power factor of which is not larger than Y , and if $p^\alpha \parallel m$, $\alpha \geq 2$, then $p^\alpha \leq Y$. From the estimation of S_Y^* we obtain that

$$\frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, A(n) \notin \mathcal{M}_Y\} \\ \limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, A(n) \notin \mathcal{M}_Y\} \leq \varepsilon_3(Y),$$

where $\varepsilon_3(Y) \rightarrow 0$ as $Y \rightarrow \infty$.

Let $\mathcal{D}_{m,k} := \{n \in \mathcal{N}_k, A(n) = m\}$ ($m \in \mathcal{M}_Y$), and let $h(n) := f(A(n))$. Thus $h(n)$ is constant on $D_{m,k}$, and from (8.2) we obtain that

$$\limsup_{x \rightarrow \infty} \sup_{\frac{k}{x_2} \in [\delta, -2\delta]} \frac{1}{N_k(x)} \#\{n \leq x \mid n \in \mathcal{N}_k, |f(n) - f(A(n))| > \lambda\} \leq \varepsilon_2(Y).$$

Now we compute the density of the set $D_{m,k}$.

Let $\mathcal{N}_k(D) = \{n \in \mathcal{N}_k \mid n \in D\}$. Starting from the generating function

$$\prod_{p \nmid D} \frac{1}{1 - \frac{z}{p^s}} = \prod_{p \mid D} \left(1 - \frac{z}{p^s}\right) \cdot \sum \frac{z^{\Omega(n)}}{n^s},$$

for $N_k(x|D) = \sum_{\substack{n \leq x \\ (n,D)=1 \\ n \in \mathcal{N}_k}} 1$ we have

$$N_k(x, D) = \sum_{d \mid D} \mu(d) N_{k-\Omega(d)} \left(\frac{x}{D}\right).$$

Let $K_Y = \prod_{p \leq Y} p$.

From the convergence of the series in (1.24) we obtain that

$$\sum \frac{f^*(p)}{p} = \sum_{\substack{Y \leq p < \sqrt{x} \\ |f(p)| < 1}} \frac{f(p)}{p} = \sum_{\substack{Y \leq p < \sqrt{x} \\ \rho_p < |f(p)| < 1}} \frac{f(p)}{p}$$

tends to zero as $x \rightarrow \infty$. The right hand side of (8.1) tends to zero as well. Applying these relations, from (8.1) we obtain

Consequently

$$\#(D_m, k) = N_{k-\Omega(m)} \left(\frac{x}{m} \mid K_Y \right) = \sum_{d \mid K_Y} N_{k-\Omega(m)-\Omega(d)} \left(\frac{x}{md} \right) \mu(d),$$

and so

$$\frac{\#(D_m, k)}{N_k(x)} = (1 + o_x(1)) \frac{\xi_{k, x_2}^{\Omega(m)}}{m} \prod_{p \mid K_Y} \left(1 - \frac{\xi_{k, x_2}}{p} \right) \quad (8.3)$$

uniformly as $\frac{k}{x_2} \in [\delta, 1 - \delta]$, $m \in \mathcal{M}_Y$ even if $Y = Y_x \rightarrow \infty$ slowly. Hence the assertion easily follows.

§9. Proof of Theorem 2

This can be carried over by a simple application of Theorem 7 and of (8.3).

Let $f(p^\alpha) = \arg g(p^\alpha) \in [-\pi, \pi]$, f be extended so that $f \in \mathcal{A}$. Then $g(n) = e^{if(n)}$.

From the convergence of $\sum \frac{1-g(p)}{p}$ we obtain that $\sum \frac{f(p)}{p}$, $\sum \frac{f^2(p)}{p}$ are convergent. For some $n \in \mathcal{N}_k$ define $g_Y(n) := g(A(n))$. First we observe that $\frac{1}{N_k(x)} \sum_{n \leq x} |g(n) - g_Y(n)| \leq \varepsilon_1(Y)$, uniformly in $\frac{k}{x_2} \in [\delta, 2 - \delta]$, where $\varepsilon_1(Y) \rightarrow 0$ if $Y \rightarrow \infty$. Furthermore

$$\sum_{m \in \mathcal{M}_Y} g_Y(m) \frac{\#(D_m, k)}{N_k(x)} = (1 + o_x(1)) \sum_{m \in \mathcal{M}_Y} \frac{g(m) \xi_{k, x_2}^{\Omega(m)}}{m} \prod_{p \mid K_Y} \left(1 - \frac{\xi_{k, x_2}}{p} \right).$$

The right hand side clearly tends to $M_{\xi_k, x_2}(g)$ defined in Theorem 4.

Since

$$\limsup_x \sup_k \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k \\ A(n) \notin \mathcal{M}_Y}} |g(n)| \rightarrow 0 \text{ as } Y \rightarrow \infty,$$

our theorem immediately follows.

§10. Proof of Theorem 6 and 7

The proof is completely analogous to that of Theorem 2 and 5. So we omit it.

ACKNOWLEDGEMENT. The first author would like to thank both referees for valuable comments and suggestions.

References

- [1] H. DELANGE, Sur les fonctions q -additives ou q -multiplicatives, *Acta Arithmetica* **21** (1972), 285–298.
- [2] J. KUBILIUS, Probabilistic Methods in the Theory of Numbers, Amer. Math. Soc. Translations of Math. Monographs, No. 11, *Providence*, 1964.
- [3] M. FRECHET and J. SHOHAT, A proof of the generalized central limit theorem, *Trans. Amer. Math. Soc.* **33** (1931), 533–543.
- [4] G. TENENBAUM, Introduction to Analytic and Probabilistic Number Theory, *Cambridge University Press*, 1995.
- [5] I. KÁTAI and M. V. SUBBARAO, Distribution of additive and q -additive functions under some conditions, *Publ. Math. Debrecen* **64** (2004), 167–187.
- [6] I. KÁTAI and M. V. SUBBARAO, Distribution of 2-additive functions under some conditions, *Annales Univ. Sci. Budapest, Sect. Comp.* **26** (2006), 137–143.
- [7] P. D. T. A. ELLIOTT, Probabilistic Number Theory, VI, *Berlin*, 1979.

IMRE KÁTAI
 DEPARTMENT OF COMPUTER ALGEBRA
 EÖTVÖS LORÁND UNIVERSITY
 PÁZMÁNY PÉTER SÉTÁNY 1/C
 H-1117 BUDAPEST
 HUNGARY

E-mail: katai@compalg.inf.elte.hu

M. V. SUBBARAO
 DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF ALBERTA
 EDMONTON, ALBERTA T6G 2G1
 CANADA

E-mail: m.v.subbarao@ualberta.ca

(Received February 20, 2007; revised November 8, 2007)